On the Foundations of Fuzzy Dynamical System Theory:
Controllability and Observability

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Abstract

The controllability and the observability of fuzzy dynamical system are studied. The uncertainty of the fuzzy dynamical system is due to the initial condition. The available information on the uncertainty can be described via a membership function. In this paper, we propose a systematic approach to investigate the controllability and the observability of fuzzy dynamical system via the use of the membership function. The concepts of the controllability and the observability in the fuzzy sense are presented via the use of the membership function. The necessary and sufficient conditions for the controllability and the observability of the linear time-variant and time-invariant uncertain system are derived. As a result, the paper lays the foundation for the control of fuzzy dynamical system.

Keywords: Fuzzy dynamical system, fuzzy set theory, controllability, observability, membership function.

1. Introduction

Fuzzy set theory, originally proposed by Zadeh [1], is considered as a rather effective approach in describing the uncertainty of dynamical system, besides the stochastic system theory. Due to the effectiveness to solve the uncertainty of dynamical system, the topic of exploring the control of the fuzzy dynamical system with fuzzy description of system uncertainty has received more and more attention in recent years [2-5]. There have been many successful applications [6-9]. However, it is obvious that many basic issues of fuzzy dynamical system theory remain to be a research blank. The controllability and the observability in the fuzzy sense, as the foundation of fuzzy dynamical system theory, are one of the most important issues that need to be explored in fuzzy dynamical system theory.

In the classical control theory [10-12], the controllability is the characteristic of the system to transfer a crisp initial state to any desired crisp state in a finite time interval by applying an appropriate control input. The observability is the characteristic of the system to estimate the crisp initial state according to the knowledge of the input and the output in a finite time interval. However, when the initial state of fuzzy dynamical system is fuzzy, not deterministic, the concepts of the controllability and the observability in the classical control theory cannot be applied for the fuzzy dynamical system. As a result, it is considered important to research the controllability and the observability of fuzzy dynamical system. However, there are little literatures that have studied the controllability and the observability of the fuzzy dynamical system. For the controllability of fuzzy dynamical system, the literatures [13, 14] investigated the controllability of the uncertain Takagi-Sugeno (T-S) fuzzy-model-based control system, and pointed out that the controllability was the characteristic of the system to transfer a crisp initial state to any desired crisp state by a control input for all fuzzy-rule models. The literature [15] investigated the controllability of the fuzzy dynamical system with the fuzzy input, and presented that the controllability was the characteristic of the system to transfer a crisp initial state to certain fuzzy state by a fuzzy input. For the observability of fuzzy dynamical system, the literatures [16, 17] investigated the observability of the T-S fuzzy-model-based control system, and pointed out that the observability was the characteristic of the system to transfer a crisp initial state according to the knowledge of the fuzzy input and the fuzzy output in a finite time interval for all fuzzy-rule models. The literature [18] investigated the observability of the fuzzy dynamical system with the fuzzy initial state, and pointed out that the observability was the characteristic of the system to estimate the range of the fuzzy initial state according to the knowledge of the fuzzy input and the fuzzy output in a finite time interval.

In this paper, we investigate the controllability and the observability of the fuzzy dynamical system with the fuzzy initial state via the use of the membership function,
which was not available earlier. The main contributions are threefold. First, considering a class of fuzzy dynamical system with the fuzzy initial state which can be prescribed by a membership function, we propose a systematic approach to analyze the controllability and the observability in the fuzzy sense via the use of the membership function. Second, the concepts of the controllability and the observability in the fuzzy sense are presented via the use of the membership function. Third, the necessary and sufficient conditions of the controllability and the observability for the linear time-variant and time-invariant uncertain system are derived. This paper provides a general guideline for engineers to follow in the control of fuzzy dynamical system.

2. Preliminaries

In this section, some fundamental concepts and definitions of fuzzy logic theory [19, 20] are summarized for later purpose.

Fuzzy set: Let \( X \) be a given universal set, \( A \) is called a fuzzy set in \( X \) if there exists a membership function \( \mu_A(x) \) maps elements of the universal set \( X \) into real numbers in \([0,1] \), that is,
\[
\mu_A: X \rightarrow [0,1] \quad (1)
\]
\( \alpha \)-cut: Given a fuzzy set \( A \) defined on \( X \) and its membership function \( \mu_A(x) \), for any number \( \alpha \in [0,1] \), the \( \alpha \)-cut, \( \alpha \)-cut, are the crisp sets
\[
\alpha \text{-cut of a fuzzy set } A = \{ x | \mu_A(x) \geq \alpha \}
\]
That is, the \( \alpha \)-cut of a fuzzy set \( A \) is the crisp set \( \alpha \)-cut that contains all the elements of the universal set \( X \) whose member grades in \( A \) are greater than or equal to the specified value \( \alpha \).

Fuzzy number: Let \( G \) be the fuzzy set in \( R \), \( G \) is called a fuzzy number if: (i) \( G \) is normal, (ii) \( G \) is convex, (iii) the support of \( G \) is bounded, (iv) all \( \alpha \)-cuts are closed intervals in \( R \).

Fuzzy arithmetic: Let \( G \) and \( H \) be two fuzzy numbers and \( \alpha \)-cut of fuzzy number \( G \) is \( \alpha \)-cut of fuzzy number \( G \) and \( \alpha \)-cut of fuzzy number \( H \) is \( \alpha \)-cut of fuzzy number \( H \), \( \alpha \in [0,1] \). The four arithmetic operations on closed intervals, are represented as follows:
\[
\alpha \text{-cut of } (G+H) = \alpha \text{-cut of } (G+H) = \{g_{\alpha} + h_{\alpha} | g_{\alpha}, h_{\alpha} \}
\]
\( \alpha \text{-cut of } (G-H) = \alpha \text{-cut of } (G-H) = \{\min(g_{\alpha} - h_{\alpha}, g_{\alpha} - h_{\alpha})
\]
\( \alpha \text{-cut of } (G \times H) = \alpha \text{-cut of } (G \times H) = \{\min(g_{\alpha} h_{\alpha}, g_{\alpha} h_{\alpha}, g_{\alpha} h_{\alpha}, g_{\alpha} h_{\alpha})
\]

\[
\frac{a G}{H} = \bigg[\min\bigg(\frac{g_{\alpha}}{h_{\alpha}}, \frac{g_{\alpha}}{h_{\alpha}}, \frac{g_{\alpha}}{h_{\alpha}}, \frac{g_{\alpha}}{h_{\alpha}}\bigg), \max\bigg(\frac{g_{\alpha}}{h_{\alpha}}, \frac{g_{\alpha}}{h_{\alpha}}, \frac{g_{\alpha}}{h_{\alpha}}, \frac{g_{\alpha}}{h_{\alpha}}\bigg)\bigg]
\]

Theorem 1 (Decomposition theorem): Define a fuzzy set \( \alpha \) \( \text{in } X \) with the membership function \( \mu_{\alpha}(x) \), where \( I_{\alpha}(x) = 1 \) if \( x \in \alpha \), \( I_{\alpha}(x) = 0 \) if \( x \notin \alpha \), \( X \). Then, the fuzzy set \( \alpha \) can be obtained as
\[
A = \bigcup_{\alpha \in [0,1]} \alpha \quad (7)
\]
where \( \cup \) is the union of the fuzzy sets.

Remark 1: Based on this theorem, after the arithmetic operation of two fuzzy numbers via their \( \alpha \)-cuts, we can rebuild the membership function of the resulting fuzzy number.

3. Controllability of Fuzzy Dynamical System

In this section, we propose a systematic approach to analyze the controllability of fuzzy dynamical system via the use of membership function. The concept of the controllability in the fuzzy sense is presented via the use of the membership function. And the necessary and sufficient conditions for the controllability of the linear time-variant and time-invariant uncertain system are derived.

A. Definition of the controllability in the fuzzy sense

Considering the following nonlinear uncertain system
\[
\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0 \quad (8)
\]
\[
y(t) = h(x(t), u(t), t)
\]
Where \( x(t) \in R^n \) is the state, \( x_0 \) is the initial state, \( y(t) \in R^n \) is the output, \( u(t) \in R^n \) is the input. Suppose that the initial state \( x_0 \) is uncertain, which can be described by a membership function. Then the state \( x(t) \) and the output \( y(t) \) are also uncertain due to the uncertainty of the initial state \( x_0 \).

Definition 1: Considering the fuzzy dynamical system (8), let \( \mu_0(x_0) \) be any known membership function of the initial state \( x_0 \) at time \( t_0 \), \( \hat{\mu}(x(t_0)) \) be the target membership function of the state \( x(t_0) \) at time \( t_0 \). We assume that \( \max(\mu_0(x)) = 1 \) and \( \max(\hat{\mu}(x)) = 1 \). The fuzzy dynamical system is said to be controllable in the sense of \( \hat{\mu}(x(t_0)) \), if there exists a finite \( t_1 > t_0 \) such that for the membership function \( \mu_0(x_0) \) of the initial state \( x_0 \), there exists an input \( u(t) \) that will transfer
the membership function \( \mu_0(x_0) \) of the initial state \( x_0 \)
to the membership function \( \mu(x(t_i)) \) of the state \( x(t_i) \),
and the relationship between \( \mu_i(x(t_i)) \) and \( \hat{\mu}(x(t_i)) \)
satisfies
\[
\mu_i(x(t_i)) \leq \hat{\mu}(x(t_i))
\]
(9)
Remark 2: Note that the membership functions in this paper are all under the constraint \( \max(\mu(x)) = 1 \).
Remark 3: From the Definition 1, we know that the controllability in the fuzzy sense is to transfer the fuzzy set of the initial state \( x_0 \) to the fuzzy set of the state \( x(t_i) \), while the controllability in the classical control theory is to transfer the crisp initial state \( x_0 \) to the crisp state \( x(t_i) \).
Remark 4: The controllability in the classical control theory can be considered as the special case of the controllability in the fuzzy sense when the fuzzy set of the initial state only contains one element. Therefore, compared with the controllability in the classical control theory, the controllability in the fuzzy sense can be applied more widely.

B. Necessary and sufficient conditions for the controllability of linear time-variant uncertain system

The linear time-variant uncertain system is a special case of the nonlinear uncertain system (8). Considering the following linear time-variant uncertain system
\[
x(t) = A(t)x(t) + B(t)u(t), x(t_0) = x_0
\]
y(t) = C(t)x(t)
(10)
where \( x(t) \in R^n \) is the state, \( x_0 \) is the fuzzy initial state, \( y(t) \in R^p \) is the output, \( u(t) \in R^n \) is the input, \( A(t) \in R^{n \times n} \), \( B(t) \in R^{n \times p} \) and \( C(t) \in R^{p \times n} \) are the time-variant matrices. We propose the necessary and sufficient conditions for the controllability of the linear time-variant uncertain system (10) in the following.

Lemma 1: Let \( f_i \in R^{b_i} \), for \( i = 1, 2, \ldots, n \), be the continuous functions defined on \([t_1, t_2]\). Let \( F \in R^{b_1} \) be the matrix with \( f_i \) as its \( i^{th} \) row. We define
\[
W(t_1, t_2) = \int_{t_1}^{t_2} F(t) F^T(t) dt.
\]
Then \( f_1, f_2, \cdots, f_n \) are linearly independent on \([t_1, t_2]\) if and only if the constant matrix \( W(t_1, t_2) \) is nonsingular.

Proof: We prove it by contradiction. Necessity: Suppose that \( f_1, f_2, \cdots, f_n \) are linearly independent on \([t_1, t_2]\), but \( W(t_1, t_2) \) is singular. Then there exists a nonzero row vector \( \alpha \in R^{b_1} \) such that
\[
\alpha W(t_1, t_2) = 0.
\]
As a result,
\[
a W(t_1, t_2) \alpha^T = \int_{t_1}^{t_2} (\alpha F) (\alpha F)^T dt = 0
\]
(11)
This implies that \( \alpha F = 0 \), for all \( t \) in \([t_1, t_2]\), that is, the rows of \( F \) are linearly dependent. This is contradicted with the assumption. Therefore, if \( f_1, f_2, \cdots, f_n \) are linearly independent on \([t_1, t_2]\), then \( W(t_1, t_2) \) is nonsingular.

Sufficiency: Suppose that \( W(t_1, t_2) \) is nonsingular, but \( f_1, f_2, \cdots, f_n \) are linearly dependent on \([t_1, t_2]\). Then there exists a nonzero constant row vector \( \alpha \in R^{b_1} \) such that \( \alpha F = 0 \) for all \( t \) in \([t_1, t_2]\). Consequently, we have
\[
a W(t_1, t_2) = \int_{t_1}^{t_2} \alpha F F^T dt = 0
\]
(12)
which contradicts the assumption that \( W(t_1, t_2) \) is nonsingular. Therefore, if \( W(t_1, t_2) \) is nonsingular, \( f_1, f_2, \cdots, f_n \) are linearly independent on \([t_1, t_2]\).

Lemma 2: Considering two fuzzy sets \( G \) and \( H \), let \( \mu_G(x) \) and \( \mu_H(x) \) denote the membership function of the fuzzy sets \( G \) and \( H \), respectively. Let \( \underline{G}, \overline{G}, \underline{H}, \overline{H} \) denote the lower bound, the upper bound of the fuzzy set \( G \), \( \overline{G}, \underline{H}, \overline{H} \) denote the lower bound, the upper bound of the fuzzy set \( H \). We assume that (i) \( \underline{G} \leq \underline{H}, \overline{G} \geq \overline{H} \), and \( \underline{G} + \overline{G} = \underline{H} + \overline{H} = \Delta \), (ii) the membership functions \( \mu_G(x) \) and \( \mu_H(x) \) are in the same shape (The shape of a membership function refers to its generic geometric nature, such as triangular, trapezoidal, bell-shaped, Gaussian-shaped, and so on. Therefore the same shape means that the membership functions share the same generic geometric nature, for example, they are both triangular), (iii) the membership function \( \mu_G(x) \) is monotone increasing function when \( \underline{G} \leq x \leq \overline{G} \), monotone decreasing function when \( \Delta \leq x \leq \Delta \), monotone decreasing function when \( \Delta \leq x \leq \Delta \), monotone increasing function when \( \underline{H} \leq x \leq \overline{H} \), and the membership function \( \mu_H(x) \) is monotone increasing function when \( \underline{H} \leq x \leq \overline{H} \), monotone decreasing function when \( \Delta \leq x \leq \Delta \), monotone decreasing function when \( \Delta \leq x \leq \Delta \). Then for all \( x \), we have
\[
\mu_G(x) \geq \mu_H(x).
\]
Proof: Since \( \mu_G(x) \) and \( \mu_H(x) \) are in the same shape, we have
\[
\mu_G(x) = \mu_H(x) \left( \frac{x - \underline{G}}{\overline{G} - \underline{G}} (\overline{H} - \underline{H}) + \underline{H} \right)
\]
(13)
\[ \mu_o(x) - \mu_i(x) \]
\[ = \mu_h \left( \frac{x - x_o}{x_o - x_h} (x_h - x_o) + x_h \right) - \mu_h(x) \]  
(14)

Since
\[ \frac{x - x_o}{x_o - x_h} (x_h - x_o) + x_h = (\Delta - x) \left( \frac{(x_o - x_h) - (x_h - x_o)}{x_o - x_h} \right) \]  
(15)

that is,
\[ \frac{x - x_o}{x_o - x_h} (x_h - x_o) + x_h \geq x \quad \text{when} \quad x \leq \Delta, \]  
and
\[ \frac{x - x_o}{x_o - x_h} (x_h - x_o) + x_h \leq x \quad \text{when} \quad x \geq \Delta. \]

Therefore, by the assumption (iii), we have, for all \( x \), \( \mu_o(x) \geq \mu_i(x) \).

**Theorem 2:** Considering the fuzzy dynamical system (10), let \( x_o \) and \( x_h \) be the upper bound and the lower bound of the initial state \( x_o \), \( \bar{x}(t) \) and \( \hat{x}(t) \) be the upper bound and the lower bound of the target state \( x(t) \). Let \( \mu_o(x_o) \) denote the membership function of the initial state \( x_o \), \( \hat{\mu}(x(t)) \) denote the target membership function of the state \( x(t) \). Let \( \phi(t,t_o) \in \mathbb{R}^{n \times n} \) denote the state transition matrix of (10). We assume that \( \mu_o(x_o) \) and \( \hat{\mu}(x(t)) \) are in the same shape, and \( \mu_o(x_o) \) is a monotone increasing function when
\[ x_o \leq x \leq \frac{x_o + x_h}{2}, \]
a monotone decreasing function when
\[ \frac{x_o + x_h}{2} \leq x \leq x_h. \]

Then the fuzzy dynamical system is controllable in the sense of \( \hat{\mu}(x(t)) \) if and only if

(i) The \( n \) rows of the matrix function \( \phi(t,t_o)B(t) \in \mathbb{R}^{n \times m} \) are linearly independent on \([t_o,t_i]\).

(ii) \( \phi(t,t_o)(x_o - x_h) \leq \bar{x}(t) - \hat{x}(t) \).

**Proof:** *Sufficiency:* Since the rows of \( \phi(t,t_o)B(t) \) are linearly independent on \([t_o,t_i]\), by using Lemma 1, the matrix
\[ W(t_o,t_i) = \int_{t_o}^{t_i} \phi(t,t_o)B(\tau)B(\tau)^T \phi(t_o,\tau)^T d\tau \]  
(16)
is nonsingular. According to the classical control theory, the solution of (10) are in the form of
\[ x(t) = \phi(t,t_o)x_o + \int_{t_o}^{t} \phi(t,\tau)B(\tau)u(\tau)d\tau \]  
(17)
\[ y(t) = C(t)\phi(t,t_o)x_o + \int_{t_o}^{t} C(t)\phi(t,\tau)B(\tau)u(\tau)d\tau \]  
(18)

Let \( \underline{x}^\alpha \) be the initial state that make the \( \alpha \)-cut of the state \( x(t_i) \) reach the upper bound \( \overline{x}(t_i) \), and \( \underline{x}^\alpha \) be the initial state that make the \( \alpha \)-cut of the state \( x(t_i) \) reach the lower bound \( \underline{x}(t_i) \), by (17), the \( \alpha \)-cut of the state \( x(t_i) \) takes the form of
\[ \underline{x}(t_i) = \phi(t_i,t_o)^\alpha \underline{x}^\alpha + \int_{t_o}^{t_i} \phi(t_i,\tau)B(\tau)u(\tau)d\tau \]  
(19)

We claim that the control input of fuzzy dynamical system (10) takes the form of
\[ u(t) = B^T(t)\phi^T(t_o,t)W^{-1}(t_o,t_i) \]
\[ x[\phi(t_o,t_i)\overline{x}(t_i) + \hat{x}(t_i) - \underline{x}^\alpha - \underline{x}^\alpha] \]  
(20)

Substituting (20) into (19) and using the Theorem 1, we obtain
\[ \underline{x}(t_i) = \phi(t_i,t_o)^\alpha \underline{x}^\alpha + \frac{\overline{x}(t_i) + \hat{x}(t_i)}{2} - \phi(t_i,t_o)^\alpha \underline{x}^\alpha + \frac{\underline{x}^\alpha + \underline{x}^\alpha}{2} \]  
(21)

As a result, we have
\[ \overline{x}(t_i) + \underline{x}(t_i) = \frac{\overline{x}(t_i) + \hat{x}(t_i)}{2} \]  
(23)

Since \( \phi(t_i,t_o)(x_o - x_h) \leq \bar{x}(t) - \hat{x}(t) \), then
\[ \overline{x}(t_i) \leq \frac{\overline{x}(t_i) - \hat{x}(t_i)}{2} + \frac{\hat{x}(t_i) + \bar{x}(t_i)}{2} = \overline{x}(t_i) \]  
(24)

Therefore, by using the Lemma 2, we have \( \mu(x(t_i)) \leq \hat{\mu}(x(t_i)) \), where \( \mu(x(t)) \) is the membership function of the state \( x(t) \) transferred from the membership function of the initial state \( x_o \). That is, if the fuzzy dynamical system (10) satisfies the conditions (i) and (ii), then the system is controllable in the sense of \( \hat{\mu}(x(t)) \).

*Necessity:* We prove it by contradiction. Firstly, the condition (i) of Theorem 2 is considered. Suppose the fuzzy dynamical system is controllable, but the rows of \( \phi(t,t_o)B(t) \) are linearly dependent on \([t_o,t_i]\). Then there exists a nonzero constant row vector \( \alpha \in \mathbb{R}^{n \times n} \) such that
\[ \alpha \phi(t_o,t_o)B(t) = 0 \]  
(25)

For the fuzzy dynamical system (10), the deterministic
initial state can be considered as the special case of the fuzzy initial state. As a result, let us choose the deterministic initial state \( x_0 = \alpha^T \). Then (17) can be described as

\[
x(t) = \phi(t, t_0)\alpha^T + \int_{t_0}^{t} \phi(t, \tau)B(\tau)u(\tau)d\tau
\]  

(26)

or

\[
\phi(t_0, t_1)x(t_1) = \alpha^T + \int_{t_0}^{t_1} \phi(t_0, \tau)B(\tau)u(\tau)d\tau
\]  

(27)

Pre-multiplying both side of (27) by \( \alpha \), we have

\[
\alpha \phi(t_0, t_1)x(t_1) = \alpha \alpha^T + \int_{t_0}^{t_1} \alpha \phi(t_0, \tau)B(\tau)u(\tau)d\tau
\]  

(28)

By hypothesis, the fuzzy dynamical system is controllable. Hence, there exists a control input \( u(t) \) such that \( x(t_1) = 0 \). Substituting (25) into (28), we obtain

\[
\alpha \alpha^T = 0
\]  

(29)

which implies that \( \alpha = 0 \). This is contradicted with the assumption. Therefore, if the fuzzy dynamical system is controllable, the rows of \( \phi(t_0, t)B(t) \) are linearly independent on \([t_0, t_1]\).

Secondly, the condition (ii) of the Theorem 2 will be considered. Suppose the fuzzy dynamical system is controllable, but \( \phi(t_0, t_0)(\xi_0 - \xi_0) \). According to (22), we have

\[
\begin{align*}
\overline{x}(t_1) &\geq \frac{\overline{x}(t_0) - \overline{x}(t_1)}{2} + \frac{\overline{x}(t_1) + \overline{x}(t_0)}{2} = \overline{x}(t_1) \\
\overline{x}(t_1) &\leq \frac{\overline{x}(t_0) - \overline{x}(t_1)}{2} + \frac{\overline{x}(t_1) + \overline{x}(t_0)}{2} = \overline{x}(t_1)
\end{align*}
\]  

(30)

By using the Lemma 2, we have \( \mu(x(t_1)) \geq \mu(x(t_1)) \), which is contradiction with the assumption that the fuzzy dynamical system is controllable. Therefore, if the fuzzy dynamical system is controllable, then \( \phi(t_0, t_0)(\xi_0 - \xi_0) \leq \overline{x}(t_1) - \overline{x}(t_1) \).

Thus, the proof is completed.

Remark 5: For the linear time-variant uncertain system (10), when the initial state \( x_0 \) is deterministic, the condition (ii) in the Theorem 2 is always true. That is, when the initial state \( x_0 \) is deterministic, the Theorem 2 is equivalent to the necessary and sufficient conditions for the controllability in the classical control theory. That is, the Theorem 2 can be used to examine the controllability in the classical control theory. Therefore, the Theorem 2 can be applied much more widely than the necessary and sufficient conditions for the controllability in the classical control theory.

Remark 6: Note that linear time-invariant (LTI) system is a special case of linear time-variant (LTV) system. Therefore, Theorem 2 can also be used to verify the controllability of LTI system. However, LTI systems have some unique features, which time-variant (LTV) systems do not share. The next section endeavors to study the controllability of LTI systems by using these unique features, which cannot be applied to LTV systems.

C. Necessary and Sufficient conditions for the controllability of linear time-invariant uncertain system

In this subsection, we will study the controllability of linear time-invariant uncertain system as follow,

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0
\]  

(31)

where \( x(t) \in R^n \) is the state, \( x_0 \) is the fuzzy initial state. As a result, let us choose the deterministic initial state, \( y(t) \in R^p \) is the output, \( u(t) \in R^n \) is the input, \( A \in R^{nxn} \), \( B \in R^{nxm} \) and \( C \in R^{pnx} \) are the constant matrices.

Lemma 3: Let \( f_i \in R^{nxn} \), for \( i = 1,2,\cdots,n \), be the continuous functions defined on \([t_1, t_2]\). Let \( F^{(k)} \) be the matrix with\( f^k \) as its \( k \)th row. Let \( F^{(k)} \) be the derivative of \( F \) and \( t_0 \) be any fixed point in \([t_1, t_2]\). Then \( f_1, f_2, \cdots, f_n \) are linearly independent on \([t_1, t_2]\) if and only if \( \text{rank}[F(t_0), F^{(1)}(t_0), \cdots, F^{(n-1)}(t_0), \cdots] = n \).

Proof: We prove it by contradiction

Sufficiency: Suppose that there exists some \( t_0 \) in \([t_1, t_2]\) such that

\[
\text{rank}[F(t_0), F^{(1)}(t_0), \cdots, F^{(n-1)}(t_0)] = n
\]  

(32)

and the \( f_i \)’s are linearly dependent on \([t_1, t_2]\). Then by definition, there exists a nonzero row vector \( \alpha \in R^{nxn} \) such that, for all \( t \) in \([t_1, t_2]\), \( \alpha F(t) = 0 \). This implies that, for all \( t \) in \([t_1, t_2]\) and \( k = 1,2,\cdots,n-1 \), \( \alpha F^{(k)}(t) = 0 \). Therefore, we have, for all \( t \) in \([t_1, t_2]\),

\[
\alpha[F(t), F^{(1)}(t), \cdots, F^{(n-1)}(t)] = 0
\]  

(33)

in particular,

\[
\alpha[F(t_0), F^{(1)}(t_0), \cdots, F^{(n-1)}(t_0)] = 0
\]  

(34)

which implies that all the \( n \) rows of \([F(t_0), F^{(1)}(t_0), \cdots, F^{(n-1)}(t_0)] \) are linearly dependent. This contradicts the hypothesis that \([F(t_0), F^{(1)}(t_0), \cdots, F^{(n-1)}(t_0)] \) has rank \( n \). Therefore, the \( f_i \)’s are linearly independent on \([t_1, t_2]\).

Necessity: Suppose that

\[
\text{rank}[F(t_0), F^{(1)}(t_0), \cdots, F^{(n-1)}(t_0)] = n
\]  

(35)

then the rows of the infinite matrix \([F(t_0), F^{(1)}(t_0), \cdots, F^{(n-1)}(t_0), \cdots] \) are linearly dependent. Consequently, there exists a nonzero row vector \( \alpha \in R^{nxn} \) such that

\[
\alpha[F(t_0), F^{(1)}(t_0), \cdots, F^{(n-1)}(t_0), \cdots] = 0
\]  

(36)

The \( f_i \)’s are analytic on \([t_1, t_2]\) by assumption.
Therefore, there exists an \( \varepsilon > 0 \) such that, for all \( t \) in \([t_0 - \varepsilon, t_0 + \varepsilon]\), \( F(t) \) can be represented as a Taylor series about the point \( t_0 \):

\[
F(t) = \sum_{n=0}^{\infty} \frac{(t - t_0)^n}{n!} F^{(n)}(t_0)
\]

Pre-multiplying \( \alpha \) on both sides of (37) and using (36), we obtain, for all \( t \) in \([t_0 - \varepsilon, t_0 + \varepsilon]\), \( \alpha F(t) = 0 \).

Since the sum of analytic functions is an analytic function, the analyticity assumption of the \( f_i \)'s implies that \( \alpha F(t) \) as a row vector function is analytic over \([t_1, t_2]\).

Consequently, the equation \( \alpha F(t) = 0 \) for all \( t \) in \([t_0 - \varepsilon, t_0 + \varepsilon]\), implies that \( \alpha F(t) = 0 \) for all \( t \) in \([t_1, t_2]\), or, equivalently, the \( f_i \)'s are linearly dependent on \([t_1, t_2]\). This is a contradiction. Therefore, if the \( f_i \)'s are linearly independent on \([t_1, t_2]\), then \([F(t_0), F^{(1)}(t_0), \ldots, F^{(n-1)}(t_0), \ldots] \) has rank \( n \).

Thus, the proof is completed.

**Theorem 3:** Considering the fuzzy dynamical system (31), let \( \tilde{x}_0 \) and \( \underline{x}_0 \) be the upper bound and the lower bound of the initial state \( x_0 \), \( \tilde{x}(t) \) and \( \underline{x}(t) \) be the upper bound and the lower bound of the target state \( x(t) \). Let \( \mu_\alpha(x_0) \) denote the membership function of the initial state \( x_0 \), \( \mu_\alpha(x(t)) \) denote the target membership function of the state \( x(t) \). We assume that \( \mu_\alpha(x_0) \) and \( \mu_\alpha(x(t)) \) are in the same shape, and \( \mu_\alpha(x_0) \) is a monotone increasing function when

\[
\frac{\tilde{x}_0 - x_0}{2} \leq x \leq \underline{x}_0,
\]

a monotone decreasing function when

\[
\frac{x_0 + \underline{x}_0}{2} \leq x \leq \tilde{x}_0.
\]

Then the fuzzy dynamical system is controllable in the sense of \( \mu_\alpha(x(t)) \) if and only if any of the following equivalent conditions is satisfied:

1. All rows of \( e^{(t-i)\lambda}B \) are linearly independent on \([t_0, \infty)\), and \( e^{(t-i)\lambda} (\tilde{x}_0 - x_0) \leq \tilde{x}(t) - \underline{x}(t) \).
2. All rows of \( (sI - A)^{-1}B \) are linearly independent on \([t_0, \infty)\), and \( e^{(t-i)\lambda} (\tilde{x}_0 - x_0) \leq \tilde{x}(t) - \underline{x}(t) \).
3. The controllability gramian \( W(t_0,t) = \int_{t_0}^t e^{A(t-s)}B e^{A(t-s)}ds \) is nonsingular for any \( t > t_0 \), and \( e^{(t-i)\lambda} (\tilde{x}_0 - x_0) \leq \tilde{x}(t) - \underline{x}(t) \).
4. The controllability matrix \( U = [B, AB, \ldots, A^{n-1}B] \) has rank \( n \), and \( e^{(t-i)\lambda} (\tilde{x}_0 - x_0) \leq \tilde{x}(t) - \underline{x}(t) \).
5. For every eigenvalue \( \lambda \) of \( A \), the complex matrix \( [\lambda I - A, B] \) has rank \( n \), and \( e^{(t-i)\lambda} (\tilde{x}_0 - x_0) \leq \tilde{x}(t) - \underline{x}(t) \).

**Proof:**

1. For the linear time-invariant system, \( \phi(t, t_0) = e^{A(t-t_0)} \), according to the Theorem 2, the statement 1 can be proved.

2. Taking the Laplace transform of \( e^{(t-i)\lambda}B \), we have \( \zeta e^{(t-i)\lambda}B = (sI - A)^{-1}B \). Since the Laplace transform is a one-to-one linear operator, if all rows of \( e^{(t-i)\lambda}B \) are linearly independent on \([t_0, \infty)\), then all rows of \( (sI - A)^{-1}B \) are linearly independent on \([t_0, \infty)\), and vice versa. Therefore, the statement 2 is equivalent to the statement 1.

3. Since \( \phi(t, t_0) e^{(t-i)\lambda} \), by using the Lemma 1, we conclude that the statement 3 is equivalent to the statement 1.

4. According to the Lemma 3, the rows of \( e^{(t-i)\lambda}B \) are linearly independent if and only if

\[
\text{rank}[e^{(t-i)\lambda}B, e^{(t-i)\lambda}AB, \ldots, e^{(t-i)\lambda}A^{n-1}B, \ldots] = n
\]

(38)

Let \( t = t_0 \), (38) can be represented as

\[
\text{rank}[B, AB, \ldots, A^{n-1}B, \ldots] = n
\]

(39)

From the Cayley-Hamilton theorem, we know that \( A^i \) with \( i \geq n \) can be written as a linear combination of \( I, A, \ldots, A^{n-1} \). Therefore, the columns of \( A^{i}B \) with \( i \geq n \) are linearly dependent on the columns of \( B, AB, \ldots, A^{n-1}B \). Consequently,

\[
\text{rank}[B, AB, \ldots, A^{n-1}B, \ldots] = \text{rank}[B, AB, \ldots, A^{n-1}B] \]

(40)

Therefore, the rows of \( e^{(t-i)\lambda}B \) are linearly independent if and only if \( \text{rank}[B, AB, \ldots, A^{n-1}B] = n \). That is, the statement 4 is equivalent to the statement 1.

5. **Necessity:** According to the Theorem 2, we know that if the fuzzy dynamical system is controllable, then \( e^{(t-i)\lambda} (\tilde{x}_0 - x_0) \leq \tilde{x}(t) - \underline{x}(t) \). Next, we will show that if the system is controllable, then \( \text{rank}[\lambda I - A, B] = n \) at every eigenvalue of \( A \). If not, then there exist an eigenvalue \( \lambda \) and a nonzero vector \( \alpha \in \mathbb{R}^n \) such that

\[
\alpha[\lambda I - A, B] = 0
\]

(41)

or

\[
\alpha \lambda = \alpha A \quad \text{and} \quad \alpha B = 0
\]

(42)

which imply \( \alpha A^i = \mathcal{A} \alpha \), for \( i = 1, 2, \ldots \). Hence, we have

\[
\{B, AB, \ldots, A^{n-1}B\} \subset \{AB, \alpha AB, \ldots, \mathcal{A}^{n-1}\alpha B\} = 0
\]

(43)

This contradicts the assumption of the controllability. Therefore, if the system is controllable, then \( \text{rank}[\lambda I - A, B] = n \) at every eigenvalue of \( A \). That is, if the system is controllable, the matrix \([\lambda I - A, B] \) has rank \( n \), and \( e^{(t-i)\lambda} (\tilde{x}_0 - x_0) \leq \tilde{x}(t) - \underline{x}(t) \).

**Sufficiency:** According to the Theorem 2, we know that
if \( e^{A(t-t_0)}(x_0-x_n) \geq \bar{x}(t_0) - \tilde{x}(t_i) \), then the system is not controllable. That is, \( e^{A(t-t_0)}(x_0-x_n) \leq \bar{x}(t_0) - \tilde{x}(t_i) \) is one of the sufficient conditions that the system is controllable. Next, we will prove that \( \text{rank}[A - \lambda I, B] = n \) is the sufficient condition that the system is controllable under the assumption of \( \lambda \) is a nonsingular, \( \lambda = \lambda B \). Now we form a vector \( \alpha = [0 \beta] \in \mathbb{R}^n \). Then we have

\[
\alpha[A - \lambda I, B] = [0 \beta] \begin{bmatrix} \lambda I - A'_{11} & -A'_{12} & B'_{1} \\ 0 & \lambda I - A'_{22} & 0 \end{bmatrix} = 0
\]

which implies

\[
0 = \alpha[P(\lambda I - A)^{-1}, PB] = \alpha[P(\lambda I - A)^{-1}, B]
\]

(46)

Since \( \alpha \neq 0 \), we have \( \alpha' = \alpha P^{-1} B \). Because \( P^{-1} \) is nonsingular, \( \alpha' = \alpha P^{-1} \). Hence we have \( \alpha'[\lambda I - A, B] = 0 \). In other words, if the system is not controllable, then \( \text{rank}[\lambda I - A, B] < n \) for some eigenvalue \( \lambda \) of \( A \). That is, if for every eigenvalue \( \lambda \) of \( A \) the complex matrix \( [\lambda I - A, B] \) has rank \( n \), and \( e^{A(t-t_0)}(x_0-x_n) \leq \bar{x}(t_0) - \tilde{x}(t_i) \), then the system is controllable.

**Remark 7:** For the linear time-invariant uncertain system (31), when the initial state \( x_0 \) is deterministic, \( e^{A(t-t_0)}(x_0-x_n) \leq \bar{x}(t_0) - \tilde{x}(t_i) \) is always true. That is, when the initial state \( x_0 \) is deterministic, the Theorem 3 is equivalent to the necessary and sufficient conditions for the controllability in the classical control theory. That is, the Theorem 3 can be used to examine the controllability in the classical control theory.

**D. First example**

In the following, we will take a linear time-invariant uncertain system for example to illustrate the controllability of the fuzzy dynamical system. Consider the fuzzy dynamical system (31), let

\[
A = \begin{bmatrix} 0.1 & 0.1 \\ 0.05 & 0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix}, \quad C = [1 1] \quad (47)
\]

We assume that the membership function \( \mu_t(x_0) = [\mu_{0t}(x), \mu_{0t}(x)] \) of the initial state at time \( t_0 = 0 \) and the target membership function \( \hat{\mu}(x(t_0)) = [\hat{\mu}_{1t}(x), \hat{\mu}_{2t}(x)] \) of the state at time \( t_i = 5 \) can be represented as

\[
\mu_{0t}(x) = \begin{cases} \frac{1}{2}(x - 1), & 1 \leq x < 3 \\ \frac{1}{2}(5 - x), & 3 \leq x < 5 \\ 0, & \text{otherwise} \end{cases}
\]

(48)

\[
\hat{\mu}_{1t}(x) = \begin{cases} \frac{1}{6}(x - 144), & 144 \leq x < 150 \\ \frac{1}{6}(156 - x), & 150 \leq x < 156 \\ 0, & \text{otherwise} \end{cases}
\]

(49)

As a result,

\[
e^{A(t-t_0)}(x_0-x_n) = \begin{bmatrix} 1.7726 & 0.5459 \\ 1.0919 & 2.8645 \end{bmatrix} = \begin{bmatrix} 9.2744 \\ 7.9128 \end{bmatrix} \quad (50)
\]

and the rows of \( e^{A(t-t_0)}B(t) \) are linearly independent. Then by using the Theorem 3, we conclude that the fuzzy dynamical system is controllable.

In order to verify the controllability of the described fuzzy dynamical system, let the control \( u(t) \) in the form of (20). We conduct the numerical simulation of the fuzzy dynamical system (31). Figure 1 shows the actual membership function \( u_t(x, t_i) \) and the target membership function \( \hat{u}_t(x, t_i) \) of the state \( x(t_i) \). Figure 2 shows the actual membership function \( u_t(x, t_i) \) and the target membership function \( \hat{u}_t(x, t_i) \) of the state \( x(t_i) \). The simulation result shows that the fuzzy dynamical system is controllable.

**Remark 8:** Controllability is the basic concept in the control of dynamic system. This paper proposes a systematic approach to analyze the controllability of the fuzzy dynamical system via the use of membership function, which was not available earlier. This provides a general guideline for engineers to follow in the control of fuzzy dynamical system.
Remark 9: The Theorem 2 and the Theorem 3 are only applied for the fuzzy dynamical system that \( u_0(x_0) \) and \( \hat{u}(x(t_i)) \) are in the same shape, where \( u_0(x_0) \) and \( \hat{u}(x(t_i)) \) are the membership functions of the initial state and the target state respectively. If \( u_0(x_0) \) and \( \hat{u}(x(t_i)) \) are in different shapes, we can transfer \( \hat{u}(x(t_i)) \) to a new membership function \( \tilde{u}(x(t_i)) \) in the same shape with \( u_0(x_0) \), and \( \tilde{u}(x(t_i)) \leq \tilde{\mu}(x(t_i)) \) for all \( x(t_i) \). After the transformation, the Theorem 2 and the Theorem 3 can be used to verify the controllability of the fuzzy dynamical system.

Figure 1. The actual and target membership function of the state \( x_i(t_i) \).

Figure 2. The actual and target membership function of the state \( x_0(t) \).

4. Observability of Fuzzy Dynamical System

Observability is the basic concept in the fuzzy dynamical system theory. In this section, we propose a systematic method to analyze the observability of fuzzy dynamical system via the use of membership function. The concept of the observability in the fuzzy sense is presented via the use of the membership function. And the necessary and sufficient conditions for the observability of the linear time-variant and time-invariant uncertain system are derived.

A. Definition of the observability in the fuzzy sense

Definition 2: Considering the fuzzy dynamical system (8), we assume that the output \( y(t) \) and the input \( u(t) \) is measurable. Let \( u_0(x_0) \) be the actual membership function of the initial state \( x_0 \) at time \( t_0 \). The fuzzy dynamical system is said to be observability in the fuzzy sense, if there exists a finite \( t_1 > t_0 \) such that for the membership function \( u_0(x_0) \) of the initial state \( x_0 \), according to the knowledge of the input \( u_0[x_{t_0},t_1] \) and the output \( y_{[t_0,t_1]} \) over the time interval \( [t_0, t_1] \), we can obtain the estimated membership function \( \bar{\mu}(x_0) \) of the initial state \( x_0 \). For all \( x_0 \), the estimated membership function \( \bar{\mu}(x_0) \) satisfies

\[
\bar{\mu}(x_0) \geq \mu_0(x_0)
\]

Remark 10: This definition suggests that as long as the estimated membership function dominates the actual membership function, the associated estimated fuzzy set subsumes the actual fuzzy set as a subset, which in turn means the estimated fuzzy set indeed captures the actual fuzzy set.

Remark 11: According to the Definition 2, we know that the observability in the fuzzy sense is to estimate the fuzzy set of the initial state \( x_0 \) by the knowledge of the output \( y(t) \) and the input \( u(t) \), while the observability in the classical control theory is to estimate the crisp initial state \( x_0 \).

Remark 12: The observability in the classical control theory can be considered as the special case of the observability in the fuzzy sense when the fuzzy set of the initial state only contains one element. Therefore, compared with the observability in the classical control theory, the concept of the observability in the fuzzy sense can be applied more widely.

B. Necessary and sufficient conditions of the observability in the fuzzy sense

In the following, we will propose the necessary and sufficient conditions for the observability of the linear time-variant uncertain system (10).

Theorem 4: Consider the linear time-variant uncertain system (10), we assume that the output \( y(t) \) and the input \( u(t) \) is measurable. Then the fuzzy dynamical system is observable at \( t_0 \) if and only if there exists a finite \( t_1 > t_0 \) such that the \( n \) columns of the matrix function \( C(t)\phi(t,t_0) \in \mathbb{R}^{n \times n} \) are linearly independent on \([t_0, t_1]\).
Proof: Sufficiency: Pre-multiplying $\phi^T(t,t_0)C(t)$ on both side of (18) and integrating from $t_0$ to $t$, we have

$$\int_{t_0}^{t} \phi^T(t,t_0)C(t)y'(t)dt = V(t_0,t_1)x_0$$

(52)

where

$$y'(t) = y(t) - C(t)I^T(\phi(t,\tau)B(\tau)u(\tau)d\tau$$

(53)

$$V(t_0,t_1) = \int_{t_0}^{t} \phi^T(t,t_0)C(t)\phi(t,t_0)dt$$

(54)

According to the Lemma 1 and the assumption that all the columns of $C(t)\phi(t,t_0)$ are linearly independent on $[t_0,t_1]$, we conclude that $V(t_0,t_1)$ is nonsingular. Therefore, (52) can be represented as

$$x_0 = V^{-1}(t_0,t_1)\int_{t_0}^{t} \phi^T(t,t_0)C(t)y'(t)dt$$

(55)

Let

$$C(t)\phi(t,t_0) = \begin{bmatrix}
d_{11}(t) & d_{12}(t) & \cdots & d_{1n}(t) \\
d_{21}(t) & d_{22}(t) & \cdots & d_{2n}(t) \\
\vdots & \vdots & \ddots & \vdots \\
d_{p1}(t) & d_{p2}(t) & \cdots & d_{pn}(t) \\
\end{bmatrix}$$

(56)

$$V^{-1}(t_0,t_1) = \begin{bmatrix}
v_{11}(t) & v_{12}(t) & \cdots & v_{1n}(t) \\
v_{21}(t) & v_{22}(t) & \cdots & v_{2n}(t) \\
\vdots & \vdots & \ddots & \vdots \\
v_{n1}(t) & v_{n2}(t) & \cdots & v_{nn}(t) \\
\end{bmatrix}$$

(57)

$$C(t)\int_{t_0}^{t} \phi(t,\tau)B(\tau)u(\tau)d\tau = \begin{bmatrix}
u_1(t) \\
u_2(t) \\
\vdots \\
u_p(t) \\
\end{bmatrix}$$

(58)

In order to estimate the fuzzy set of the initial state $x_0$, the fuzzy set of the output $y(t)$ should be solved first. We assume that the actual membership function of the initial state is $u_0(x_0)$, the lower bound and the upper bound of the $\alpha$-cut of the initial state are $\alpha^{-}x_0$ and $\alpha^{+}x_0$ respectively. Then by (18), the $\alpha$-cut of the output $y(t)$ can be represented as

$$\alpha^{-}y_i(t) = \sum_{j=1}^{p} \max(d_{ij}(t)^{\alpha^{-}}x_{0j},d_{ij}(t)^{\alpha^{+}}x_{0j}) + u_i(t)$$

(59)

$$\alpha^{+}y_i(t) = \sum_{j=1}^{p} \min(d_{ij}(t)^{\alpha^{-}}x_{0j},d_{ij}(t)^{\alpha^{+}}x_{0j}) + u_i(t)$$

for $i = 1, 2, \cdots, p$. As a result, the $\alpha$-cut of $y(t)$ take the form of, for $i = 1, 2, \cdots, p$,

$$\alpha^{-}y_i(t) = \sum_{j=1}^{p} \max(d_{ij}(t)^{\alpha^{-}}x_{0j},d_{ij}(t)^{\alpha^{+}}x_{0j})$$

(60)

Next, we will use (59) and (60) for the estimation of the fuzzy initial state $\tilde{x}_0$. According to (55), the estimated initial state $\tilde{x}_0$ can be represented as

$$\tilde{x}_0 = \begin{bmatrix}
v_{11}(t) & v_{12}(t) & \cdots & v_{1n}(t) \\
v_{21}(t) & v_{22}(t) & \cdots & v_{2n}(t) \\
\vdots & \vdots & \ddots & \vdots \\
v_{n1}(t) & v_{n2}(t) & \cdots & v_{nn}(t) \\
\end{bmatrix}\times$$

$$\begin{bmatrix}
d_{11}(t) & d_{12}(t) & \cdots & d_{1n}(t) \\
d_{21}(t) & d_{22}(t) & \cdots & d_{2n}(t) \\
\vdots & \vdots & \ddots & \vdots \\
d_{n1}(t) & d_{n2}(t) & \cdots & d_{nn}(t) \\
\end{bmatrix}$$

(61)

Hence, for $l = 1, 2, \cdots, n$, the $\alpha$-cut of the estimated initial state $\tilde{x}_0$ is in the form of

$$\alpha^{-}\tilde{x}_0 = \sum_{l=1}^{n} \max(v_{l1}(t),t)\int_{t_0}^{t} \sum_{k=1}^{p} d_{lk}(t)^{\alpha^{-}}y_k(t)dt$$

(62)

$$\alpha^{+}\tilde{x}_0 = \sum_{l=1}^{n} \max(v_{l1}(t),t)\int_{t_0}^{t} \sum_{k=1}^{p} d_{lk}(t)^{\alpha^{+}}y_k(t)dt$$

(63)

Substituting (60) into (62) and (63), for $l = 1, 2, \cdots, n$, we have

$$\alpha^{-}\tilde{x}_0 = \sum_{l=1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \max(v_{l1}(t),t)\int_{t_0}^{t} d_{lk}(t)d_{kj}(t)d_{ij}(t)^{\alpha^{-}}x_{0j}dt$$

(64)

$$\alpha^{+}\tilde{x}_0 = \sum_{l=1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \min(v_{l1}(t),t)\int_{t_0}^{t} d_{lk}(t)d_{kj}(t)d_{ij}(t)^{\alpha^{+}}x_{0j}dt$$

if $v_{l1}(t)\int_{t_0}^{t} d_{lk}(t)d_{kj}(t)d_{ij}(t)dt \geq 0$

if $v_{l1}(t)\int_{t_0}^{t} d_{lk}(t)d_{kj}(t)d_{ij}(t)dt \leq 0$
According to (64) and (65), we assume that
\[ v_i(t) \int_0^t d_{ij}(t)d_{ij}(t)dt \leq 0 \quad \text{when} \quad i = i_k, k = k_j, j = j_i, \] and
\[ i_k, j_i \] are the subsets of \( \{1, 2, \cdots, n\} \), \( k_i \) is the subset of \( \{1, 2, \cdots, p\} \).

Since \( V^{-1}(t_0, t_1)V(t_0, t_1) = I \), then
\[
\begin{bmatrix}
 v_{i1}(t) & v_{i2}(t) & \cdots & v_{in}(t) \\
 v_{21}(t) & v_{22}(t) & \cdots & v_{2n}(t) \\
 \vdots & \vdots & \ddots & \vdots \\
 v_{n1}(t) & v_{n2}(t) & \cdots & v_{nn}(t)
\end{bmatrix}
\begin{bmatrix}
 d_{11}(t) & d_{12}(t) & \cdots & d_{1p}(t) \\
 d_{21}(t) & d_{22}(t) & \cdots & d_{2p}(t) \\
 \vdots & \vdots & \ddots & \vdots \\
 d_{p1}(t) & d_{p2}(t) & \cdots & d_{pp}(t)
\end{bmatrix}
dt = I \tag{66}
\]
That is, for \( \alpha = 1, 2, \cdots, n \), and \( \beta = 1, 2, \cdots, n \),
\[
\sum_{i=1}^{n} v_{i\alpha}(t) \int_0^t d_{ij}(t)d_{ij}(t)dt = \sum_{i=1}^{n} v_{i\alpha}(t) \int_0^t d_{ij}(t)d_{ij}(t)dt
= \begin{cases} 1, & \text{for } \alpha = \beta \\ 0, & \text{for } \alpha \neq \beta \end{cases} \tag{67}
\]
As a result, by using (67), for \( l = 1, 2, \cdots, n \), (64) and (65) can be described as
\[
\begin{align*}
\bar{a} \leq \bar{x}_0 &= \bar{x}_0 - \sum_{i=1, k=1, j=1}^{n} v_i(t) \int_0^t d_{ij}(t)d_{ij}(t)dt(\bar{a} \bar{x}_0 - \bar{a} \bar{x}_0), \\
\bar{a} \geq \bar{x}_0 &= \bar{x}_0 - \sum_{i=1, k=1, j=1}^{n} v_i(t) \int_0^t d_{ij}(t)d_{ij}(t)dt(\bar{a} \bar{x}_0 - \bar{a} \bar{x}_0).
\end{align*} \tag{68}
\]
Since
\[
\sum_{i=1, k=1, j=1}^{n} v_i(t) \int_0^t d_{ij}(t)d_{ij}(t)dt(\bar{a} \bar{x}_0 - \bar{a} \bar{x}_0) \leq 0 \tag{69}
\]
Then, by (68), we have, for \( l = 1, 2, \cdots, n \),
\[
\begin{align*}
\bar{a} \leq \bar{x}_0 &\leq \bar{x}_0, \\
\bar{a} \geq \bar{x}_0 &\geq \bar{x}_0, \\
\bar{a} \bar{x}_0 + \bar{a} \bar{x}_0 &= \bar{x}_0 + \bar{x}_0, \\
\frac{\bar{a} \bar{x}_0 + \bar{a} \bar{x}_0}{2} &= \frac{\bar{x}_0 + \bar{x}_0}{2} \tag{70}
\end{align*}
\]
By using the Theorem 1, we obtain
\[
\bar{x}_0 + \bar{x}_0 \leq \bar{x}_0 + \bar{x}_0 \tag{71}
\]
\[
\bar{x}_0 + \bar{x}_0 = \bar{x}_0 + \bar{x}_0 \tag{72}
\]
As a result, by using the Lemma 2, we conclude that \( \bar{a}(\bar{x}_0) \geq \bar{a}(\bar{x}_0) \). Therefore, if all the columns of \( C(t)\phi(t, t_0) \) are linearly independent on \( [t_0, t_1] \), the fuzzy dynamical system is observable.

Necessity: We prove it by contradiction. Suppose the fuzzy dynamical system is observable, but the columns of \( C(t)\phi(t, t_0) \) are linearly dependent on \( [t_0, t_1] \). Then there exists an nonzero constant vector \( \alpha \in R^m \) such that
\[
C(t)\phi(t, t_0)\alpha = 0 \tag{73}
\]
For the fuzzy dynamical system (10), the deterministic initial state can be considered as the special case of the fuzzy initial state. Let us choose the initial state \( x_0 = \alpha \), then
\[
y'(t) = C(t)\phi(t, t_0)\alpha = 0 \tag{74}
\]
As a result, the initial state \( x_0 \) cannot be obtained. This is contradicted with the assumption. Therefore, if the fuzzy dynamical system is observable, all the columns of \( C(t)\phi(t, t_0) \) are linearly independent on \( [t_0, t_1] \).

Thus, the proof is completed.

Remark 13: For the linear time-variant system, the necessary and sufficient conditions for the observability in the fuzzy sense are equivalent to that in the classical control theory. That is, this paper broadens the application range of the necessary and sufficient conditions of the observability in the classical control theory.

C. Necessary and Sufficient conditions for the observability of linear time-invariant uncertain system

Considering the linear time-invariant uncertain system (31), we propose the necessary and sufficient conditions for the observability of this system in this subsection.

Theorem 5: Consider the linear time-invariant uncertain system (31), we assume that the output \( y(t) \) and the input \( u(t) \) are measurable. Then the fuzzy dynamical system is observable if and only if any of the following equivalent conditions is satisfied:

(1) All columns of \( Ce^{A(t-t_0)} \) are linearly independent.
(2) All columns of \( C(sI - A)^{-1} \) are linearly independent.
(3) The observability grammian
\[
V(t_0, t_1) = \int_{t_0}^{t_1} e^{A(t-t_0)}C^T Ce^{A(t-t_0)} dt \tag{75}
\]
is nonsingular for any $t > t_0$.

(4) The observability matrix

$$L = \begin{bmatrix} C & CA & \cdots & CA^{n-1} \end{bmatrix}$$

has rank $n$.

(5) For every eigenvalue $\lambda$ of $A$, the matrix

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$$

has rank $n$.

Proof: (1) For the linear time-invariant uncertain system (31), $\phi(t,t_0) = e^{t(t-t_0)}$. According to the Theorem 4, the statement 1 can be proved.

(2) Taking the Laplace transform of $Ce^{t(t-t_0)}$, we have $\mathcal{L}[Ce^{t(t-t_0)}] = C(sI - A)^{-1}$. Since the Laplace transform is a one-to-one linear operator, if the columns of $Ce^{t(t-t_0)}$ are linearly independent on $[t_0, \infty)$, so are all the columns of $C(sI - A)^{-1}$, and vice versa. Therefore, the statement 2 is equivalent to the statement 1.

(3) Since $\phi(t,t_0) = e^{t(t-t_0)}$, by using the Lemma 1, we conclude that the statement 3 is equivalent to the statement 1.

(4) Since $\text{rank}(L) = \text{rank}(L^r)$, then we will prove $\text{rank}(L^r) = n$. The Lemma 3 implies that the columns of $Ce^{t(t-t_0)}$ are linearly independent on $[t_0, \infty)$ if and only if

$$\text{rank}[(Ce^{t(t-t_0)})^T, (CAe^{t(t-t_0)})^T, \cdots, (CA^{n-1}e^{t(t-t_0)})^T, \cdots] = n$$

Let $t = t_0$, then (77) can be represented as

$$\text{rank}[C^T, (CA)^T, \cdots, (CA^{n-1})^T, \cdots] = n$$

From the Cayley-Hamilton theorem, we know that $A'$ with $i \geq n$ can be written as a linear combination of $I, A, \cdots, A^{n-1}$. Therefore, the columns of $(CA)^T$ with $i \geq n$ are linearly dependent on the columns of $(CA)^T, (CA)^T, \cdots, (CA^{n-1})^T$. Consequently,

$$\text{rank}[C^T, (CA)^T, \cdots, (CA^{n-1})^T, \cdots] = \text{rank}[C^T, (CA)^T, \cdots, (CA^{n-1})^T] = n$$

Therefore, the columns of $Ce^{t(t-t_0)}$ are linearly independent on $[t_0, \infty)$ if and only if

$$\text{rank}[C^T, (CA)^T, \cdots, (CA^{n-1})^T] = n$$. That is, the statement 4 is equivalent to the statement 1.

(5) Necessity: Now we show that if the system is observable, then

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$$

has rank $n$ at every eigenvalue of $A$. If not, then there exist an eigenvalue $\lambda$ and a nonzero vector $\alpha \in \mathbb{R}^n$ such that

$$\alpha[(\lambda I - A)^T, C^T] = 0$$

or

$$\alpha\lambda = \alpha A^T$$

and

$$\alpha C^T = 0$$

which imply $\alpha(A^i) = \lambda^i \alpha$, for $i = 1, 2, \cdots$. Hence, we have

$$\alpha[C^T, (CA)^T, \cdots, (CA^{n-1})^T] = [\alpha C^T, \lambda \alpha C^T, \cdots, \lambda^{n-1} \alpha C^T] = 0$$

This contradicts the assumption of the observability. Therefore, if the system is observable, then

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$$

has rank $n$ at every eigenvalue of $A$.

Sufficiency: Now we prove that if (76) has rank $n$, then the system is observable. If the system is not observable, there exists an equivalence transformation $P$ that transforms the matrices $A$ and $C$ into $A'$ and $C'$ with

$$A' = PA^TP^{-1} = \begin{bmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{bmatrix}$$

$$C' = PC'^T = \begin{bmatrix} C'_{1} \\ 0 \end{bmatrix}$$

Let $\lambda$ be an eigenvalue of $A'_{22}$. We choose $\beta \neq 0$ such that $\beta A'_{22} = \lambda \beta$. Now we form a vector $\alpha = [0 \beta]$. Then we have

$$\alpha[(\lambda I - A'^T, C'^T] = 0 \beta \begin{bmatrix} \lambda I - A'_{11} & -A'_{12} & C'_{1} \\ 0 & \lambda I - A'_{22} & 0 \end{bmatrix} = 0$$

which implies

$$0 = \alpha[P(\lambda I - A')P^{-1}PC'^T] = \alpha P[(\lambda I - A')P^{-1}C'^T]$$

Since $\alpha \neq 0$, we have $\alpha' = \alpha P \neq 0$. Because $P^{-1}$ is nonsingular, $\alpha'(\lambda I - A') = 0$ implies $\alpha'(\lambda I - A') = 0$. Therefore, we have $\alpha'[\lambda I - A' C'^T] = 0$. In other word, if the system is not observable, then (76) does not have a rank of $n$ for some eigenvalue of $A$. Therefore, if (76) has rank $n$, then the system is observable.

Remark 14: For the linear time-invariant system, the necessary and sufficient conditions for the observability in the fuzzy sense are equivalent to that in the classical control theory. That is, this paper broadens the application range of the necessary and sufficient conditions of the observability in the classical control theory.

D. Second example

In this part, we will take a linear time-invariant uncertain system for example to illustrate the observability of the fuzzy dynamical system. Consider the fuzzy dynamical system (31), let
We assume that the membership function \( \mu_i(x_i) \) of the initial state at time \( t_0 = 0 \) can be represented as
\[
\begin{align*}
\mu_{b_1}(x) &= \begin{cases} 
\frac{1}{2}(x-1), & 1 \leq x \leq 3 \\
\frac{1}{2}(5-x), & 3 \leq x \leq 5 \\
0, & \text{otherwise}
\end{cases} \\
\mu_{b_2}(x) &= \begin{cases} 
x - 2, & 2 \leq x \leq 3 \\
4 - x, & 3 \leq x \leq 4 \\
0, & \text{otherwise}
\end{cases}
\end{align*}
\]

Since all the columns of \( C(t)e^{At} \) are linearly independent, according to the Theorem 5, we can conclude that the fuzzy dynamical system is observable.

In order to verify the observability of the described fuzzy dynamical system, we conduct the numerical simulation of the fuzzy dynamical system. Figure 3 shows the estimated membership function \( \tilde{\mu}_{i_1}(x_{i_1}) \) and the actual membership function \( \mu_{b_1}(x_{i_1}) \) of the initial state \( x_{i_1} \). Figure 4 shows the estimated membership function \( \tilde{\mu}_{i_2}(x_{i_2}) \) and the actual membership function \( \mu_{b_2}(x_{i_2}) \) of the initial state \( x_{i_2} \). The simulation result shows that the fuzzy dynamical system is observable.

Remark 15: Observability is a basic concept in the control of dynamical system. This paper proposes a systematic method to analyze the observability of the fuzzy dynamical system via the use of membership function. This was not available earlier. This lays a foundation in the fuzzy dynamical system theory.

Remark 16: We anticipate the result will be extremely important and present significant contributions toward this newly established fuzzy dynamical system theory. This can be divided into three levels. First, at the conceptual level, it lays an important foundation for the development of a newly established fuzzy dynamical system theory. The parallel to this is the introduction of controllability and observability by R.E. Kalman in the 60’s for classical (crisp) dynamical systems, which was fundamental and turned out to be significant. This paper should help to establish control and observer design for fuzzy dynamical systems. Second, at the illustration level, two examples (pp. 15-17 and 26-28) have been used to demonstrate the results are applied. Third, at the implementation level, the theory works the best if fuzzy quantities are involved in prescribing uncertainty characteristics and system performance. One such case would be Heating, Ventilation, and Air Conditioning (HVAC) system control design [21]. There, the human body size (large/medium/small) and weight (heavy/medium/light) are possible heat source, which are uncertain and best prescribed by fuzzy membership functions. Furthermore, the desirable room comfort level performance, which is related to human subjective feeling, is best described via fuzzy description as well.

5. Conclusions

The controllability and the observability are two basic concepts in the control of dynamic system. In this paper, we propose a systematic approach to analyze the controllability and the observability of fuzzy dynamical system via the use of the membership function. The concepts of the controllability and the observability in the fuzzy sense are presented via the use of the membership function. The necessary and sufficient conditions for the controllability and the observability of linear time-variant and time-invariant uncertain system are derived. This paper broadens the application scope of the controllability and the observability, and lays the foundation for the control of the fuzzy dynamical system.
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References


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