Generalized Hesitant Fuzzy Prioritized Einstein Aggregation Operators and Their Application in Group Decision Making

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Abstract

In this paper, a hesitant fuzzy multiple attribute group decision making problem where there exists prioritization relationships over the attributes and decision makers is studied. First, some Einstein operations on hesitant fuzzy elements and their properties are presented. Then, several generalized hesitant fuzzy prioritized Einstein aggregation operators, including the generalized hesitant fuzzy prioritized Einstein weighted averaging operator and the generalized hesitant fuzzy prioritized Einstein weighted geometric operator, are introduced. Moreover, some desirable properties and special cases are investigated. It is shown that some existing hesitant fuzzy aggregation operators are the special cases of the proposed operators. Further, a new approach of hesitant fuzzy group decision making is developed based on the proposed operators. Finally, a practical example is provided to illustrate the developed approach.

Keywords: Aggregation operator, Einstein operations, generalized prioritized Einstein aggregation operator, hesitant fuzzy set, group decision making.

1. Introduction

Decision making is one of the most significant human activities in many fields, such as industry, service, business, and so on. A decision making problem is to find a desirable solution from a finite alternatives. In order to choose a desirable solution, decision maker often provide his/her preference information which needs to be aggregated into an overall one by using a proper aggregation technique. The study on information aggregation is an important topic in decision making. A variety of operators have been developed in the past few decades, among which generalized mean operator introduced by Dyckhoff [1] is the widely used one. As the generalized mean operator has an alterable parameter, it is usually used to generalize the existing operators [2-9].

In classical decision making, the assessments of alternatives are precisely known. However, due to the objects being complex and uncertain, it is difficult for the decision maker to provide a clear assessment level. For example, in a group decision making process, some decision makers may provide 0.3, some provide 0.5, and the others provide 0.6, and these three parts can not persuade each other. The existing tools including fuzzy set, intuitionistic fuzzy set, interval-valued fuzzy set and type-2 fuzzy set, can not effectively express this kind of information. Fortunately, hesitant fuzzy set [10, 11], which permits the membership of an element to a given set having a few different values, is a good selection to deal with this case. The hesitant fuzzy element \{0.3, 0.5, 0.6\} can describe this dilemma vividly and it is more convenient and reasonable than the crisp number 0.3 (or 0.6), or the interval-valued fuzzy number [0.3, 0.6], or the intuitionistic fuzzy number (0.3, 0.4), because the scores are just three possible values 0.3, 0.5 and 0.6.

Since its appearance, hesitant fuzzy set has been investigated from different points of view by many authors [12-28], especially in aggregation operators. For example, Xia and Xu [20] presented some hesitant fuzzy operational laws and proposed a series of hesitant fuzzy aggregation operators. Xia et al. [21] developed some confidence induced aggregation operators for hesitant fuzzy information. Xia et al. [22] gave several series of hesitant fuzzy aggregation operators with the help of quasi-arithmetic means. By combining the Bonferroni mean and the geometric mean, Zhu et al. [23] investigated some geometric Bonferroni mean operators under hesitant fuzzy environment. Zhang [24] extended the power aggregation operator to the hesitant fuzzy power aggregation operators. Wei [25] extended prioritized aggregation operator to deal with the hesitant fuzzy information and developed two hesitant fuzzy prioritized aggregation operators. And he applied them to hesitant fuzzy decision making.

On the one hand, the above aggregation operators for hesitant fuzzy information are assuming that the attributes are at the same priority level although some of them...
consider the correlation phenomena between attributes. However, in some real decision making problems, this kind of compensation between attributes is not feasible. Besides, they have not considered the prioritization relationship over decision makers. Hence, it is necessary to propose new methods to overcome these difficulties. On the other hand, all aggregation operators introduced previously are based on the algebraic product and algebraic sum of hesitant fuzzy values. In fact, Einstein operations include Einstein product and Einstein sum are also good alternatives for structuring aggregation operators, and they have been used to aggregate the intuitionistic fuzzy values by many authors [29-31]. Thus, it is meaningful to use Einstein operations to aggregate hesitant fuzzy information. In this paper, motivated by Wei [25], we extend the prioritized aggregation operator to the generalized hesitant fuzzy prioritized Einstein aggregation operators with the help of Einstein operations, and apply them to group decision making under hesitant fuzzy environment.

The remainder of this paper is organized as follows. The following section recalls briefly some basic concepts and notions. In Section 3, based on the generalized aggregation operator, the prioritized aggregation operator and the Einstein operations, we propose the generalized hesitant fuzzy prioritized Einstein weighted averaging operator and the generalized hesitant fuzzy prioritized Einstein weighted geometric operator. Section 4 develops an approach to multiple attribute group decision making with hesitant fuzzy information based on the proposed operators. An example is given to demonstrate the practicality and effectiveness of the proposed approach in Section 5. Finally, Section 6 concludes the paper.

2. Preliminaries

A. Hesitant fuzzy set

As a new generalization of fuzzy set [32], hesitant fuzzy set (HFS) was first introduced by Torra [10, 11].

Definition 1: Let $X$ be a reference set, an HFS on $X$ is in terms of a function that when applied to $X$ returns a subset of $[0,1]$, which can be represented as

$$H = \left\{ \frac{h_{xy}(x)}{x} \bigg| x \in X \right\},$$

where $h_{xy}(x)$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set $H$.

For convenience, Xu and Xia [17] called $h_{xy}(x)$ a hesitant fuzzy element (HFE). Torra [10, 11] defined the union, intersection and complementation of HFEs as follows.

Definition 2: Let $h_1$, $h_2$ and $h_3$ be three HFEs, then

1. $h_1 \cup h_2 = \bigcup_{\gamma_{i1},\gamma_{i2} \in \gamma} \left\{ \max(\gamma_{i1}, \gamma_{i2}) \right\}$;
2. $h_1 \cap h_2 = \bigcup_{\gamma_{i1},\gamma_{i2} \in \gamma} \left\{ \min(\gamma_{i1}, \gamma_{i2}) \right\}$;
3. $h' = \bigcup_{\gamma \in \gamma} \{1 - \gamma\}$.

Xia and Xu [20] also defined some operations on the HFEs.

Definition 3: Let $\alpha > 0, h_1$ and $h_2$ be three HFEs, then

1. $h_1 \oplus h_2 = \bigcup_{\gamma_{i1},\gamma_{i2} \in \gamma} \left\{ \gamma_{i1} + \gamma_{i2} - \gamma_{i1}\gamma_{i2} \right\}$,
2. $h_1 \otimes h_2 = \bigcup_{\gamma_{i1},\gamma_{i2} \in \gamma} \left\{ \gamma_{i1}\gamma_{i2} \right\}$,
3. $\alpha h = \bigcup_{\gamma \in \gamma} \left\{ \gamma^\alpha \right\}$,
4. $h'^\alpha = \bigcup_{\gamma \in \gamma} \left\{ 1 - (1 - \gamma)^\alpha \right\}$.

To compare the HFEs, Xia and Xu [20] defined the score function of HFEs and gave the comparison laws.

Definition 4: Let $h$ be an HFE. $s(h) = \frac{1}{n(h)} \sum_{x \in X} x$ is called the score function of $h$, where $n(h)$ is the number of values of $h$. For two HFEs $h_1$ and $h_2$, if $s(h_1) > s(h_2)$, then $h_1 \succ h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

B. Einstein operations of hesitant fuzzy elements

It is well known that the $t$-norms and $t$-conorms are the general concepts satisfying the requirements of the conjunction and disjunction operators. Einstein operations includes the Einstein sum $\oplus_e$ and Einstein product $\otimes_e$, which are examples of $t$-norms and $t$-conorms, respectively. They are defined by Klement [33] as follows:

$$x \oplus_e y = \frac{xy}{1 + (1 - x)(1 - y)},$$

$$x \otimes_e y = \frac{x + y}{1 + xy}, x, y \in [0,1].$$

Based on the above Einstein operations, we give some new operations on HFEs as below:

Definition 5: Let $\alpha > 0, h_1$ and $h_2$ be three HFEs, then

1. $h_1 \oplus_e h_2 = \bigcup_{\gamma_{i1},\gamma_{i2} \in \gamma} \left\{ \frac{\gamma_{i1} + \gamma_{i2}}{1 + \gamma_{i1}\gamma_{i2}} \right\}$,
2. $h_1 \otimes_e h_2 = \bigcup_{\gamma_{i1},\gamma_{i2} \in \gamma} \left\{ \frac{\gamma_{i1}\gamma_{i2}}{1 + (1 - \gamma_{i1})(1 - \gamma_{i2})} \right\}$,
3. $\alpha_e h = \bigcup_{\gamma \in \gamma} \left\{ \left( 1 + \gamma \right)^\alpha - (1 - \gamma)^\alpha \right\}.$
(4) \( h^\gamma_{\gamma^a} = \bigcup_{\gamma, \gamma^a} \left\{ \frac{2\gamma^a}{(2-\gamma^a)\gamma + \gamma^a} \right\} \).

Based on the operational laws (1)-(4) in Definition 5, we can easily obtain the following properties.

**Theorem 1**: Let \( \alpha > 0, \alpha_i > 0, \alpha_i > 0 \), \( h, h_1 \) and \( h_2 \) be three HFES, then

1. \( h_1 \otimes h_2 = h_2 \otimes h_1 \);
2. \( (h_1 \otimes h_2) \otimes h_3 = h_1 \otimes (h_2 \otimes h_3) \);
3. \( \alpha \cdot (h_1 \otimes h_2) = (\alpha \cdot h_1) \otimes (\alpha \cdot h_2) \);
4. \( \alpha_{i+1} \cdot (\alpha_i \cdot h) = (\alpha_{i+2} \cdot h) \);
5. \( \alpha \cdot h = h \);
6. \( (h_1 \otimes h_2) \otimes h_3 = h_1 \otimes (h_2 \otimes h_3) \);
7. \( \alpha \cdot h = h \).

**Proof**: (1), (2), (5) and (6) are trivial, (7) and (8) are similar to (3) and (4), respectively, so we only prove (3) and (4) in the following.

(3) Since \( h_1 \otimes h_2 = \bigcup_{\gamma, \gamma^a} \left\{ \frac{\gamma_1 + \gamma_2}{1 + \gamma_1 \gamma_2} \right\} \). By the operational law (3) in Definition 5, we have

\[ \alpha \cdot (h_1 \otimes h_2) = \bigcup_{\gamma, \gamma^a} \left\{ \frac{(1 + \gamma_1 + \gamma_2)\gamma^a - (1 - \gamma_1 + \gamma_2)}{1 + \gamma_1 \gamma_2} \right\} \]

\[ = \bigcup_{\gamma, \gamma^a} \left\{ \frac{(1 + \gamma_1 + \gamma_2)\gamma^a + (1 - \gamma_1 + \gamma_2)}{1 + \gamma_1 \gamma_2} \right\} \]

Since \( \alpha \cdot h = \bigcup_{\gamma, \gamma^a} \left\{ \frac{(1 + \gamma_1)\gamma^a - (1 - \gamma_1)}{1 + \gamma_1 \gamma_2} \right\} \) and
\[ (\alpha \cdot h_1) \otimes (\alpha \cdot h_2) = \bigcup_{\gamma, \gamma^a} \left\{ \frac{(1 + \gamma_1)\gamma^a - (1 - \gamma_1)}{1 + \gamma_1 \gamma_2} \right\} \]
\[ = \bigcup_{\gamma, \gamma^a} \left\{ \frac{(1 + \gamma_1)\gamma^a - (1 - \gamma_1)\gamma^a}{1 + \gamma_1 \gamma_2} \right\} \]

Thus \( \alpha \cdot (h_1 \otimes h_2) = (\alpha \cdot h_1) \otimes (\alpha \cdot h_2) \).

(4) Since \( \alpha \cdot h = \bigcup_{\gamma, \gamma^a} \left\{ \frac{(1 + \gamma_1)\gamma^a - (1 - \gamma_1)\gamma^a}{1 + \gamma_1 \gamma_2} \right\} \), then
\[ \alpha_{i+1} \cdot (\alpha_i \cdot h) = \bigcup_{\gamma, \gamma^a} \left\{ \frac{(1 + \gamma_1)\gamma^a - (1 - \gamma_1)\gamma^a}{1 + \gamma_1 \gamma_2} \right\} \]

**Remark 1**: Let \( \alpha > 0, \alpha_i > 0 \) and \( h \) be an HF. It is worth noting that \( (\alpha \cdot h) \otimes (\alpha \cdot h) = (\alpha_i + \alpha_j) \cdot h \) and \( h^\gamma_{\gamma^a} \otimes h^\gamma_{\gamma^a} = h^\gamma_{\gamma^a} \) do not hold necessarily in general. To illustrate this case, we give a simple example as follows.

**Example 1**: Let \( h = (0.3, 0.5) \), \( \alpha_i = \alpha_j = 1 \), then
\[ \alpha \cdot h \otimes (\alpha \cdot h) = h \otimes h = \bigcup_{\gamma, \gamma^a} \left\{ \frac{1 + \gamma_1}{1 + \gamma_1 \gamma_2} \right\} \]
\[ = \{0.5505, 0.6957, 0.6957, 0.8\} \]

Clearly,
\[ s((\alpha \cdot h) \otimes (\alpha_i \cdot h)) = 0.6854 > 0.6752 = s((\alpha_i + \alpha_j) \cdot h) \]

Thus \( (\alpha_i + \alpha_j) \cdot h \otimes (\alpha_i + \alpha_j) \cdot h \geq (\alpha_i + \alpha_j) \cdot h \).

Similarly, we have
\[ s(h^\gamma_{\gamma^a} \otimes h^\gamma_{\gamma^a}) = 0.1238 < 0.1302 = s(h^\gamma_{\gamma^a} \otimes h^\gamma_{\gamma^a}) \]

Therefore, \( h^\gamma_{\gamma^a} \otimes h^\gamma_{\gamma^a} = h^\gamma_{\gamma^a} \).

However, if number of the values in \( h \) is only one, i.e., \( HF \) \( h \) is reduced to a fuzzy value, then the above result holds.

**Proposition 1**: Let \( \alpha_i > 0, \alpha_j > 0 \) and \( h \) be an HF, in which number of the values is only one, i.e., \( h = \{\gamma_1\} \), then
1. \( (\alpha_i \cdot h) \otimes (\alpha_i \cdot h) = (\alpha_i + \alpha_i) \cdot h \);
2. \( h^\gamma_{\gamma^a} \otimes h^\gamma_{\gamma^a} = h^\gamma_{\gamma^a} \).

**Proof**: We only prove (1), (2) can be proved using similar technology.

Since \( \alpha_i \cdot h = \bigcup_{\gamma, \gamma^a} \left\{ \frac{(1 + \gamma_1)\gamma^a - (1 - \gamma_1)\gamma^a}{1 + \gamma_1 \gamma_2} \right\} \) and
\[ \alpha_i \cdot h = \bigcup_{\gamma, \gamma^a} \left\{ \frac{(1 + \gamma_1)\gamma^a - (1 - \gamma_1)\gamma^a}{1 + \gamma_1 \gamma_2} \right\} \]

Thus \( \alpha_{i+1} \cdot (\alpha_i \cdot h) = (\alpha_i \cdot h) \otimes (\alpha_i \cdot h) \).

(4) Since \( \alpha \cdot h = \bigcup_{\gamma, \gamma^a} \left\{ \frac{(1 + \gamma_1)\gamma^a - (1 - \gamma_1)\gamma^a}{1 + \gamma_1 \gamma_2} \right\} \), then
\[ \alpha_{i+1} \cdot (\alpha_i \cdot h) = \bigcup_{\gamma, \gamma^a} \left\{ \frac{(1 + \gamma_1)\gamma^a - (1 - \gamma_1)\gamma^a}{1 + \gamma_1 \gamma_2} \right\} \]
C. Generalized mean operator

The generalized mean operator was developed by Dyckhoff and Pedrycz [1].

**Definition 6:** A generalized mean (GM) operator of dimension \( n \) is a mapping \( GM: (R^+)^n \rightarrow R^+ \), which has the following form:

\[
GM(a_1, a_2, \cdots, a_n) = \left( \sum_{j=1}^{n} w_j a_j^\lambda \right)^{1/\lambda}
\]

where \( \lambda > 0, w = (w_1, w_2, \cdots, w_n) \) is the weight vector of the arguments \( a_j (j = 1, 2, \cdots, n) \) with \( w_j \geq 0 \) and \( \sum_{j=1}^{n} w_j = 1 \), and \( R^+ \) is the set of all nonnegative real numbers.

D. Prioritized average operator

Yager [34] firstly introduced the prioritized average operator, which is shown as follows.

**Definition 7:** Let \( C = \{ C_1, C_2, \cdots, C_n \} \) be a collection of attributes and that there is a prioritization between the attributes expressed by the linear ordering \( C_1 \succ C_2 \succ \cdots \succ C_n \), indicate attribute \( C_j \) has a higher priority than \( C_k \), if \( j < k \). The value \( C_j(x) \) is the performance of any alternative \( x \) under attribute \( C_j \), and satisfies \( C_j(x) \in [0,1] \). If

\[
PA(C_j(x)) = \frac{\sum_{j=1}^{n} T_j C_j(x)}{\sum_{j=1}^{n} T_j}
\]

where \( T_j = \prod_{k=1}^{j-1} C_k(x) (j = 2, 3, \cdots, n) \) and \( T_1 = 1 \), then \( PA \) is called the prioritized average (PA) operator.

Based on the operational laws (1)-(4) in Definition 3, Wei [25] extend the \( PA \) operators to accommodate the situations where the input arguments are hesitant fuzzy information, and proposed two aggregation operators for \( HFEs \) as listed below.

Let \( h_j (j = 1, 2, \cdots, n) \) be a collection of \( HFEs \), then the hesitant fuzzy prioritized averaging (HFPA) operator

\[
HFPA(h_1, h_2, \cdots, h_n) = \bigcup_{j=1}^{n} \left\{ 1 - \prod_{i=1}^{n} (1 - y_{ij})^{1/T_j} \right\}
\]

(2) The hesitant fuzzy prioritized geometric (HFPG) operator

\[
HFPG(h_1, h_2, \cdots, h_n) = \bigcup_{j=1}^{n} \left\{ \prod_{i=1}^{n} y_{ij}^{1/T_j} \right\}
\]

3. Generalized Hesitant Fuzzy Prioritized Einstein Aggregation Operators

With the help of Einstein operations on \( HFEs \), in this section, we propose two generalized hesitant fuzzy prioritized Einstein aggregation operators, then we study their special cases and discuss some desirable properties.

A. Generalized hesitant fuzzy prioritized Einstein weighted averaging operator

We combine the \( GM \) operator and the \( PA \) operator, and introduce the generalized hesitant fuzzy prioritized Einstein weighted averaging operator.

**Definition 8:** Let \( h_j (j = 1, 2, \cdots, n) \) be a collection of \( HFEs \). A generalized hesitant fuzzy prioritized Einstein weighted averaging \( (GHFPEWA^\lambda_j) \) operator is defined as follows:

\[
GHFPEWA^\lambda_j(h_1, h_2, \cdots, h_n) = \left( \left( \prod_{j=1}^{n} w_j h_j^{\lambda \lambda} \right) \right)^{1/(\lambda \lambda)}
\]

where \( w_j = W_j / T_j \) for \( j = 1, 2, \cdots, n \), \( T_j = \prod_{i=1}^{j-1} s(h_i) (j = 2, 3, \cdots, n) \), \( s(h_j) \) is the score values of \( h_j \).

**Remark 2:** From Definition 8, we can see that the \( GHFPEWA^\lambda_j \) operator has the following advantages. First, the \( GHFPEWA^\lambda_j \) operator can accommodate situations in which the input arguments are hesitant fuzzy information. Second, the \( GHFPEWA^\lambda_j \) operator is based on the arithmetic average, which is one of the basic aggregation techniques and which focuses on the group opinion. Third, the \( GHFPEWA^\lambda_j \) operator is based on Einstein operators, so it is a good alternative for an overall evaluation. Fourth, the \( GHFPEWA^\lambda_j \) operator considers the prioritization relationships among...
the attributes and decision makers. And the weights of attributes and decision makers can be only determined by the source decision information. Finally, the \textit{GHFPEWA}_k operator has an additional parameter \( \lambda \) that controls the power to which the argument values are raised. When we use different choices for the parameter \( \lambda \), we obtain some special cases.

Based on Einstein operations of the HFES, we can drive Theorem 2.

\textbf{Theorem 2:} Let \( h_j (j=1,2,\cdots,n) \) be a collection of HFES, then their aggregated value by using the \textit{GHFPEWA}_k operator is also an HFES, and

\begin{align*}
\text{GHFPEWA}_k (h_1, h_2, \cdots, h_n) = \bigg( \bigg( \prod_{j=1}^n A_j^{\gamma_j} - \prod_{j=1}^n B_j^{\gamma_j} \bigg)^{1/k} \bigg)
\end{align*}

then, when \( n = k + 1 \), by the operational laws (3) and (4) in Definition 5, we have

\begin{align*}
T_{k+1} = \bigg( \prod_{j=1}^k A_j^{\gamma_j} - \prod_{j=1}^k B_j^{\gamma_j} \bigg)
\end{align*}

By the operational laws (1) in Definition 5, we have

\begin{align*}
\sum_{j=1}^k T_j = \bigg( \prod_{j=1}^k A_j^{\gamma_j} - \prod_{j=1}^k B_j^{\gamma_j} \bigg)
\end{align*}

and

\begin{align*}
\prod_{j=1}^k A_j^{\gamma_j} + \prod_{j=1}^k B_j^{\gamma_j}
\end{align*}

i.e. Eq. (4) holds for \( n = k + 1 \). Thus, Eq. (4) holds for all \( n \).

For convenience, let \( \Delta = \prod_{j=1}^n A_j^{\gamma_j} - \prod_{j=1}^n B_j^{\gamma_j} \) and

\begin{align*}
\sum_{j=1}^n T_j = w_j = \sum_{j=1}^n T_j. \quad \Delta = \prod_{j=1}^n A_j^{\gamma_j} - \prod_{j=1}^n B_j^{\gamma_j}
\end{align*}

\begin{align*}
\prod_{j=1}^n A_j^{\gamma_j} + \prod_{j=1}^n B_j^{\gamma_j}
\end{align*}

Thus, Eq. (4) holds for \( n = 1 \).

(2) Suppose that Eq. (4) holds for \( n = k \), that is

\begin{align*}
\bigg( \prod_{j=1}^k A_j^{\gamma_j} - \prod_{j=1}^k B_j^{\gamma_j} \bigg) = \bigg( \prod_{j=1}^k A_j^{\gamma_j} - \prod_{j=1}^k B_j^{\gamma_j} \bigg)
\end{align*}

which completes the proof of the theorem.

Especially, if \( \lambda \to 0 \), then \textit{GHFPEWA}_k operator becomes

\begin{align*}
\text{GHFPEWA}_k (h_1, h_2, \cdots, h_n) = \bigg( \prod_{j=1}^n A_j^{\gamma_j} - \prod_{j=1}^n B_j^{\gamma_j} \bigg)
\end{align*}

If \( k = 1 \), then \( A_j = 2(1 + \gamma_j) \), \( B_j = 2(1 - \gamma_j) \), and the \textit{GHFPEWA}_k operator reduces to the hesitant fuzzy prioritized Einstein weighted averaging (HFPEWA) op-
operator is defined as
\[ \operatorname{HFPEWA}(h_1, h_2, \ldots, h_n) = \bigcup_{\gamma_j, \chi_j, j=1,2,\ldots,n} \left\{ \prod_{j=1}^{n} \left(1 + \gamma_j \right) \frac{\tau_j / \sum_{j} \tau_j - \prod_{j=1}^{n} \left(1 - \gamma_j \right) \frac{\tau_j / \sum_{j} \tau_j}{j} \right) \right\} \]

Furthermore, when the priority level of the aggregated arguments reduces to the same level, the \( \operatorname{GHFPEWA}^\lambda \) operator reduces to the hesitant fuzzy Einstein weighted averaging (\( \operatorname{HEWA} \)) operator
\[ \operatorname{HEWA}(h_1, h_2, \ldots, h_n) = \bigcup_{\gamma_j, \chi_j, j=1,2,\ldots,n} \left\{ \prod_{j=1}^{n} \left(1 + \gamma_j \right)^{\lambda_j} \prod_{j=1}^{n} \left(1 - \gamma_j \right)^{\lambda_j} \right\}, \]

which was developed by Yu [35].

Based on (1) in Proposition 1, we easily get the following property.

**Proposition 2:** If all \( h_j (j=1,2,\ldots,n) \) are equal and the number of values in \( h_j \) is only one, i.e. \( h_j = \{ \gamma \} \) for all \( j = 1,2,\ldots,n \), then
\[ \operatorname{GHFPEWA}^\lambda(h_1, h_2, \ldots, h_n) = h. \]

In general, however, the \( \operatorname{GHFPEWA}^\lambda \) operator is not idempotent. The following example is given to illustrate this case.

**Example 2:** Let \( h_1 = h_2 = h_3 = h = (0.4, 0.6) \), then \( T_1 = 1 \), \( T_2 = 0.5 \), and \( T_3 = 0.25 \). If \( \lambda = 1 \), then \( \operatorname{GHFPEWA}^\lambda(h_1, h_2, h_3) = \{0.4, 0.5068, 0.4656, 0.5644, 0.4443, 0.5457, 0.5068, 0.6\} \).

If \( \lambda = 2 \), then \( \operatorname{GHFPEWA}^\lambda(h_1, h_2, h_3) = \{0.4, 0.5068, 0.4656, 0.5644, 0.4443, 0.5457, 0.5068, 0.6\} \).

By Definition 4, we have \( s(\operatorname{GHFPEWA}^\lambda(h_1, h_2, h_3)) = 0.5042 \), \( s(\operatorname{GHFPEWA}^\lambda(h_1, h_2, h_3)) = 0.5042 \), and \( s(h) = 0.5 \). Therefore \( \operatorname{GHFPEWA}^\lambda(h_1, h_2, h_3) > h \) and \( \operatorname{GHFPEWA}^\lambda(h_1, h_2, h_3) > h \).

**Lemma 1 [36]:** Let \( \gamma_j > 0, w_j > 0, j = 1,2,\ldots,n \) and \( \sum_{j=1}^{n} w_j = 1 \). Then \( \prod_{j=1}^{n} \gamma_j^{w_j} \leq \sum_{j=1}^{n} w_j \gamma_j \) with equality if and only if \( \gamma_1 = \gamma_2 = \cdots = \gamma_n \).

**Theorem 3:** Let \( h_j (j=1,2,\ldots,n) \) be a collection of HFEs, then
\[ \operatorname{HFPEWA}(h_1, h_2, \ldots, h_n) \leq \operatorname{HFW}(h_1, h_2, \ldots, h_n), \]

where the equality holds if and only if \( h_j = \{ \gamma \} \) for all \( j = 1,2,\ldots,n \).

**Proof:** For any \( \gamma_j \in h_j (j=1,2,\ldots,n) \), by Lemma 1, we have
\[ \prod_{j=1}^{n} \left(1 + \gamma_j \right)^{\lambda_j} / \sum_{j=1}^{n} \left(1 + \gamma_j \right) \]
\[ \leq \sum_{j=1}^{n} \left(1 + \gamma_j \right)^{\lambda_j} / \sum_{j=1}^{n} \left(1 + \gamma_j \right) \]
\[ = 1 - \prod_{j=1}^{n} \left(1 - \gamma_j \right)^{\lambda_j} / \sum_{j=1}^{n} \left(1 - \gamma_j \right)^{\lambda_j} \]

It follows that \( s(\operatorname{HFPEWA}(h_1, h_2, \ldots, h_n)) \leq s(\operatorname{HFW}(h_1, h_2, \ldots, h_n)) \), which completes the proof of the theorem.

Theorem 3 shows that the values obtained by the \( \operatorname{HFPEWA} \) operator are not more than the ones obtained by the \( \operatorname{HFW} \) operator proposed by Wei [25] (i.e. Eq. (1)). That is to say, the \( \operatorname{HFPEWA} \) operator shows the decision maker's more pessimistic attitude in aggregation process.

**B. Generalized hesitant fuzzy prioritized Einstein weighted geometric operator**

On the basis of the \( \operatorname{GHFPEWA}^\lambda \) operator and the geometric mean, in what follows, we develop a generalized hesitant fuzzy prioritized Einstein weighted geometric operator.

**Definition 9:** Let \( h_j (j=1,2,\ldots,n) \) be a collection of HFEs. A generalized hesitant fuzzy prioritized Einstein weighted geometric \( \operatorname{GHFPEWG}^\lambda \) operator is defined as follows:
\[ \operatorname{GHFPEWG}^\lambda(h_1, h_2, \ldots, h_n) = \frac{1}{\lambda} \left( \lambda^* \cdot \left( \lambda^* \cdot \left( \lambda^* \cdot \left( \lambda^* \cdot \left( \lambda^* \cdot h_i \right)^{\omega_i} \right)^{\omega_i} \right)^{\omega_i} \right)^{\omega_i} \right). \]
where \( w_j = T_j / \sum_{j=1}^{n} T_j, T_j = \prod_{j=1}^{n} s(h_j) (j = 2, 3, \ldots, n), \lambda > 0, \) 
\( T_1 = 1 \) and \( s(h_j) \) is the score values of \( h_j \).

Remark 3: From Definition 9, we can see that the \( GHFPEWG^\delta \) operator has the following benefits. First, the \( GHFPEWG^\delta \) operator can accommodate situations in which the input arguments are hesitant fuzzy information. Second, the \( GHFPEWG^\delta \) operator is based on the geometric average, which gives more importance to individual opinions. Third, the \( GHFPEWG^\delta \) operator is based on Einstein operators, thus it is a good alternative for an overall evaluation. Fourth, the \( GHFPEWG^\delta \) operator considers the prioritization relationships among the attributes and decision makers. And the weights of attributes and decision makers are absolutely depended on the source decision information. Fifth, the \( GHFPEWG^\delta \) operator has an additional parameter \( \lambda \) that controls the power to which the argument values are raised. With the change of the parameter, the \( GHFPEWG^\delta \) operators can be evolved into many special aggregation operators.

Based on Einstein operations of the HFEs, we can drive Theorem 4.

Theorem 4: Let \( h_j (j = 1, 2, \ldots, n) \) be a collection of HFEs, then their aggregated value by using the \( GHFPEWG^\delta \) operator is also an HFE, and

\[
GHFPEWG^\delta (h_1, h_2, \ldots, h_n) = \bigcup_{j, k, \ldots, \lambda, \mu} \left\{ \left( \prod_{j=1}^{n} C_j^{\lambda_j} + \prod_{j=1}^{n} D_j^{\mu_j} \right)^{1/\lambda_k} \left[ \prod_{j=1}^{n} C_j^{\lambda_j} - \prod_{j=1}^{n} D_j^{\mu_j} \right]^{1/\lambda_k}, \right. \\
where \( w_j = T_j / \sum_{j=1}^{n} T_j, \lambda > 0, \quad C_j = (1 + \gamma_j)^4 + 3(1 - \gamma_j)^4, \quad D_j = (1 + \gamma_j)^4 - (1 - \gamma_j)^4, \quad T_j = \prod_{j=1}^{n} s(h_j) (j = 2, 3, \ldots, n), \quad T_1 = 1 \) and \( s(h_j) \) is the score values of \( h_j \).

Proof: The proof is similar to Theorem 2.

Especially, if \( \lambda = 1 \), then \( C_j = 2(2 - \gamma_j), D_j = 2\gamma_j, \) and the \( GHFPEWG^\delta \) operator reduces to the hesitant fuzzy prioritized Einstein weighted geometric (\( HFPEWG^\delta \)) operator

\[
HFPEWG^\delta (h_1, h_2, \ldots, h_n) = \bigcup_{j, k, \ldots, \lambda, \mu} \left\{ \left( \prod_{j=1}^{n} C_j^{\lambda_j} + \prod_{j=1}^{n} D_j^{\mu_j} \right)^{1/\lambda_k} \left[ \prod_{j=1}^{n} C_j^{\lambda_j} - \prod_{j=1}^{n} D_j^{\mu_j} \right]^{1/\lambda_k}, \right. \\
\]

Furthermore, when the priority level of the aggregated arguments reduced to the same level, the \( GHFPEWG^\delta \) operator reduces to the hesitant fuzzy Einstein weighted geometric (\( HFEWG \)) operator

\[
HFEWG (h_1, h_2, \ldots, h_n) = \bigcup_{j, k, \ldots, \lambda, \mu} \left\{ \left( \prod_{j=1}^{n} C_j^{\lambda_j} + \prod_{j=1}^{n} D_j^{\mu_j} \right)^{1/\lambda_k} \left[ \prod_{j=1}^{n} C_j^{\lambda_j} - \prod_{j=1}^{n} D_j^{\mu_j} \right]^{1/\lambda_k}, \right. \\
\]

which was introduced by Zhou [37].

From (2) in Proposition 1, we can get the following result.

Proposition 3: If all \( h_j (j = 1, 2, \ldots, n) \) are equal and the number of values in \( h_j \) is only one, i.e. \( h_j = \{ \gamma_j \} \) for all \( j = 1, 2, \ldots, n \), then \( HFPEWG^\delta (h_1, h_2, \ldots, h_n) = h \).

Theorem 5: Let \( h_j (j = 1, 2, \ldots, n) \) be a collection of HFEs, then

\[
HFPEWG^\delta (h_1, h_2, \ldots, h_n) \geq HFP(G(h_1, h_2, \ldots, h_n)), \\
\]

where the equality holds if and only if \( h_j = \{ \gamma_j \} \) for all \( j = 1, 2, \ldots, n \).

Proof: For any \( \gamma_j \in h_j (j = 1, \ldots, n) \), by Lemma 1, we have

\[
\prod_{j=1}^{n} \frac{1}{(2 - \gamma_j)^4} + \prod_{j=1}^{n} \frac{1}{\gamma_j^4} \leq \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} (2 - \gamma_j) + \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} \gamma_j = 2. \\
\]

Then

\[
2 \prod_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} = 2 \prod_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} \left( \frac{1}{(2 - \gamma_j)^4} + \frac{1}{\gamma_j^4} \right) \leq \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} (2 - \gamma_j) + \sum_{j=1}^{n} \frac{T_j}{\sum_{j=1}^{n} T_j} \gamma_j = 2. \\
\]

It follows that

\[
s(HFPEWG^\delta (h_1, h_2, \ldots, h_n)) \geq s(HFP(G(h_1, h_2, \ldots, h_n))), \\
\]

which completes the proof of Theorem 5.

Theorem 5 tells us the result that the \( HFPEWG^\delta \) operator shows the decision maker's more optimistic attitude than the \( HFP(G) \) operator proposed Wei [25] (i.e. Eq. (2)) in aggregation process.
Theorem 6: Let \( h_j (j = 1, 2, \ldots, n) \) be a collection of HFEs, then
\[
GHFPEWA^T_h (h_1^j, h_2^j, \ldots, h_n^j)
\]
(1)
\[
= (GHFPEWG^T_h (h_1^j, h_2^j, \ldots, h_n^j))^r;
\]
(2)
\[
GHFPEWG^T_h (h_1^j, h_2^j, \ldots, h_n^j)
\]
\[
= (GHFPEWA^T_h (h_1^j, h_2^j, \ldots, h_n^j))^r.
\]
Proof: We observe that (2) is similar to (1). Next, we only prove (1).

Let \( w_j = \frac{1}{\sum_{j=1}^{n} w_j} \) and \( \lambda > 0 \). By Eq. (3) and Eq. (8), we have
\[
A_j = (2 - \gamma_j)^x + 3\gamma_j^x, \quad B_j = (2 - \gamma_j)^x - \gamma_j^x,
\]
\[
C_j = (1 + \gamma_j)^x + 3(1 - \gamma_j)^x = [2 - (1 - \gamma_j)]^x + 3(1 - \gamma_j)^x,
\]
\[
D_j = (1 + \gamma_j)^x - (1 - \gamma_j)^x = [2 - (1 - \gamma_j)]^x - (1 - \gamma_j)^x.
\]
Then \( GHFPEWA^T_h (h_1^j, h_2^j, \ldots, h_n^j) \)
\[
= \bigcup_{j=1,2,\ldots,n} \left\{ \frac{2}{(\prod_{j=1}^{n} C_j^x - (\prod_{j=1}^{n} D_j^x))^{1/2}} \left[ \begin{array}{c}
\prod_{j=1}^{n} C_j^y + 3\prod_{j=1}^{n} D_j^y \\
\prod_{j=1}^{n} C_j^y - 3\prod_{j=1}^{n} D_j^y
\end{array} \right] \right\}^{1/2}
\]
\[
= \bigcup_{j=1,2,\ldots,n} \left\{ \frac{1}{(\prod_{j=1}^{n} C_j^y + 3\prod_{j=1}^{n} D_j^y) - (\prod_{j=1}^{n} C_j^y - 3\prod_{j=1}^{n} D_j^y)} \right\}^{1/2}
\]
\[
= (GHFPEWG^T_h (h_1^j, h_2^j, \ldots, h_n^j))^r.
\]

4. An Approach to Multiple Attribute Group Decision Making with Hesitant Fuzzy Information

In this section, we shall utilize the proposed aggregation operators to develop an approach for multiple attribute group decision making with hesitant fuzzy information.

Let \( Y = \{Y_1, Y_2, \ldots, Y_m\} \) be a set of alternatives, \( G = \{G_1, G_2, \ldots, G_k\} \) be a collection of attributes. Different attributes may have different degrees of importance, so we assume that there is a prioritization between the attributes expressed by the linear ordering \( G_1 \succ G_2 \succ \cdots \succ G_k \), which indicate attribute \( G_i \) has a higher priority than \( G_s \) if \( s < t \). Let \( D = \{d_1, d_2, \ldots, d_l\} \) be the set of decision makers. As decision makers may have different impact, we suppose that decision makers are in different priority levels expressed by the linear ordering \( d_1 \succ d_2 \succ \cdots \succ d_l \). Due to the complicate properties of modern society, it is very difficult or impossible for decision makers to show their evaluation with mathematical precision. Hesitant fuzzy set is an effective tool to use when decision makers hesitate between several different values of evaluation [20-29]. Suppose that \( h_i^{(k)} = (h_i^{(k)})_{\text{mean}} \) (\( k = 1, 2, \ldots, l \)) is the hesitant fuzzy decision making matrix, where \( h_i^{(k)} \) is an attribute value, which take the form of hesitant fuzzy element, given by the decision maker \( d_i \in D \), for the alternative \( Y_j \in Y \) with respect to the attribute \( G_j \in G \).

Then, we utilize the proposed operators to develop an approach to solve multiple attribute group decision making problems with hesitant fuzzy information, which can be described as below:

Step 1: Calculate the values of \( T_p^{(k)} (k = 1, 2, \ldots, l) \) as
\[
T_p^{(k)} = \prod_{j=1}^{n} s(h_j^{(k)}) \quad (p = 2, 3, \ldots, l),
\]
\[
T_1^{(k)} = 1.
\]

Step 2: Utilize the decision information given in matrix \( R^{(k)} (k = 1, 2, \ldots, l) \), and the \( GHFPEWA^T_h \) operator
\[
h_j = GHFPEWA^T_h (h_1^{(k)}, h_2^{(k)}, \ldots, h_n^{(k)})
\]
\[
= \left\{ \frac{2}{(\prod_{j=1}^{n} A_j^{(k)} - (\prod_{j=1}^{n} B_j^{(k)}))^{1/2}} \right\}
\]
\[
= \bigcup_{j=1,2,\ldots,n} \left\{ \frac{1}{(\prod_{j=1}^{n} A_j^{(k)}) + 3(\prod_{j=1}^{n} B_j^{(k)})} \right\}^{1/2}
\]
\[
= (GHFPEWG^T_h (h_1^{(k)}, h_2^{(k)}, \ldots, h_n^{(k)}))^r.
\]

where
\[
A_j^{(k)} = (2 - \gamma_j^{(k)})^x + 3(\gamma_j^{(k)})^x, \quad B_j^{(k)} = (2 - \gamma_j^{(k)})^x - (\gamma_j^{(k)})^x,
\]
\[
w_j^{(k)} = \frac{T_p^{(k)}}{\sum_{j=1}^{n} T_j^{(k)}}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n,
\]
or the \( GHFPEWG^T_h \) operator
\[
h_j = GHFPEWG^T_h (h_1^{(k)}, h_2^{(k)}, \ldots, h_n^{(k)})
\]
\[
= \left\{ \frac{2}{(\prod_{j=1}^{n} C_j^{(k)} + 3(\prod_{j=1}^{n} D_j^{(k)}))^{1/2}} \right\}
\]
\[
= \bigcup_{j=1,2,\ldots,n} \left\{ \frac{1}{(\prod_{j=1}^{n} C_j^{(k)}) + 3(\prod_{j=1}^{n} D_j^{(k)})} \right\}^{1/2}
\]
\[
= (GHFPEWA^T_h (h_1^{(k)}, h_2^{(k)}, \ldots, h_n^{(k)}))^r.
\]
where \( C_y^{(k)} = (1 + \gamma_y^{(k)})^2 + 3 \left(1 - \gamma_y^{(k)}\right)^2 \),
\[ D_y^{(k)} = \left(1 + \gamma_y^{(k)}\right)^2 - \left(1 - \gamma_y^{(k)}\right)^2, \]
\[ w_y^{(k)} = \frac{T_y^{(k)}}{\sum_{i=1}^{n} T_y^{(k)}, i=1,2,\ldots,m, j=1,2,\ldots,n}, \]
to aggregate all the individual decision matrices \( R^{(k)} = (k=1,2,\ldots,l) \) into the collective decision matrix \( R = (h)^{max} \).

Step 3: Calculate the values of \( T_y^{(i)} (i=1,2,\ldots,m) \) as
\[ T_y^{(i)} = \frac{\sum_{q=1}^{\eta} s(h_y^{(q)})}{\eta = 2,3,\ldots,n}, \quad (14) \]

Step 4: Aggregate all hesitant fuzzy preference value \( h_y (j=1,2,\ldots,n) \) by using the \( GHFPEW_{\alpha}^k \) operator
\[ h = GHFPEW_{\alpha}^k (h_1,h_2,\ldots,h_n) \]
\[ = \bigcup_{j=1}^{2} \left( \frac{\left( \sum_{j=1}^{n} A_{ij}^{(k)} + 3 \left( \sum_{j=1}^{n} B_{ij}^{(k)} \right) \right)^{1/\alpha}}{\left( \sum_{j=1}^{n} A_{ij}^{(k)} + 3 \left( \sum_{j=1}^{n} B_{ij}^{(k)} \right) \right)^{1/\alpha}} \right), \quad (15) \]
where \( A_{ij} = (2 - \gamma_y^{(i)})^2 + 3 \gamma_y^{(i)}, B_{ij} = (2 - \gamma_y^{(i)})^2 - \gamma_y^{(i)} \),
\[ w_y^{(i)} = \frac{T_y^{(i)}}{\sum_{j=1}^{n} T_y^{(i)}, i=1,2,\ldots,m, j=1,2,\ldots,n}, \]

or the \( GHFPEWG_{\alpha}^k \) operator
\[ h = GHFPEWG_{\alpha}^k (h_1,h_2,\ldots,h_n) \]
\[ = \bigcup_{j=1}^{2} \left( \frac{\left( \sum_{j=1}^{n} C_{yj}^{(k)} + 3 \left( \sum_{j=1}^{n} D_{yj}^{(k)} \right) \right)^{1/\alpha}}{\left( \sum_{j=1}^{n} C_{yj}^{(k)} + 3 \left( \sum_{j=1}^{n} D_{yj}^{(k)} \right) \right)^{1/\alpha}} \right), \quad (16) \]
where \( C_y = (1 + \gamma_y)^2 + 3(1-\gamma_y)^2, D_y = (1 + \gamma_y)^2 - (1 - \gamma_y)^2, \)
\[ w_y^{(i)} = \frac{T_y^{(i)}}{\sum_{j=1}^{n} T_y^{(i)}, i=1,2,\ldots,m, j=1,2,\ldots,n}, \]
to derive the overall hesitant fuzzy preference value \( h_y (i=1,2,\ldots,m) \) of the alternative \( Y_i \).

Step 5: Calculate the score values by Definition 4.

Step 6: Rank all the alternatives \( Y_i (i=1,2,\ldots,m) \) and select the best one(s) in accordance with the score values.

### 5. Illustrative Example

In this section, we present an example (adapted from Yu [38]) to illustrate the application of the proposed method.

The talented people are the knowledge economy time most valued asset. Suppose there are five possible candidates: \( Y_1, Y_2, Y_3, Y_4 \) and \( Y_5 \). After careful consideration, the decision makers \( d_i (k=1,2,3) \) select three attributes: \( G_1 \) is work attitude; \( G_2 \) is working capability; \( G_3 \) is learning skill. It is well known that the work attitude is the most important feature of the recruitment. If someone owns the negative work attitude, then he should not be employed no matter how good performance he received in working capability \( G_2 \), learning skill \( G_3 \). To put it in another way, there exist prioritization relationships between these attributes and denoted by \( G_1 > G_2 > G_3 \). As three decision makers are from different classes, they have different impact, which is denoted by \( d_1 > d_2 > d_3 \). The decision makers \( d_i (k=1,2,3) \) evaluate the alternatives \( Y_i (i=1,2,3,4,5) \) with respect to the attributes \( G_j (j=1,2,3) \) and construct the following hesitant fuzzy decision matrices \( R^{(k)} \) (see Table 1-3).

| Table 1. Hesitant fuzzy decision making matrix \( R^{(1)} \). |
|---|---|---|
| \( Y_1 \) | \( G_1 \) | \( G_2 \) | \( G_3 \) |
| \( Y_1 \) | \( 0.4 \) | \( 0.3 \) | \( 0.5 \) |
| \( Y_2 \) | \( 0.2,0.3 \) | \( 0.3 \) | \( 0.5,0.6 \) |
| \( Y_3 \) | \( 0.4,0.6,0.7 \) | \( 0.4,0.5 \) | \( 0.5,0.7 \) |
| \( Y_4 \) | \( 0.2,0.4 \) | \( 0.7,0.8,0.9 \) | \( 0.6,0.8 \) |
| \( Y_5 \) | \( 0.5,0.7 \) | \( 0.6,0.7,0.8 \) | \( 0.7,0.9 \) |

| Table 2. Hesitant fuzzy decision making matrix \( R^{(2)} \). |
|---|---|---|
| \( Y_1 \) | \( 0.6,0.7 \) | \( 0.5 \) | \( 0.2,0.4 \) |
| \( Y_2 \) | \( 0.4,0.5,0.7 \) | \( 0.4,0.5 \) | \( 0.7,0.9 \) |
| \( Y_3 \) | \( 0.4,0.6 \) | \( 0.6,0.7,0.8 \) | \( 0.2,0.3 \) |
| \( Y_4 \) | \( 0.3,0.4,0.5 \) | \( 0.5,0.7 \) | \( 0.4,0.6 \) |
| \( Y_5 \) | \( 0.4,0.5 \) | \( 0.3,0.5 \) | \( 0.6,0.7,0.8 \) |

| Table 3. Hesitant fuzzy decision making matrix \( R^{(3)} \). |
|---|---|---|
| \( Y_1 \) | \( 0.7 \) | \( 0.4,0.5 \) | \( 0.5,0.6 \) |
| \( Y_2 \) | \( 0.4,0.7 \) | \( 0.5,0.6,0.7 \) | \( 0.3,0.4 \) |
| \( Y_3 \) | \( 0.5,0.6 \) | \( 0.3,0.5 \) | \( 0.2,0.3,0.4 \) |
| \( Y_4 \) | \( 0.5,0.7,0.8 \) | \( 0.2,0.4 \) | \( 0.5,0.6 \) |
| \( Y_5 \) | \( 0.4,0.6 \) | \( 0.1,0.3,0.5 \) | \( 0.6,0.8 \) |

In order to select the best candidate, we utilize the \( GHFPEW_{\alpha}^k \) operator to develop an approach to multiple attribute group decision making problems with hesitant fuzzy information, which can be described as below:
Step 1: Utilize Eq. (11) to calculate the \( T_y^{(k)} (k=1,2,3) \) as
\[
T_y^{(1)} = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix},
\]
\[
T_y^{(2)} = \begin{bmatrix}
0.4 & 0.35 & 0.5 \\
0.25 & 0.4 & 0.4 \\
0.5667 & 0.45 & 0.6 \\
0.3 & 0.8 & 0.7 \\
0.6 & 0.7 & 0.8 \\
\end{bmatrix}, \quad T_y^{(3)} = \begin{bmatrix}
0.26 & 0.175 & 0.15 \\
0.1333 & 0.18 & 0.32 \\
0.2833 & 0.315 & 0.15 \\
0.12 & 0.48 & 0.35 \\
0.27 & 0.28 & 0.56 \\
\end{bmatrix}.
\]

Step 2: Utilize the decision information given in matrices \( R^{(k)} (k=1,2,3) \), and the \( GHFPEWA_i \) operator (i.e. Eq. (12)) to aggregate all the individual decision matrices \( R^{(k)} \) into the collective decision matrix \( R=(h_{iy})_{5 \times 3} \).

Take candidate \( Y_i \) for an example, and let \( \lambda = 1 \), we have
\[
h_1 = GHFPEWA_i \left( h_1^{(1)}, h_1^{(2)}, h_1^{(3)} \right) = HFPWEA_i \left( (0.4),(0.6,0.7),(0.7) \right) = \{0.6588,0.6792\},
\]
and \( h_3 = \{0.4933,0.5517,0.4933,0.5517\} \). Other results can be obtained similarly.

Step 3: Utilize Eq. (14) to calculate the values of \( T_y^{(i)} (i=1,2,\cdots,5, j=1,2,3) \) as
\[
T_y = \begin{bmatrix}
1 & 0.6687 & 0.3186 \\
1 & 0.5791 & 0.3263 \\
1 & 0.6088 & 0.4074 \\
1 & 0.6159 & 0.4819 \\
1 & 0.6061 & 0.4149 \\
\end{bmatrix}.
\]

Step 4: Aggregate all hesitant fuzzy preference value \( h_i (j=1,2,3) \) by using the \( GHFPEWA_i \) operator (i.e. Eq. (15)) to derive the overall hesitant fuzzy preference values \( h_i (1,2,\cdots,5) \) of the candidates \( Y_i \). Take candidate \( Y_i \) for an example, and let \( \lambda = 1 \), we have
\[
h_i = GHFPEWA_i (h_i, h_2, h_3) = HFPWEA_i (0.6588,0.6792),
\]
\[
(0.4729,0.4803,0.4731,0.4804),(0.4933,0.5517,0.4933,0.5517)) = \{0.4542,0.4554,0.4606,0.4618,0.4576,0.4589,0.4640,0.4652,
\]
\[
0.4719,0.4731,0.4782,0.4794,0.4753,0.4765,0.4815,0.4827,0.4732,
\]
\[
0.4745,0.4795,0.4807,0.4766,0.4778,0.4828,0.4840,0.4906,0.4918,
\]
\[
0.4967,0.4978,0.4939,0.4951,0.4999,0.5011\}.
\]

As the parameter \( \lambda \) changes, we can get different results for each candidate. Here we will not list them for vast amounts of data.

Step 5: Compute the score values \( s(h_i) (i=1,2,\cdots,5) \) of \( h_i (i=1,2,\cdots,5) \) by Definition 4. The score values for the candidates are shown in Table 4.

Step 6: By ranking \( s(h_i) (i=1,2,3,4,5) \), we can get the priorities of the candidates \( Y_i (i=1,2,\cdots,5) \) as the parameter \( \lambda \) changes, which are listed in Table 4.

From Table 4, we can find that the score values obtained by the \( GHFPEWA_i \) operator become bigger as the parameter \( \lambda \) increases for the same aggregation arguments, and the ranking of candidates are different with different values of the parameter \( \lambda \).

To investigate the variation trends of the scores and the rankings of the candidates with the change of the values of the parameter \( \lambda \), we use figure to illustrate these issues. Fig. 1 gives the scores of the candidates obtained by the \( GHFPEWA_i \) operator as \( \lambda \) is assigned different values. We can find that

1. when \( \lambda \in (0,2.3] \), the ranking is \( Y_5>Y_4>Y_3>Y_2>Y_1 \);
2. when \( \lambda \in (2.3,4.8] \), the ranking is \( Y_4>Y_5>Y_3>Y_2>Y_1 \);
3. when \( \lambda \in (4.8,7.15] \), the ranking is \( Y_3>Y_2>Y_5>Y_4>Y_1 \);
4. when \( \lambda \in (7.15,20] \), the ranking is \( Y_3>Y_2>Y_1>Y_4>Y_5 \).

Moreover, it can be easily seen that the scores of candidates increase as the parameter \( \lambda \) increases. Thus, the decision makers with optimistic attitude should use a larger parameter when they utilize the \( GHFPEWA_i \) operator in real decision making problems.

Table 4. Score values and the rankings of candidates based on \( \lambda \)

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( F_1 )</th>
<th>( F_2 )</th>
<th>( F_3 )</th>
<th>( F_4 )</th>
<th>( F_5 )</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>( GHFPEWA_i )</td>
<td>0.4779</td>
<td>0.3870</td>
<td>0.5311</td>
<td>0.4863</td>
<td>0.5961</td>
<td>( Y_3&gt;Y_1&gt;Y_2&gt;Y_4&gt;Y_5 )</td>
</tr>
<tr>
<td>( GHFPEWA_i )</td>
<td>0.5169</td>
<td>0.4783</td>
<td>0.5636</td>
<td>0.5856</td>
<td>0.6371</td>
<td>( Y_3&gt;Y_2&gt;Y_5&gt;Y_4&gt;Y_1 )</td>
</tr>
<tr>
<td>( GHFPEWA_i )</td>
<td>0.5574</td>
<td>0.5608</td>
<td>0.5955</td>
<td>0.6568</td>
<td>0.6784</td>
<td>( Y_3&gt;Y_2&gt;Y_5&gt;Y_4&gt;Y_1 )</td>
</tr>
<tr>
<td>( GHFPEWA_i )</td>
<td>0.6139</td>
<td>0.6681</td>
<td>0.6423</td>
<td>0.7319</td>
<td>0.7395</td>
<td>( Y_3&gt;Y_2&gt;Y_5&gt;Y_4&gt;Y_1 )</td>
</tr>
<tr>
<td>( GHFPEWA_i )</td>
<td>0.6538</td>
<td>0.7340</td>
<td>0.6785</td>
<td>0.7758</td>
<td>0.7855</td>
<td>( Y_3&gt;Y_2&gt;Y_5&gt;Y_4&gt;Y_1 )</td>
</tr>
</tbody>
</table>

Figure 1. Scores for candidates obtained by \( GHFPEWA_i \) operator.
In Step 2 and Step 4, if we use the $GHFPEWG^a_\lambda$ operator instead of the $GHFPEWA^a_\lambda$ operator, i.e. utilize Eq. (13) and Eq. (16) to aggregation the values of the candidates, respectively, then the corresponding results can be obtained and they are listed in Table 5.

The variation trends of the scores and the rankings of the candidates with the change of the values of the parameter $\lambda$ are shown in Fig. 2. We can see that

1. when $\lambda \in (0, 2.3]$, the ranking is $Y_5 > Y_1 > Y_3 > Y_4 > Y_2$;
2. when $\lambda \in (2.3, 4.1]$, the ranking is $Y_5 > Y_1 > Y_3 > Y_4 > Y_2$;
3. when $\lambda \in (4.1, 5.5]$, the ranking is $Y_5 > Y_1 > Y_3 > Y_4 > Y_2$;
4. when $\lambda \in (5.5, 7.8]$, the ranking is $Y_5 > Y_1 > Y_3 > Y_4 > Y_2$;
5. when $\lambda \in (7.8, 12.3]$, the ranking is $Y_5 > Y_1 > Y_3 > Y_4 > Y_2$;
6. when $\lambda \in (12.3, 20]$, the ranking is $Y_5 > Y_1 > Y_3 > Y_4 > Y_2$.

Furthermore, we can find that the scores of candidates decrease as the value of the parameter $\lambda$ increases. So the decision makers with pessimistic attitude should use a larger parameter when they utilize the $GHFPEWG^a_\lambda$ operator in real decision making problems.

Table 5. Score values and the rankings of candidates based on $GHFPEWG^a_\lambda$.

<table>
<thead>
<tr>
<th>$Y_1$</th>
<th>$Y_2$</th>
<th>$Y_3$</th>
<th>$Y_4$</th>
<th>$Y_5$</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>$GHFPEWG^a_\lambda$</td>
<td>0.4556</td>
<td>0.3442</td>
<td>0.4479</td>
<td>0.4062</td>
<td>$Y_5 &gt; Y_1 &gt; Y_3 &gt; Y_4 &gt; Y_2$</td>
</tr>
<tr>
<td>$GHFPEWG^a_\lambda$</td>
<td>0.4290</td>
<td>0.3166</td>
<td>0.3973</td>
<td>0.3616</td>
<td>$Y_5 &gt; Y_1 &gt; Y_3 &gt; Y_4 &gt; Y_2$</td>
</tr>
<tr>
<td>$GHFPEWG^a_\lambda$</td>
<td>0.4065</td>
<td>0.2953</td>
<td>0.3534</td>
<td>0.3335</td>
<td>$Y_5 &gt; Y_1 &gt; Y_3 &gt; Y_4 &gt; Y_2$</td>
</tr>
<tr>
<td>$GHFPEWG^a_\lambda$</td>
<td>0.3687</td>
<td>0.2720</td>
<td>0.3018</td>
<td>0.2899</td>
<td>$Y_5 &gt; Y_1 &gt; Y_3 &gt; Y_4 &gt; Y_2$</td>
</tr>
<tr>
<td>$GHFPEWG^a_\lambda$</td>
<td>0.3286</td>
<td>0.2561</td>
<td>0.2665</td>
<td>0.2275</td>
<td>$Y_5 &gt; Y_1 &gt; Y_3 &gt; Y_4 &gt; Y_2$</td>
</tr>
</tbody>
</table>

Figure 2. Scores for candidates obtained by $GHFPEWG^a_\lambda$ operator.

Remark 4: In the following, we compare our operators and methods with existing aggregation operators and methods to demonstrate the advantages of the operators and methods proposed.

1. We compare the new operators and methods with non-fuzzy (e.g., crisp aggregation) and non hesitant (e.g., traditional fuzzy aggregation) operators and the corresponding methods. As the hesitant fuzzy set is a generalization of the crisp set and the traditional fuzzy set, the proposed operators and methods can be applied to deal with decision making problems in which the attribute values are crisp or fuzzy values. Conversely, the classical aggregation operators and traditional fuzzy aggregation operators cannot be applied to the decision making problems where the attribute values are given in the form of hesitant fuzzy information.

2. In Section 2, we list two existing hesitant fuzzy prioritized operators introduced by Wei [25]. In the following, we compare our operators and methods with the existing hesitant fuzzy prioritized operators and methods:
   i) It is well known that the algebraic operations focus on the average opinion while the Einstein operations focus on overall opinion. Therefore, the $HFP A_\lambda$ (i.e. Eq. (1)) and $HFP \lambda G$ (i.e. Eq. (2)) operators, based on algebraic operations, are good selections to an average evaluation. The proposed $GHFPEWA^a_\lambda$ and $GHFPEWG^a_\lambda$ operators are good alternatives for an overall evaluation. ii) We introduce an alterable parameter into the $GHFPEWA^a_\lambda$ and $GHFPEWG^a_\lambda$ operators. With the change of the parameter, the proposed operators can be evolved into lots of different aggregation operators, which make decision making more flexible and can meet the needs of different types of decision makers. But the $HFP A_\lambda$ (or $HFP \lambda G$) operator has not alterable parameter, so they can only satisfy the demand of a type of decision makers. To further compare them, we use the $HFP A_\lambda$ and $HFP \lambda G$ operators to aggregate hesitant fuzzy information in Step 2 and Step 4, respectively. Then the corresponding score values and rankings of candidates can be obtained and they are shown in Table 6. For comparison purposes, we copy the first line in Table 4 and Table 5 to the second line and the fourth line in Table 6, respectively. Table 6 indicates that the score values obtained by the $HFPPEWA^a_\lambda$ (or $HFPPEWG^a_\lambda$) operator are always not bigger (or smaller) than the values obtained by the $HFP A_\lambda$ (or $HFP \lambda G$) operator for the same aggregation operators and the corresponding problems, as required by Theorem 3 (or Theorem 5).

Table 6. Score values and the rankings of candidates based on $HFP \lambda G$ and $HFP \lambda G W$.

<table>
<thead>
<tr>
<th>$Y_1$</th>
<th>$Y_2$</th>
<th>$Y_3$</th>
<th>$Y_4$</th>
<th>$Y_5$</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>$HFP A_\lambda$</td>
<td>0.4836</td>
<td>0.3933</td>
<td>0.5367</td>
<td>0.5047</td>
<td>$Y_5 &gt; Y_1 &gt; Y_3 &gt; Y_4 &gt; Y_2$</td>
</tr>
<tr>
<td>$HFPPEWA^a_\lambda$</td>
<td>0.4779</td>
<td>0.3870</td>
<td>0.5311</td>
<td>0.4863</td>
<td>$Y_5 &gt; Y_1 &gt; Y_3 &gt; Y_4 &gt; Y_2$</td>
</tr>
<tr>
<td>$HFP \lambda G$</td>
<td>0.1767</td>
<td>0.1085</td>
<td>0.1317</td>
<td>0.1222</td>
<td>$Y_5 &gt; Y_1 &gt; Y_3 &gt; Y_4 &gt; Y_2$</td>
</tr>
<tr>
<td>$HFPPEWG^a_\lambda$</td>
<td>0.4556</td>
<td>0.3442</td>
<td>0.4479</td>
<td>0.4062</td>
<td>$Y_5 &gt; Y_1 &gt; Y_3 &gt; Y_4 &gt; Y_2$</td>
</tr>
</tbody>
</table>
(3) We compare the new operators and methods with existing hesitant fuzzy aggregation operators and the corresponding methods. Recently, many operators for aggregating hesitant fuzzy information, such as the hesitant fuzzy weighted averaging (HFWA), hesitant fuzzy weighted geometric, hesitant fuzzy ordered weighted averaging, hesitant fuzzy ordered weighted geometric, HFEWA and HFEWG operators [20, 35, 37], have been developed. However, these existing hesitant fuzzy aggregation operators cannot deal with the multiple attribute decision making problems where the attributes and decision makers have different priority levels. And the weights of attributes and decision makers must be given in advance, which are not depended on the source decision information. The new hesitant fuzzy prioritized Einstein aggregation operators can account for the prioritization relationships among the attributes and decision makers. And the weights of attributes and decision makers can be uniquely determined by the source decision information. Thus, the proposed method can overcome the difficulties because the weighting vectors of the attributes and the decision makers can be obtained by the source decision information. Therefore, the proposed method is more objective and reasonable than Xia’s method [20].

6. Conclusions

HFS has been successfully used as new mathematical tools to deal with uncertainties. Many aggregation operators and decision making approaches under hesitant fuzzy environment have been explored. All these existing hesitant fuzzy aggregation techniques are limited to the algebraic operational laws of HFEs. In view of the fact that Einstein operations typically give the same results, their special cases have given and the corresponding results can be obtained by Xia’s method. And they are listed in Table 7.

Table 7. Score values and the rankings of candidates based on HFWA.

<table>
<thead>
<tr>
<th>HFWA</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y1</td>
<td>Y2</td>
</tr>
<tr>
<td>(\omega = (0.5,0.3,0.2)^T) 0.4638 0.4244 0.5420 0.5848 0.6478</td>
<td>(Y_5 &gt; Y_4 &gt; Y_3 &gt; Y_2 &gt; Y_1)</td>
</tr>
<tr>
<td>(\omega = (0.2,0.3,0.5)^T) 0.5931 0.5209 0.5269 0.5546 0.5324</td>
<td>(Y_2 &gt; Y_1 &gt; Y_3 &gt; Y_4 &gt; Y_5)</td>
</tr>
<tr>
<td>(w = (0.7,0.2,0.1)^T) 0.5427 0.5361 0.4930 0.5616 0.5822</td>
<td>(Y_4 &gt; Y_3 &gt; Y_2 &gt; Y_5 &gt; Y_1)</td>
</tr>
<tr>
<td>(\omega = (0.4,0.3,0.3)^T) 0.5778 0.5278 0.5112 0.5580 0.5610</td>
<td>(Y_3 &gt; Y_5 &gt; Y_4 &gt; Y_2 &gt; Y_1)</td>
</tr>
<tr>
<td>(w = (0.2,0.3,0.5)^T) 0.6022 0.6064 0.5822 0.5580 0.5610</td>
<td>(Y_2 &gt; Y_1 &gt; Y_3 &gt; Y_4 &gt; Y_5)</td>
</tr>
<tr>
<td>(\omega = (0.5,0.3,0.2)^T) 0.4638 0.4244 0.5420 0.5848 0.6478</td>
<td>(Y_5 &gt; Y_4 &gt; Y_3 &gt; Y_2 &gt; Y_1)</td>
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<tr>
<td>(\omega = (0.2,0.3,0.5)^T) 0.5931 0.5209 0.5269 0.5546 0.5324</td>
<td>(Y_2 &gt; Y_1 &gt; Y_3 &gt; Y_4 &gt; Y_5)</td>
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<tr>
<td>(w = (0.7,0.2,0.1)^T) 0.5427 0.5361 0.4930 0.5616 0.5822</td>
<td>(Y_4 &gt; Y_3 &gt; Y_2 &gt; Y_5 &gt; Y_1)</td>
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<tr>
<td>(w = (0.2,0.3,0.5)^T) 0.6022 0.6064 0.5822 0.5580 0.5610</td>
<td>(Y_2 &gt; Y_1 &gt; Y_3 &gt; Y_4 &gt; Y_5)</td>
</tr>
</tbody>
</table>

From Table 7, we can find that the rankings of candidates are different when the weighting vectors of the attributes and the decision makers are different. In other words, the result depends on the weighting vectors of the attributes and the decision makers rather than the source decision information. However, our method can overcome the difficulties because the weighting vectors of the attributes and the decision makers can be obtained by the source decision information. Thus, the proposed method is more objective and reasonable than Xia’s method [20].
levels. Furthermore, we have given a numerical example to illustrate the hesitant fuzzy multiple attribute group decision making process and advantages of the proposed operators and methods. The results have shown that the proposed approach not only consider the prioritization relationships over attributes and between decision makers but also provide more selections for decision makers according to their preferences.

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