Adaptive Fuzzy Estimators in Control Charts for Short Run Production Processes

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Abstract

In this research, we consider a fuzzy decision rule of out-of-control states as a criterion in statistical process control. It is used for deciding if a process has exceeded its natural specifications limits. We propose a new framework for modeling control charts based on adaptive (non-asymptotic) fuzzy estimators. The adaptive fuzzy estimators introduced are for dealing with short production run processes. These approaches assume that the process parameters, i.e. the process mean and control parameters are fuzzy numbers estimated from statistical data. In particular, we develop: (i) a fuzzy p-chart, where the proportion of non-conforming units is fuzzy number, and (ii) a fuzzy decision rule as a way for flexible inspection that effectively allows the detection of the real out-of-control states. An application example of the proposed chart is illustrated. Moreover, the differences with prior linguistic-based and \( \alpha \)-level fuzzy control charts are discussed.

Keywords: Control charts, fuzzy estimation, process control, short production run processes.

1. Introduction

Pioneered by Shewhart [17], the use of control charts in Statistical Process Control (SPC) aims at monitoring and detecting changes in a process. This ensures that the products or services being made conform to defined standards [16].

There are various different types of control charts e.g. \( p \)-chart, \( np \)-chart, \( c \)-chart, and \( u \)-chart. There are also control charts for variable data, as well as extensions of them. The fuzzy control charts are suitable when the numerical data we deal with are imprecise, uncertain, vague, incomplete or even linguistic [24]. Several improvements and extensions of control charts have been proposed in the literature based on fuzzy sets [2, 5, 7-10, 12, 18, 23].

However, sometimes it is difficult to accurately estimate the control parameters. This happens when the process provides few data by itself for an adequate statistical estimation. Most importantly, when the process is at an initial stage or it has just restarted after process redesign [4, 14], then the use of fuzzy estimators is imperative. As Elam and Case [4] showed that in these cases, short-run control charts should be used. In this paper we address how to deal with both fuzziness and short-run problems. A brief reference to prior related work is given.

First fuzzy control chart extensions proposed the use of linguistic variables for qualitative judgments. They concerned the control limits and process data [12, 23]. Wang and Raz [23] adopted linguistic terms to express the quality characteristics. The membership functions of the various linguistic terms were defined to describe the quality characteristics associated with each linguistic term. Based on Wang and Raz approach, Kanagawa et al. [12] proposed a new linguistic control chart. The process average and process variability were based on the estimation of probability distribution according to linguistic data. Gulbay et al. [8-10] introduced the \( \alpha \)-Cut Fuzzy Control Charts (\( \alpha \)-FCC) for linguistic data to deal with cases where the data cannot be recorded or collected precisely. They provided a way to determine the tightness of inspection by selecting a suitable \( \alpha \)-level; the higher \( \alpha \), the tighter the inspection. Cheng [2] used fuzzy numbers instead of linguistic terms. He proposed control charts with fuzzy quality ratings in which possibility theory is used to check out-of-control conditions. The control charts he proposed monitor the central tendency of the process and indicate its degree of fuzziness. In a recent paper, Senturk and Erginel [18] proposed the use of fuzzy control limits for individual and moving range control charts. Their idea has been that if a sample mean is too close to the control limits and the used measurement system is not so sensitive, the decision may be faulty. In this case, fuzzy control limits provide a more accurate and flexible evaluation.

In short-run cases, the usual three standard deviations rule of process control is inadequate [4]. Instead, a two-stage procedure for determining the control parameters has been suggested. Two factors are used for
the first and second stages respectively. Each factor determined is based on a value that represents the desired probability of false alarms (Type I Error). Following this idea and extending it in fuzzy control charts, Fonseca et al. [7] proposed an extension of α-FCC. They introduced two stages. In the first stage, a small size of the data set is used to estimate the parameters of the process. In the second stage, in their words, the subgroups that were not deleted from the data set, after review of the stage 1 results, are used to set the parameters for the process, given the points plotted during this stage. Their method is based on linguistic attribute data but the resulting control limits are not fuzzy derived through defuzzification on the linguistic terms set.

When a control limit is crisp rather than fuzzy and when this control limit is exceeded by a very small amount, the process should not necessarily be judged as out-of-control. The particular problem stems from the fact that Shewhart control charts are sensitive to small changes in control parameters due to the sharpness of their tolerance limits. To solve this problem, fuzzy control charts which retain the fuzziness of the original observations are used. They have the ability to detect a degree of inspection, thus leading to decisions less prone to errors.

In this paper we present control charts based on an adaptive fuzzy estimator either for process modeling or control. The introduced adaptive fuzzy estimators, also called non-asymptotic fuzzy estimators [19, 20, 22], are ideal for short production run processes. In particular we address two issues: (a) how to accurately estimate the control parameters, given a set of observations, and (b) how to accomplish process control when both the observations and control limits are fuzzy.

Both the process characteristics and the process mean value are fuzzy numbers estimated from statistical data. The control limits are fuzzy numbers obtained through fuzzy arithmetic. We provide an appropriate and explicit way to obtain the process out-of-control conditions. Our methodology for process control entails a fuzzy comparison about whether a limit has being exceeded or not. Thus the uncertainty and imprecise character of the recorded process data is retained. The ability to detect intermediate levels of inspection during process control is also achieved.

The paper is organized as follows. In Section 2 we provide terminology and preliminary concepts. In Section 3 we present fuzzy estimators, which are to be used in control charts. Our proposed methodology of process control based on fuzzy estimators is provided in Section 4. Section 5 is a numerical example on a real data set to illustrate the methodology. Conclusions are given in the last section.

2. Basic Terminology

Let $X$ be a universal set. Every function of the form $A : X \rightarrow [0,1]$ is called a fuzzy set or a fuzzy subset of $X$, where $A(x)$ is interpreted as the “membership degree” of $x$ in the fuzzy set $A$. One of the most important concepts of fuzzy sets is that of an α-cut. Given a fuzzy set $A$ defined on $X$ and any number $\alpha \in (0,1]$ the $\alpha$-cut ($^\alpha A$) is the crisp set given in Equation (1).

$$ ^\alpha A = \{x \in \Re : A(x) \geq \alpha \} $$

Moreover, it is known that the $\alpha$-cuts define a fuzzy set $A$. The support of a fuzzy set, denoted $0 A$, is the closure in the topology of $A$ of the union of all the $\alpha$-cuts [3], that is

$$ 0 A = \bigcup_{\alpha \in [0,1]}^\alpha A = \{x : A(x) > 0\}. $$

We say that a fuzzy set $A$ is a fuzzy number if and only if the following conditions are met:

- $A$ is normal, that is there exists $x \in \Re$ such that $A(x) = 1$.
- $A$ is a convex fuzzy set, that is for every $t \in [0,1]$ and $x_1, x_2 \in \Re$ we have $A\{(1-t)x_1 + tx_2\} \geq \min\{A(x_1), A(x_2)\}$.
- $A$ is upper semi-continuous.
- The support of $A$, $0 A = \bigcup_{\alpha \in [0,1]}^\alpha A = \{x : A(x) > 0\}$, is compact.

We denote a fuzzy number using Nguyen’s intervals form [15]:

$$ ^\alpha A = [A_1(\alpha), A_2(\alpha)] $$

where

- $A_1(\alpha)$ is non-decreasing and left continuous,
- $A_2(\alpha)$ is non-increasing and left continuous,
- $A_1(\alpha) \leq A_2(\alpha)$.

Arithmetic operations between fuzzy numbers are defined according to Zadeh’s extension principle. Any crisp function $f : X \rightarrow Y$ induces a fuzzy valued function $F : P(X) \rightarrow P(Y)$ where $P(\cdot)$ denotes a fuzzy power set. A fuzzy binary arithmetic operation is defined as

$$ C(z) = \max_{z = + \Theta \in [\Lambda, B]} \left[ A(x) \wedge B(y) \right] , \quad (x, y, z) \in \Re^3 $$

where $\Theta \in \{+, -, \ast, \div\}$. Under the extension principle arithmetic operations are convolution-like operations on the support elements of fuzzy numbers and logical operations [6]. A more efficient method is to consider arithmetic operations between fuzzy numbers as
arithmetic operations on basis of closed bounded intervals on real numbers by employing their \( \alpha \)-cuts representation [13].

The \( \alpha \)-cut operations of the fuzzy arithmetic addition (\( \oplus \)), subtraction (\( \ominus \)), multiplication (\( \otimes \)) and division (\( \oslash \)) are

\[
\begin{align*}
\alpha \text{-cut addition } & : (A \oplus B) = [A^\alpha \oplus B^\alpha, A^\alpha + B^\alpha] \\
\alpha \text{-cut subtraction } & : (A \ominus B) = [A^\alpha \ominus B^\alpha, A^\alpha - B^\alpha] \\
\alpha \text{-cut multiplication } & : (A \otimes B) = \left[ \min(A^\alpha \otimes B^\alpha, A^\alpha \otimes B^\alpha, A^\alpha \otimes B^\alpha), \max(A^\alpha \otimes B^\alpha, A^\alpha \otimes B^\alpha, A^\alpha \otimes B^\alpha) \right] \\
\alpha \text{-cut division } & : (A \oslash B) = \left[ \min\left(\frac{A^\alpha}{B^\alpha}, A^\alpha / B^\alpha, A^\alpha / B^\alpha\right), \max\left(\frac{A^\alpha}{B^\alpha}, A^\alpha / B^\alpha, A^\alpha / B^\alpha\right)\right]
\end{align*}
\]

where \( A^\alpha = A \) and \( B^\alpha = B \) for short and \( 0 \notin \alpha \) for the division.

3. Fuzzy Estimation of the Process Mean and Control Limits

Adaptive fuzzy estimators are constructed directly from the data with the help of confidence intervals. This constitutes an extension of the statistical confidence interval estimations which retains all the fuzziness of the original measurements. A typical form of a fuzzy estimator for the mean value is presented in Fig. 1.

![Graphical representation of a fuzzy mean estimator.](image)

The main proposition is stated as follows: Let \( X_1, X_2, ..., X_n \) be a random sample of size \( n \) from a distribution of unknown parameter \( \theta \) and let \( \left[h_1(\alpha), h_2(\alpha)\right] \) denote the \((1-\alpha)100\%\) confidence intervals for \( \theta \). Using any monotonic increasing, continuous and onto function

\[
h_\gamma(\alpha): (0,1) \rightarrow \left[\frac{Y}{2}, 0.5\right], \gamma \in (0,1)
\]

then a fuzzy estimator \( \tilde{\theta}_\gamma \) is derived by the following superimposed intervals

\[
\alpha \tilde{\theta}_\gamma = \left[h_1(2\alpha(\alpha)), h_2(2\alpha(\alpha))\right], \alpha \in (0,1)
\]  (6)

In standard control charts, three standard deviations (3\( \sigma \)) limits are not suitable for short run cases and therefore different factors should be used. Nedumaran and Leon [14] suggested two parameters, named \( k_1 \) and \( k_2 \), for the first and second stages respectively. Their computation is well explained in the literature [14]. Each factor was determined based on a value that represented the desired probability of false alarms (Type I Error). Their respective values were \( a_1 \) and \( a_2 \) for the first and second stage, while for a value of 0.0027 the factors for both first and second stage approach the value of 3 as the subgroup size (m) increases. The formulas for the computation were created taking for granted the probability that a point will plot outside the control limits as false alarm, when the process is actually in control.

Let us consider the case of p-charts. The fraction of nonconforming for each subgroup \( i \) and the fraction nonconforming for the data set are given by Equations (7) and (8):

\[
p_i = \frac{D_i}{n}
\]  (7)

\[
\overline{p} = \frac{\sum_{i=1}^{m} p_i}{m}
\]  (8)

where \( n \) is the total number of units and \( D_i \) is the number of non-conforming units.

In the proposed fuzzy control charts, each subgroup \( m \) is considered to have a fuzzy estimator for the proportion of nonconforming units. The \( \alpha \)-cuts of the fuzzy process Central Line (\( \tilde{CL} \)) becomes

\[
\alpha \tilde{CL} = \left[p_{L}(\alpha), p_{R}(\alpha)\right] - \frac{1}{m} \sum_{i=1}^{m} P_i(\alpha), \frac{1}{m} \sum_{i=1}^{m} P_i(\alpha)
\]  (9)

We place a tilde over fuzzy numbers symbols so as to distinguish them from crisp numbers. The fuzzy control upper (\( \overline{UCL} \)) and lower (\( LCL \)) limits for the first and second stage are given in Equations (10) - (15):

**First stage:**

\[
\alpha \overline{UCL} = \left[p_{L}(\alpha), p_{R}(\alpha)\right] - k_i \sigma_i
\]  (10)

\[
\overline{CL} = \tilde{p}
\]  (11)

\[
\alpha \overline{UCL} = \left[p_{L}(\alpha) + k_i \sigma_i, p_{R}(\alpha) + k_i \sigma_i\right]
\]  (12)

**Second stage:**
with \( 2(n-x+1) \) numerator degrees of freedom and \( 2x \) denominator degrees of freedom.

The corresponding formulas for the \( \alpha \)-cuts of the fuzzy estimators are derived from the probability distribution, e.g. normal, t-student etc. In our application example of the next section we use the confidence intervals of the t-student distribution. The \( \alpha \)-cuts of this fuzzy number are

\[
\tilde{\alpha} = \left[ \tilde{\alpha}, \tilde{\alpha} \right], \alpha \in (0, 1)
\]

where

\[
\tilde{\alpha} = \frac{\hat{p} - \tilde{p}}{\tilde{\sigma}}
\]

\[
\tilde{\sigma} = \sqrt{\frac{\hat{p} (1-\hat{p})}{n}}
\]

from which \( \tilde{\sigma} \) is evaluated. \( \hat{p} \) for the second stage is evaluated similarly.

Let \( \hat{\alpha} \) be the fuzzy probability of a false alarm Type I, then the fuzzy probability that a point will plot within the control limits is

\[
\tilde{p} \left\{ -k_1 \leq \left( \frac{\hat{p} - \tilde{p}}{\tilde{\sigma}} \right) \leq k_2 \right\} = 1 - \hat{\alpha}
\]

An important characteristic of the fuzzy control charts produced is that the central line, the control limits as well as the data used are of fuzzy nature. In this section, a new approach is developed which provides the ability of determining the tightness of inspection, based on estimations of control parameters from statistical data. Let \( \phi \in (0, 1) \) be a control parameter used to decide if an exceeding sample should indeed be rejected or not. We will give an explicit measure of whether a sample value has exceeded a control limit. Value of \( \phi \) can be set according to how strict the results are desired to be.

Let \( \tilde{\Delta} \) be a non asymptotic fuzzy estimator for the mean value of a subgroup \( m \) of size \( n \). The sample's distribution is considered random e.g. binomial and the \( \alpha \)-cuts of \( \tilde{\Delta} \) can be constructed as shown in previous section. Thus we obtain all closed intervals

\[
\tilde{\Delta} = \left[ \Delta_{\alpha} (\alpha), \Delta_{\delta} (\alpha) \right], \alpha \in (0, 1)
\]

that needs to be compared against fuzzy upper control limits (Eq. (12), Eq. (15)) and fuzzy lower control limits (Eq. (10), Eq. (13)). For this purpose, we produce a fuzzy measure inspired from fuzzy hypothesis testing [1] taken a step further in order to be applied in fuzzy control charts. The accept/reject decision is made according to the following rule:

- \( H_{\theta} \): the sample is in-control if \( A_{\theta} / A_{\theta} \leq \phi \), therefore accepted
- \( H_{\gamma} \): the sample is out-of-control if \( A_{\theta} / A_{\theta} > \phi \), therefore rejected

\( A_{\gamma} \) is calculated as the complete area of fuzzy number \( \tilde{\Lambda} \):

\[
A_{\gamma} = \int_{\Delta_{\alpha} (\gamma)}^{\Delta_{\delta} (\gamma)} \tilde{\Lambda}(u) \, du
\]
where \([\Delta_1(\gamma), \Delta_r(\gamma)]\) is the support level \(^a\bar{\Delta}.\) \(A_r\) is the area of \(\bar{\Delta}\) that lies beyond the fuzzy number used to define the upper or lower control limit. To calculate \(A_r\) two cases must be considered. Let \(\delta, \ UCL, \ LCL\) denote the central points of fuzzy sets \(\bar{\Delta}, \ U\bar{C}L\) and \(L\bar{C}L\) respectively (i.e. crisp elements with membership equal to 1).

If \(\delta \geq UCL\) the required \(A_r\) area is calculated by Equation (22):

\[
A_r = \int_{\delta - UCL}^{\delta - UCL} \Delta_r(\gamma) \geq UCL(\gamma) d\gamma - \int_{\delta - UCL}^{\delta - UCL} \Delta_r(\gamma) < UCL(\gamma) d\gamma
\]

\[
U = \int_{\delta - UCL}^{\delta - UCL} (\bar{\Delta} \cap U\bar{C}L)(u) du
\]

\[
E_1 = \int_{\delta - UCL}^{\delta - UCL} \bar{\Delta}(u) du, \ E_2 = \int_{\delta - UCL}^{\delta - UCL} \bar{\Delta}(u) du
\]

while if \(\delta \leq UCL\), \(A_r\) is calculated by Equation (23):

\[
A_r = \begin{cases} 
\int_{\delta - UCL}^{\delta - UCL} \Delta_r(\gamma) \geq UCL(\gamma) d\gamma - \int_{\delta - UCL}^{\delta - UCL} \Delta_r(\gamma) < UCL(\gamma) d\gamma, & \text{if } \delta \leq UCL \\
0, & \text{if } \delta > UCL
\end{cases}
\]

\[
E_1 = \int_{\delta - UCL}^{\delta - UCL} U\bar{C}L(u) du,
\]

where

\[
\tau = \bar{\Delta}^{-1}(\alpha_r), \ \alpha_r = \sup_{\alpha} \min \{\bar{\Delta}(u), U\bar{C}L(u)\}
\]

and

\[
q\left(\bar{\Delta} \cap U\bar{C}L\right) = \left[\left(\bar{\Delta} \cap U\bar{C}L\right)_{\tau}(\alpha), \left(\bar{\Delta} \cap U\bar{C}L\right)_{\alpha}(\alpha)\right]
\]

Fig. 2 illustrates the required areas of the fuzzy comparison.

![Figure 2. Illustration of the fuzzy comparison areas](image)

Correspondingly, if \(\delta \leq LCL\), \(A_r\) area is calculated by Equation (24):

\[
A_r = \begin{cases} 
\int_{\delta - LCL}^{\delta - LCL} \Delta_r(\gamma) \leq LCL(\gamma) d\gamma - \int_{\delta - LCL}^{\delta - LCL} \Delta_r(\gamma) > LCL(\gamma) d\gamma, & \text{if } \delta \leq LCL \\
0, & \text{if } \delta > LCL
\end{cases}
\]

\[
U = \int_{\delta - LCL}^{\delta - LCL} (\bar{\Delta} \cap L\bar{C}L)(u) du
\]

\[
E_1 = \int_{\delta - LCL}^{\delta - LCL} \bar{\Delta}(u) du, \ E_2 = \int_{\delta - LCL}^{\delta - LCL} \bar{\Delta}(u) du
\]

while if \(\delta \geq LCL\), \(A_r\) is calculated by Equation (25):

\[
A_r = \begin{cases} 
\int_{\delta - UCL}^{\delta - UCL} \Delta_r(\gamma) \leq UCL(\gamma) d\gamma - \int_{\delta - UCL}^{\delta - UCL} \Delta_r(\gamma) > UCL(\gamma) d\gamma, & \text{if } \delta \geq LCL \\
0, & \text{if } \delta < LCL
\end{cases}
\]

\[
E_1 = \int_{\delta - UCL}^{\delta - UCL} \bar{\Delta}(u) du, \ E_2 = \int_{\delta - UCL}^{\delta - UCL} \bar{\Delta}(u) du
\]

Based on the above formulation, the process control is made as follows: a control parameter \(\phi\) is compared to the index \(A_r / \bar{\Delta}\). If the process is new e.g. short run, \(\phi < 0.10\) can be chosen for the evaluation, which would lead to severely strict results. Such a parameter is problem dependent. In cases that a process has just been initiated or is being monitored after it has been brought into statistical control or the study is of a process that produces very little data. On the contrary, when the process has been stabilized \(\phi\) is suggested to be equal or greater than 0.30.

5. Application

For comparison’ sake, we used the same data set provided by Taleb and Limam [21] regarding Tunisian Porcelain. Taleb and Limam categorized products into four categories with respect to the quality, as in Table 1. The same data and classification were also sustained in the research conducted by Gulbay et al. [8] and Fonseca et al. [7].

When a product has no default or a visible unimportant default, it is classified as standard (S). If it presents a visible unimportant default that does not affect its use, it is classified as second choice (SC). When a visible important default is presented but the product’s use is not affected, it is denoted as third choice (TC). Finally, when a visible unimportant default that does not affect its use, it is classified as second choice (SC).

To create a p-chart, the percentage of the items in each category is necessary for the calculation of the sample mean \(M_j\), which stands for the weighted average of the data. In Equation (26) \(k_{ij}\) stands for the number of data within the category \(i\) for sample \(j\), \(r_j\) is the membership value for the particular category and \(n_j\) is the size of sample \(j\).

\[
M_j = \frac{\sum_{i=1}^{30} k_{ij} r_j}{n_j}, \quad j = 1, \ldots, 30
\]

To produce fuzzy CL, \(M_j\) is necessary to be taken for a set of data, where \(j = 1, \ldots, 30\). Using fuzzy estimation, the estimator of CL is produced (Fig. 3). The support range of the standard deviation \(SD_j\) should be calculated. To determine \(SD_j\) for each sample Equation (27) is used.
The results are presented in Table 3.

\[ SD_j = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} k_j (r_j - M_j)^2} \quad (27) \]

Table 1. Representative values of linguistic terms.

<table>
<thead>
<tr>
<th>Linguistic term</th>
<th>Representative value ( r_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>0</td>
</tr>
<tr>
<td>SC</td>
<td>0.25</td>
</tr>
<tr>
<td>TC</td>
<td>0.5</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
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Table 2. Data of the porcelain process.

<table>
<thead>
<tr>
<th>Sample</th>
<th>S</th>
<th>SC</th>
<th>TC</th>
<th>C</th>
<th>( M_j )</th>
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<td>15</td>
<td>6</td>
<td>0.1135</td>
</tr>
<tr>
<td>29</td>
<td>110</td>
<td>15</td>
<td>9</td>
<td>1</td>
<td>0.0685</td>
</tr>
<tr>
<td>30</td>
<td>112</td>
<td>37</td>
<td>28</td>
<td>11</td>
<td>0.1822</td>
</tr>
</tbody>
</table>

Table 3. Determined values of SDj.

<table>
<thead>
<tr>
<th>Sample</th>
<th>SDj</th>
<th>Sample</th>
<th>SDj</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.202</td>
<td>16</td>
<td>0.206</td>
</tr>
<tr>
<td>2</td>
<td>0.198</td>
<td>17</td>
<td>0.177</td>
</tr>
<tr>
<td>3</td>
<td>0.220</td>
<td>18</td>
<td>0.240</td>
</tr>
<tr>
<td>4</td>
<td>0.240</td>
<td>19</td>
<td>0.254</td>
</tr>
<tr>
<td>5</td>
<td>0.244</td>
<td>20</td>
<td>0.260</td>
</tr>
<tr>
<td>6</td>
<td>0.237</td>
<td>21</td>
<td>0.192</td>
</tr>
<tr>
<td>7</td>
<td>0.247</td>
<td>22</td>
<td>0.232</td>
</tr>
<tr>
<td>8</td>
<td>0.336</td>
<td>23</td>
<td>0.242</td>
</tr>
<tr>
<td>9</td>
<td>0.195</td>
<td>24</td>
<td>0.207</td>
</tr>
<tr>
<td>10</td>
<td>0.212</td>
<td>25</td>
<td>0.273</td>
</tr>
<tr>
<td>11</td>
<td>0.211</td>
<td>26</td>
<td>0.253</td>
</tr>
<tr>
<td>12</td>
<td>0.206</td>
<td>27</td>
<td>0.261</td>
</tr>
<tr>
<td>13</td>
<td>0.232</td>
<td>28</td>
<td>0.220</td>
</tr>
<tr>
<td>14</td>
<td>0.207</td>
<td>29</td>
<td>0.163</td>
</tr>
<tr>
<td>15</td>
<td>0.217</td>
<td>30</td>
<td>0.274</td>
</tr>
</tbody>
</table>

The fuzzy mean value of samples 1 to 30 is produced by considering a set of n observations based on the membership values for each category regarding the linguistic variables. Then the fuzzy mean value of samples 1-30 is produced and the fuzzy p control chart is created (Figs. 4-6).

By applying the proposed method of Section 4, we observe that Sample 8 is exceeding UCL without a doubt as shown in Fig. 7. This brings us in a position of questioning the prior method of \( \alpha \)-cut fuzzy control charts for linguistic data [8] since in that approach Sample 8 can be taken as in control and therefore it can be a point of false alarm creation. Moreover, applying \( \alpha \)-cut fuzzy control charts for linguistic data, sample 8 starts being out of control when increasing \( \alpha \)-cut for \( \alpha \geq 0.33 \).

On the other hand, Sample 29 partially exceeds LCL (Fig. 8). For \( A_T \) used to describe the total area of Sample 8 and \( A_R \) the part of the sample that exceeds the control limit, index \( A_R/A_T \) can be computed.

\[ A_R = \frac{109.57}{462.02} = 0.2372 \]

Therefore, the fuzzy decision that can be used in order to decide if an exceeding sample should indeed be rejected or not is:
- If \( \varphi \geq 0.2372 \) then Sample 29 should be considered as in control.
- If \( \varphi \leq 0.2372 \) then Sample 29 should be considered as out of control which is generalization of classical process control where the parameter \( \varphi \) determines a plausible degree of out-of-control condition.
Figure 3. Fuzzy estimator-based LCL, CL, and UCL of the Porcelain example for p-chart.

Figure 4. Fuzzy p-chart for the Porcelain data (Samples 1-10).

Figure 5. Fuzzy p-chart for the Porcelain data (Samples 11-20).

Figure 6. Fuzzy p-chart for the Porcelain data (Samples 21-30).

Figure 7. Sample 8 (out of control) of example.

Figure 8. Sample 29.
6. Conclusion

Non-asymptotic fuzzy estimators are applied in this paper for estimating the control chart parameters. This enhances classical control charts with all the extra benefits for adaptive capabilities. We introduced a fuzzy p-chart to show the modeling methodology and the relevant process control. This framework has two distinguishing characteristics: (a) both control chart parameters, process mean and control limits, are fuzzy estimators constructed from confidence intervals; thereby the shape of fuzzy estimators is determined statistically; (b) the process control methodology is also fuzzy, making it appropriate to avoid sampling errors of the original observations. This is particular important in cases of very few data for a full statistical approach, the so called short-run cases.

The required formulation for implementing the proposed fuzzy methodology in quality control was given. Overall, our approach is easily implementable and a useful tool for control engineers who face the above problem. A numerical example was provided to clarify all aspects of the theory. A potential area for additional research would be to study the integration of time and cost performance aspects of fuzzy estimator based control charts as well as different types of assignable cause through fuzzy pattern recognition.

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References


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