Some Generalized Neutrosophic Number Hamacher Aggregation Operators and Their Application to Group Decision Making

Peide Liu, Yanchang Chu, Yanwei Li, and Yubao Chen

Abstract

The neutrosophic set (NS) can be better to express the incomplete, indeterminate and inconsistent information, and Hamacher aggregation operators extended the Algebraic and Einstein aggregation operators and the generalized aggregation operators can generalize the arithmetic, geometric, and quadratic aggregation operators. In this paper, we combined Hamacher operations and generalized aggregation operators to NS, and proposed some new aggregation operators. Firstly, we presented some new operational laws for neutrosophic numbers (NNs) based on Hamacher operations and studied their properties. Then, we proposed the generalized neutrosophic number Hamacher weighted averaging (GNNHWA) operator, generalized neutrosophic number Hamacher ordered weighted averaging (GNNHOWA) operator, and generalized neutrosophic number Hamacher hybrid averaging (GNNHHHA) operator, and explored some properties of these operators and analyzed some special cases of them. Furthermore, we gave a new method based on these operators for multiple attribute group decision making problems with neutrosophic numbers, and the operational steps were illustrated in detail. Finally, an application example is given to verify the proposed method and to demonstrate its effectiveness.

Keywords: Group decision-making, neutrosophic set (NS), Hamacher aggregation operator, generalized aggregation operator.

1. Introduction

In real decision making, the decision information is often incomplete, indeterminate and inconsistent information. Zadeh [1] firstly proposed the fuzzy set theory which is a very useful tool to process fuzzy information. However, it has a shortcoming that it only has a membership function, and cannot express non-membership function. Then Atanassov [2, 3] proposed the intuitionistic fuzzy set (IFS) by adding a non-membership function, i.e., the intuitionistic fuzzy sets consider both membership (or called truth-membership) $A^{T}(x)$ and non-membership (or called falsity-membership) $A^{F}(x)$ and satisfy the conditions $0 \leq A^{T}(x) + A^{F}(x) \leq 1$. IFSs can only handle incomplete information not the indeterminate information and inconsistent information. In IFSs, the indeterminacy (or called Hesitation degree) is $1 - (A^{T}(x) + A^{F}(x))$. Further, on the basis of IFS, Smarandache [4] proposed the neutrosophic set (NS) by adding an independent indeterminacy-membership $A^{I}(x)$, which is a generalization of IFSs. When $A^{I}(x) = 1 - A^{T}(x) - A^{F}(x)$, NS will become the IFS, i.e., IFS is a special case of NS. Because the indeterminacy is quantified explicitly, and truth-membership, indeterminacy membership, and false-membership are completely independent, NSs can handle the incomplete, indeterminate and inconsistent information. When $T_{A}(x) + I_{A}(x) + F_{A}(x) < 1$, it shows this is indeterminate information, and when $T_{A}(x) + I_{A}(x) + F_{A}(x) > 1$, it is inconsistent information.

Recently, NSs have caused the widespread concerns and made some applications. Wang et al. [5] proposed a single valued neutrosophic set (SVNS) by adding the conditions $0 \leq T_{A}(x), I_{A}(x), F_{A}(x) \leq 0.1$, and $0 \leq T_{A}(x) + I_{A}(x) + F_{A}(x) \leq 1$. Obviously, the SVNS is an instance of the neutrosophic set. Ye [6] proposed the correlation coefficient and weighted correlation coefficient of SVNSs, and proved that the cosine similarity degree is a special case of the correlation coefficient in SVNS. Further, a comparison method for SVNSs based on the correlation coefficient was proposed. Similar to extension from IFS to interval-valued intuitionistic fuzzy set (IVIFS) [7, 8], Wang et al. [9] defined interval neu-
Neutrosophic sets (INSs) in which the truth-membership, indeterminacy-membership, and false-membership were extended to interval numbers, and discussed various properties of INSs. INSs can easily express the incomplete, indeterminate and inconsistent information. Ye [10] defined the similarity measures between INSs on the basis of the Hamming and Euclidean distances, and proposed a multiple attribute decision-making method based on the similarity degree. Guo et al. [11], Guo and Cheng [12] applied neutrosophic sets to process the images with noise and proposed a new neutrosophic approach for image segmentation. However, so far, there has been no research on aggregation operators for INSs.

The information aggregation operators are the important research areas, which are receiving wide attentions [13-29]. Yager [13] proposed the ordered weighted average (OWA) operator which weighted the inputs according to the ranking position of them. Xu [14], Xu and Yager [15] proposed some arithmetic aggregation operators and geometric aggregation operators for intuitionistic fuzzy information. Zhao [16] extended generalized aggregation operators to intuitionistic fuzzy sets and proposed generalized weighted operator, generalized hybrid operator and distance operator for intuitionistic fuzzy information. The generalized aggregation operators can generalize arithmetic and geometric aggregation operators. All above aggregation operators are based on the algebraic operational rules of intuitionistic fuzzy numbers (IFNs) which are one type of operations that can be chosen to model the intersection and union of IFNs. In general, a general t-norm and t-conorm can always be used to model the intersection and union of IFNs [17, 18]. Wang and Liu [19] proposed the some Einstein aggregation operators, which have the same smooth approximations as the algebraic operators. Further, they were extended to intuitionistic fuzzy sets. Hamacher t-conorm and t-norm are the generalization of algebraic and Einstein t-conorm and t-norm [19], Liu [21] proposed some Hamacher aggregation operators for the interval-valued intuitionistic fuzzy numbers. However, until to now, there is no research about neutrosophic number aggregation operators based on Hamacher t-conorm and t-norm. Because the NS can be better to express the incomplete, indeterminate and inconsistent information, and Hamacher aggregation operators extend the algebraic and Einstein aggregation operators and the generalized aggregation operators can generalize the arithmetic, geometric and quadratic aggregation operators. So, it is meaningful to research the aggregation operators based on Frank operations and the generalized aggregation operators for SVNSs, and apply them to MAGDMD problems with neutrosophic information.

The remainder of this paper is shown as follows. In Section 2, we briefly review some basic concepts and operational rules of SVNS, propose some new operational laws for neutrosophic numbers based on Hamacher t-conorm and t-norm and discuss some properties. In Section 3, we propose the generalized neutrosophic number Hamacher weighted averaging (GNNHWGA) operator, generalized neutrosophic number Hamacher ordered weighted averaging (GNNHOWA) operator, and generalized neutrosophic number Hamacher hybrid average (GNNHHA) operator, discussed various desirable properties of these operators and analyze some special cases of them. In Section 4, we propose a new method based on these operators for multi-attribute group decision making with SVNSs. In Section 5, we give an example to show the application of proposed method, and compare the developed method with the existing methods. In Section 6, we end this paper with some conclusions.

2. Preliminaries

A. The single valued neutrosophic set

Definition 1 [4]: Let \(X\) be a universe of discourse, with a generic element in \(X\) denoted by \(x\). A neutrosophic set \(A\) in \(X\) is

\[
A = \{(T(x), I(x), F(x)) | x \in X\}
\]

where, \(T(x)\) is the truth-membership function, \(I(x)\) is the indeterminacy-membership function, and \(F(x)\) is the falsity-membership function. \(T(x), I(x), F(x)\) are real standard or nonstandard subsets of \([0, 1]\) with the proposed method, and compare the developed method with the existing methods. In Section 6, we end this paper with some conclusions.

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and then

1. \( sc(x) = T_1 + 1 - I_1 + 1 - F_i \);
2. \( ac(x) = T_2 - F_i \)

where \( sc(x) \) and \( ac(x) \) represent the score function and accuracy function of the SVNN, respectively.

**Definition 7** [16]: Let \( x = (T_1, I_1, F_1) \) and \( y = (T_2, I_2, F_2) \) be two SVNNs. The comparison approach can be defined as follows.

1. if \( sc(x) > sc(y) \), then \( x \) is greater than \( y \), and denoted by \( x \succ y \).
2. if \( sc(x) = sc(y) \) and \( ac(x) > ac(y) \), then \( x \) is greater than \( y \) and denoted by \( x \succ y \).
3. if \( sc(x) = sc(y) \) and \( ac(x) = ac(y) \), then \( x \) is equal to \( y \), and denoted by \( x \asymp y \).

**B. 2.2 GHWA operator**

**Definition 5** [16]: A GWA operator of dimension \( n \) is a mapping GWA: \( (R^n) \rightarrow R^+ \). Such that,

\[
GWA(a_1, a_2, \ldots, a_n) = \left( \sum_{j=1}^{n} w_j a_j^2 \right)^{1/2}
\]

where \( w=(w_1, w_2, \ldots, w_n)^T \) is the weight vector of \((a_1, a_2, \ldots, a_n)\) with the conditions \( w \in [0,1]^n \) and \( \sum_{j=1}^{n} w_j = 1 \). \( w \) can be obtained by AHP method proposed Saaty [32]. \( \lambda \) is a parameter such that \( \lambda \in (-\infty, 0) \cup (0, +\infty) \), and \( R^+ \) is the set of all nonnegative real numbers.

**Definition 6** [16]: A GOWA operator of dimension \( n \) is a mapping GOWA: \( (R^n) \rightarrow R^+ \). Such that,

\[
GOWA(a_1, a_2, \ldots, a_n) = \left( \sum_{j=1}^{n} w_j a_j^2 \right)^{1/2}
\]

where \( w=(w_1, w_2, \ldots, w_n)^T \) is the weight vector which is associated with GOWA, and with the conditions \( w \in [0,1]^n \) and \( \sum_{j=1}^{n} w_j = 1 \); \( b_j \) is the jth largest of real numbers \( a_k(k=1,2,\ldots,n) \). \( \lambda \) is a parameter such that \( \lambda \in (-\infty, 0) \cup (0, +\infty) \), and \( R^+ \) is the set of all nonnegative real numbers.

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GHWA(a_1, a_2, \ldots, a_n) = \left( \sum_{j=1}^{n} w_j a_j^2 \right)^{1/2}
\]

where \( w=(w_1, w_2, \ldots, w_n)^T \) is the weight vector which is associated with GOWA, and satisfying \( \omega_j \in [0,1](j=1,2,\ldots,n) \) and \( \sum_{j=1}^{n} \omega_j = 1 \); \( b_j \) is the jth largest of real numbers \( a_k(k=1,2,\ldots,n) \) and \( R^+ \) is the set of all nonnegative real numbers.

\[ \omega_j \in [0,1](j=1,2,\ldots,n) \text{ and } \sum_{j=1}^{n} \omega_j = 1 \text{ ; } b_j \text{ is the jth largest of real numbers } a_k(k=1,2,\ldots,n) \]

C. **Hamacher operators**

The \( t \)-operators are in fact Union and Intersection operators in fuzzy set theory which are symbolized by \( T \)-conorm (\( \Gamma^+ \)) and \( T \)-norm (\( \Gamma \)), respectively [33]. Based on the \( T \)-norm and \( T \)-conorm, a generalized union and a generalized intersection for intuitionistic fuzzy sets were introduced by Deschrijver and Kerre [34], and a generalized union and a generalized intersection of the single valued neutrosophic numbers were introduced by Smaardache and Vlădăreanu [35].

**Definition 8** [35]: Let \( x = (T_1, I_1, F_1) \) and \( y = (T_2, I_2, F_2) \) are any two single valued neutrosophic numbers, then, the generalized intersection and union are defined as follows:

\[ x \cap_{\Gamma^+} y = \left( \Gamma(T_1, T_2), \Gamma(I_1, I_2), \Gamma(F_1, F_2) \right) \]
\[ x \cup_{\Gamma^+} y = \left( \Gamma^+(T_1, T_2), \Gamma(I_1, I_2), \Gamma^+(F_1, F_2) \right) \]

where \( \Gamma \) denotes a \( T \)-norm and \( \Gamma^+ \) a \( T \)-conorm.

Some special examples of \( T \)-norms and \( T \)-conorms are listed as follows [19]:

1. Algebraic \( T \)-norm and \( T \)-conorm
   \[ \Gamma(x,y) = x \vee y \text{ and } \Gamma^+(x,y) = x + y - xy \]
2. Einstein \( T \)-norm and \( T \)-conorm [19]
   \[ \Gamma(x,y) = \frac{x \wedge y}{1 + (1-x)(1-y)} \text{ and } \Gamma^+(x,y) = \frac{x + y}{1 + x \wedge y} \]
3. Hamacher \( T \)-norm and \( T \)-conorm [36].
   \[ \Gamma_\lambda(x,y) = \frac{xy}{\gamma + (1-\gamma)(x+y-xy)}, \gamma > 0 \]
   \[ \Gamma_{\lambda}^+(x,y) = \frac{x+y-xy-(1-\gamma)xy}{1-(1-\gamma)xy}, \gamma > 0 \]

Especially, when \( \gamma = 1 \), then Hamacher \( T \)-norm and \( T \)-conorm will reduce to \( \Gamma(x,y) = xy \) and \( \Gamma^+(x,y) = x + y - xy \) which are the Algebraic \( T \)-norm and \( T \)-conorm respectively; when \( \gamma = 2 \), then Hamacher \( T \)-norm and \( T \)-conorm will reduce to \( \Gamma(x,y) = \frac{xy}{1+(1-x)(1-y)} \) and \( \Gamma^+(x,y) = \frac{x+y}{1+xy} \) which are the Einstein \( T \)-norm and \( T \)-conorm respectively [19].
Hamacher weighted aggregation operators.

Definition 9: Let \( \tilde{a}_i = (T_i, I_i, F_i) \) (\( i = 1, 2, \ldots, n \)) be a collection of the SVNNs, and \( GNNHWA: \Omega' \rightarrow \Omega \), if

\[
GNNHWA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left( \sum_{j=1}^{n} w_j \tilde{a}_j^\gamma \right)^{\frac{1}{\gamma}}
\]

where \( \Omega \) is the set of all SVNNs, and \( \lambda > 0 \), \( w = (w_1, w_2, \ldots, w_n) \) is weight vector of \( (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \), such that \( w_j \geq 0 \) and \( \sum_{j=1}^{n} w_j = 1 \). Then GNNHWA is called the generalized neutrosophic number Hamacher weighted averaging operator.

Based on the Hamacher operational rules of the SVNNs, we can get the result aggregated from Definition 9 shown as theorem 2.

Theorem 2: Let \( \tilde{a}_i = (T_i, I_i, F_i) \) (\( i = 1, 2, \ldots, n \)) be a collection of the SVNNs and \( \lambda > 0 \), then the result aggregated from Definition 9 is still an SVNN, and even

\[
GNNHWA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)
\]

is an SVNN.

D. The operational rules of SVNNs based on Hamacher T-norm and T-conorm

Based on the Definition 8, we can establish the operational rules of SVNNs.

Let \( \tilde{a}_1 = (T_1, I_1, F_1) \) and \( \tilde{a}_2 = (T_2, I_2, F_2) \) be two SVNNs, and \( \gamma, n > 0 \), then the operational rules based on Hamacher T-norm and T-conorm are defined as follows.

\[
\tilde{a}_1 \otimes_\gamma \tilde{a}_2 = \left( \frac{T_1 + T_2 - T_1 T_2}{1 - (1 - \gamma) T_1 T_2}, \frac{I_1 + I_2 - I_1 I_2}{1 - (1 - \gamma) I_1 I_2}, \frac{F_1 + F_2 - F_1 F_2}{1 - (1 - \gamma) F_1 F_2} \right)
\]

and

\[
\tilde{a}_1 \oplus_\gamma \tilde{a}_2 = \left( \frac{\gamma (1 - T_1) T_2}{1 - (1 - \gamma) (T_1 - T_2)}, \frac{\gamma (1 - I_1) I_2}{1 - (1 - \gamma) (I_1 - I_2)}, \frac{\gamma (1 - F_1) F_2}{1 - (1 - \gamma) (F_1 - F_2)} \right)
\]

for any two SVNNs, and \( \gamma > 0 \), then

(1) \( \tilde{a}_1 \otimes_\gamma \tilde{a}_2 = \tilde{a}_2 \otimes_\gamma \tilde{a}_1 \)

(2) \( \tilde{a}_1 \oplus_\gamma \tilde{a}_2 = \tilde{a}_2 \oplus_\gamma \tilde{a}_1 \)

(3) \( \eta \tilde{a}_1 \otimes_\gamma \tilde{a}_2 = \eta \tilde{a}_2 \otimes_\gamma \eta \tilde{a}_1, \eta \geq 0 \)

(4) \( \eta \tilde{a}_1 \oplus_\gamma \eta \tilde{a}_2 = (\eta \tilde{a}_1) \oplus_\gamma \eta \tilde{a}_2, \eta \geq 0 \)

(5) \( \tilde{a}_1 \otimes_\gamma \eta \tilde{a}_2 = (\eta \tilde{a}_1) \otimes_\gamma \tilde{a}_2, \eta \geq 0 \)

(6) \( \tilde{a}_1 \otimes_\gamma \eta \tilde{a}_2 = (\tilde{a}_1 \otimes_\gamma \tilde{a}_2) \otimes_\gamma \eta, \eta \geq 0 \)

It is easy to prove the formulas in Theorem 1, omitted in here.

3. Some Generalized Neutrosophic Number Hamacher Weighted Aggregation Operators

In this section, we will combine Hamacher operations and the generalized aggregation operators to SVNNs, and develop some generalized neutrosophic number
(2) Calculate $w_i \alpha_j^i$, and get

$$w_i \alpha_j^i = \left( \frac{(1+(y-1)(1-l_j))^i + (y-1)^{i+1}l_j^i}{(1+(y-1)(1-l_j))^i - (1+(y-1)(1-l_j))^i} \right) \times \left( \frac{(1+(y-1)(1-l_j))^i + (y-1)^{i+1}l_j^i}{(1+(y-1)(1-l_j))^i - (1+(y-1)(1-l_j))^i} \right)$$

And for the right-hand side of the equation (24), we have

$$\gamma \prod_{j=1}^{n} \alpha_j^i = \left( \frac{\prod_{j=1}^{n} x_j^i - \prod_{j=1}^{n} y_j^i}{\prod_{j=1}^{n} x_j^i + (y-1) \prod_{j=1}^{n} y_j^i} \right)$$

(3) Calculate $\bigoplus_{j=1}^{n} (w_i \alpha_j^i)$.

For convenience, let

$$x_i = (1+(y-1)(1-l_i))^i + (y-1)^{i+1}l_i^i,$$

$$y_i = (1+(y-1)(1-l_i))^i - l_i^i,$$

$$z_i = (1+(y-1)l_i^i)^i + (y-1)^{i+1}(1-l_i)^i, \quad t_i = (1+(y-1)l_i^i)^i - (1-l_i)^i,$$

$$u_i = (1+(y-1)(1-l_i))^i + (y-1)^{i+1}l_i^i, \quad v_i = (1+(y-1)(1-l_i))^i - (1-1)^i.$$

Then

$$w_i \alpha_j^i = \left( \frac{x_i^j - y_i^j}{x_i^j + (y-1)y_i^j} \right) \times \left( \frac{y_i^j}{x_i^j + (y-1)y_i^j} \right)$$

In the following, by Mathematical induction, we can prove

$$\bigoplus_{j=1}^{n} (w_i \alpha_j^i) = \left( \frac{\prod_{j=1}^{n} x_j^i - \prod_{j=1}^{n} y_j^i}{\prod_{j=1}^{n} x_j^i + (y-1) \prod_{j=1}^{n} y_j^i} \right)$$

(24)

(i) When $n=1$,

'$^\ast$$ w_i = 1$,

For the left-hand side of the equation (24),

$$\bigoplus_{j=1}^{n} (w_i \alpha_j^i) = \alpha_i^i = \left( \frac{\prod_{j=1}^{n} x_j^i - \prod_{j=1}^{n} y_j^i}{\prod_{j=1}^{n} x_j^i + (y-1) \prod_{j=1}^{n} y_j^i} \right)$$

and for the right-hand side of the equation (24), we have

$$\gamma \prod_{j=1}^{n} \alpha_j^i = \left( \frac{\prod_{j=1}^{n} x_j^i - \prod_{j=1}^{n} y_j^i}{\prod_{j=1}^{n} x_j^i + (y-1) \prod_{j=1}^{n} y_j^i} \right)$$

(ii) Assume Equation (24) holds for $n=k$,

we have

$$\bigoplus_{j=1}^{n} (w_i \alpha_j^i) = \left( \frac{\prod_{j=1}^{n} x_j^i - \prod_{j=1}^{n} y_j^i}{\prod_{j=1}^{n} x_j^i + (y-1) \prod_{j=1}^{n} y_j^i} \right)$$

When $n=k+1$,

$$\bigoplus_{j=1}^{n} (w_i \alpha_j^i) = \bigoplus_{j=1}^{n} (w_i \alpha_j^i) \oplus_n (w_{k+1} \alpha_{k+1}^i)$$

So, when $n=k+1$, Equation (24) holds.

(iii) According to steps (i) and (ii), we can get Equation (24) holds for any $n$.

$$\bigoplus_{j=1}^{n} (w_i \alpha_j^i)$$

So,
(4) Calculate \( \bigoplus_{j=1}^{n} \left( w_{j} \tilde{a}_{j}^{n} \right) \), we can get

\[
\left( \bigoplus_{j=1}^{n} \left( w_{j} \tilde{a}_{j}^{n} \right) \right)^{\gamma} = 
\frac{\prod x_{j}^{\gamma} - \prod z_{j}^{\gamma}}{\prod x_{j}^{\gamma} + (y-1) \prod z_{j}^{\gamma}} + \frac{\prod y_{j}^{\gamma} - \prod x_{j}^{\gamma}}{\prod y_{j}^{\gamma} + (y-1) \prod x_{j}^{\gamma}}
\]

(2) Theorem 4 (Idempotency):
Let \( \tilde{a}_{j} = \tilde{a}, j = 1,2,\ldots,n \), then \( \text{GNNHWA}(\tilde{a}, \tilde{a}, \ldots, \tilde{a}) = \tilde{a} \).

(3) Theorem 5 (Bounded):
Let \( \tilde{a}_{j} = \tilde{a}, j = 1,2,\ldots,n \), then the GNNHWA operator lies between the maximum and minimum of \( (\tilde{a}_{1}, \tilde{a}_{2}, \ldots, \tilde{a}_{n}) \), i.e.,

\[
\min(\tilde{a}_{1}, \tilde{a}_{2}, \ldots, \tilde{a}_{n}) \leq \text{GNNHWA}(\tilde{a}_{1}, \tilde{a}_{2}, \ldots, \tilde{a}_{n}) \leq \max(\tilde{a}_{1}, \tilde{a}_{2}, \ldots, \tilde{a}_{n})
\]

where min and max represent the maximum and minimum of \( (\tilde{a}_{1}, \tilde{a}_{2}, \ldots, \tilde{a}_{n}) \), respectively.

In the following, we will discuss some special cases of the GNNHWA operator with respect to the parameters \( \alpha \) and \( \gamma \).

(1) If \( \alpha = 1 \), then the GNNHWA operator defined by (22) will be reduced to the neutrosophic number Hamacher weighted averaging (NNWA) operator which is defined as follows:

\[
\text{NNWA}(\tilde{a}_{1}, \tilde{a}_{2}, \ldots, \tilde{a}_{n}) = \frac{\prod_{j=1}^{n} x_{j}^{\gamma} - \prod_{j=1}^{n} z_{j}^{\gamma}}{\prod_{j=1}^{n} x_{j}^{\gamma} + (y-1) \prod_{j=1}^{n} z_{j}^{\gamma}} + \frac{\prod_{j=1}^{n} y_{j}^{\gamma} - \prod_{j=1}^{n} x_{j}^{\gamma}}{\prod_{j=1}^{n} y_{j}^{\gamma} + (y-1) \prod_{j=1}^{n} x_{j}^{\gamma}}
\]

According to (23), we can get

\[
\text{NNWA}(\tilde{a}_{1}, \tilde{a}_{2}, \ldots, \tilde{a}_{n}) = \frac{\prod_{j=1}^{n} x_{j}^{\gamma} - \prod_{j=1}^{n} z_{j}^{\gamma}}{\prod_{j=1}^{n} x_{j}^{\gamma} + (y-1) \prod_{j=1}^{n} z_{j}^{\gamma}} + \frac{\prod_{j=1}^{n} y_{j}^{\gamma} - \prod_{j=1}^{n} x_{j}^{\gamma}}{\prod_{j=1}^{n} y_{j}^{\gamma} + (y-1) \prod_{j=1}^{n} x_{j}^{\gamma}}
\]

(26)

Further,

(i) When \( \gamma = 1 \), the formula (26) will be reduced to the neutrosophic number weighted averaging (NNWA) operator which is shown as follows:

\[
\text{NNWA}(\tilde{a}_{1}, \tilde{a}_{2}, \ldots, \tilde{a}_{n}) = \frac{\prod_{j=1}^{n} x_{j}^{\gamma} - \prod_{j=1}^{n} z_{j}^{\gamma}}{\prod_{j=1}^{n} x_{j}^{\gamma} + (y-1) \prod_{j=1}^{n} z_{j}^{\gamma}} + \frac{\prod_{j=1}^{n} y_{j}^{\gamma} - \prod_{j=1}^{n} x_{j}^{\gamma}}{\prod_{j=1}^{n} y_{j}^{\gamma} + (y-1) \prod_{j=1}^{n} x_{j}^{\gamma}}
\]

(ii) When \( \gamma = 2 \), the formula (26) will be reduced to the neutrosophic number Einstein weighted averaging (NNEWA) operator which is shown as follows:

\[
\text{NNEWA}(\tilde{a}_{1}, \tilde{a}_{2}, \ldots, \tilde{a}_{n}) = \frac{\prod_{j=1}^{n} x_{j}^{\gamma} - \prod_{j=1}^{n} z_{j}^{\gamma}}{\prod_{j=1}^{n} x_{j}^{\gamma} + (y-1) \prod_{j=1}^{n} z_{j}^{\gamma}} + \frac{\prod_{j=1}^{n} y_{j}^{\gamma} - \prod_{j=1}^{n} x_{j}^{\gamma}}{\prod_{j=1}^{n} y_{j}^{\gamma} + (y-1) \prod_{j=1}^{n} x_{j}^{\gamma}}
\]

The proof ends.

It is easy to prove that the GNNHWA operator has the following properties.

(1) Theorem 3 (Monotonicity):
Let \( (\tilde{a}_{1}, \tilde{a}_{2}, \ldots, \tilde{a}_{n}) \) and \( (\tilde{a}_{1}, \tilde{a}_{2}, \ldots, \tilde{a}_{n}) \) be two collections of SVNNs, if \( \tilde{a}_{j} \leq \tilde{a}_{j} \) for all \( j = 1,2,\ldots,n \), then

\[ \text{GNNHWA}(\tilde{a}_{1}, \tilde{a}_{2}, \ldots, \tilde{a}_{n}) \leq \text{GNNHWA}(\tilde{a}_{1}, \tilde{a}_{2}, \ldots, \tilde{a}_{n}). \]
(2) If $\lambda \to 0$, then the GNNHWA operator defined by (22) will be reduced to the neutrosophic number Hamacher weighted geometric (NNHWG) operator which is defined as follows:

$$\text{NNHWG}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a}_1^{j_1} \otimes \tilde{a}_2^{j_2} \otimes \cdots \otimes \tilde{a}_n^{j_n}$$

According to (23), we get

$$\text{NNHWG}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left( \prod_{j=1}^{n} (1 + (\gamma - 1)(1 - T_j) y_j) \right)^{j_1} \prod_{j=1}^{n} (1 + (\gamma - 1) I_j) y_j,$$

$$\prod_{j=1}^{n} (1 + (\gamma - 1) F_j) y_j - \prod_{j=1}^{n} (1 - I_j) y_j \prod_{j=1}^{n} (1 + (\gamma - 1) F_j) y_j + (\gamma - 1) \prod_{j=1}^{n} (1 - I_j) y_j$$

$$\prod_{j=1}^{n} (1 + (\gamma - 1) F_j) y_j - \prod_{j=1}^{n} (1 - I_j) y_j \prod_{j=1}^{n} (1 + (\gamma - 1) F_j) y_j + (\gamma - 1) \prod_{j=1}^{n} (1 - I_j) y_j$$

(32)

Further, (i) When $\gamma = 1$, the formula (30) will be reduced to the neutrosophic number weighted geometric (NNWG) operator which is defined as follows:

$$\text{NNWG}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left( \prod_{j=1}^{n} (2 - T_j) y_j + \prod_{j=1}^{n} (1 - I_j) y_j \right)^{j_1} \prod_{j=1}^{n} (1 + I_j) y_j - \prod_{j=1}^{n} (1 - F_j) y_j$$

$$\prod_{j=1}^{n} (1 + I_j) y_j + \prod_{j=1}^{n} (1 - I_j) y_j \prod_{j=1}^{n} (1 + I_j) y_j - \prod_{j=1}^{n} (1 - F_j) y_j$$

(33)

(3) If $\gamma = 1$, then the GNNHWA operator defined by (22) will be reduced to the general neutrosophic number weighted averaging operator (GNNWA) which is defined as follows:

$$\text{GNNWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left( \prod_{j=1}^{n} (1 - T_j) y_j \right)^{j_1},$$

$$1 - \left( \prod_{j=1}^{n} (1 - (1 - I_j) y_j) \right)^{j_1},$$

$$1 - \left( \prod_{j=1}^{n} (1 - (1 - F_j) y_j) \right)^{j_1}$$

(34)

(4) If $\gamma = 2$, then the GNNHWA operator defined by (22) will be reduced to the generalized neutrosophic number Einstein weighted averaging operator (GNNEWA) which is defined as follows:

$$\text{GNNEWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left\{ \left( \prod_{j=1}^{n} (1 + (\gamma - 1) F_j) y_j - \prod_{j=1}^{n} (1 - I_j) y_j \right)^{j_1} \prod_{j=1}^{n} (1 + I_j) y_j - \prod_{j=1}^{n} (1 - F_j) y_j$$

$$\prod_{j=1}^{n} (1 + I_j) y_j + \prod_{j=1}^{n} (1 - I_j) y_j \prod_{j=1}^{n} (1 + I_j) y_j - \prod_{j=1}^{n} (1 - F_j) y_j$$

(35)

According to (23), we get

$$\text{GNNWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left( \prod_{j=1}^{n} (1 - T_j) y_j \right)^{j_1},$$

$$1 - \left( \prod_{j=1}^{n} (1 - (1 - I_j) y_j) \right)^{j_1},$$

$$1 - \left( \prod_{j=1}^{n} (1 - (1 - F_j) y_j) \right)^{j_1}$$

(34)
tion of the SVNNs, then, the result aggregated from Definition 10 is still an SVNN, and even 

\[ \text{GNNHOWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left\{ \begin{array}{ll}
\gamma \left( \prod_{j=1}^{n} x_{\tilde{a}_j} - \prod_{j=1}^{n} y_{\tilde{a}_j} \right)_{\tilde{a}_j} \\
\left( \prod_{j=1}^{n} x_{\tilde{a}_j} + (\gamma^2 - 1) \prod_{j=1}^{n} y_{\tilde{a}_j} \right)_{\tilde{a}_j} + (\gamma - 1) \left( \prod_{j=1}^{n} x_{\tilde{a}_j} - \prod_{j=1}^{n} y_{\tilde{a}_j} \right)_{\tilde{a}_j} \\
\left( \prod_{j=1}^{n} x_{\tilde{a}_j} + (\gamma^2 - 1) \prod_{j=1}^{n} y_{\tilde{a}_j} \right)_{\tilde{a}_j} + (\gamma - 1) \left( \prod_{j=1}^{n} x_{\tilde{a}_j} - \prod_{j=1}^{n} y_{\tilde{a}_j} \right)_{\tilde{a}_j} \\
\left( \prod_{j=1}^{n} x_{\tilde{a}_j} + (\gamma^2 - 1) \prod_{j=1}^{n} y_{\tilde{a}_j} \right)_{\tilde{a}_j} + (\gamma - 1) \left( \prod_{j=1}^{n} x_{\tilde{a}_j} - \prod_{j=1}^{n} y_{\tilde{a}_j} \right)_{\tilde{a}_j} \\
\end{array} \right. \]

(38)

where

\[ x_j = (1 + (\gamma - 1)(1 - T_j))^d + (\gamma^2 - 1)T_j^d, \]
\[ y_j = (1 + (\gamma - 1)(1 - T_j))^d - T_j^d, \]
\[ z_j = (1 + (\gamma - 1)I_j)^d + (\gamma^2 - 1)(1 - I_j)^d, \]
\[ t_j = (1 + (\gamma - 1)I_j)^d - (1 - I_j)^d, \]
\[ u_j = (1 + (\gamma - 1)F_j)^d + (\gamma^2 - 1)(1 - F_j)^d, \]
\[ v_j = (1 + (\gamma - 1)F_j)^d - (1 - F_j)^d, \]

are some special cases, it can also be determined by some mathematical methods. For example, Xu [37] proposed a method shown as follows

\[ \omega_j = \frac{e^{-(j-\theta_a)^2}}{\sum_{j=1}^{n} e^{-(j-\theta_a)^2}} \quad (j = 1, 2, \ldots, n-1) \]

(39)

where \( \theta_{a-\frac{1}{2}} \) and \( \theta_{a+\frac{1}{2}} \) are the mean value and the standard deviation of the collection of \( 1, 2, \ldots, n-1 \), respectively. \( \theta_{a-\frac{1}{2}} \) and \( \theta_{a+\frac{1}{2}} \) was calculated by the following formulas, respectively.

\[ \theta_{a-\frac{1}{2}} = \frac{n}{2} \]
\[ \theta_{a+\frac{1}{2}} = \frac{1}{n-1} \sum_{j=1}^{n} (j - \theta_{a-\frac{1}{2}})^2 \]\n
(40)

(41)

The GNNHOWA operator has the following properties:

1. **Theorem 7 (Monotonicity):**

   Let \( (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \) and \( (\tilde{a}_1', \tilde{a}_2', \ldots, \tilde{a}_n') \) be two collections of SVNNs, if \( \tilde{a}_j \leq \tilde{a}_j' \) for all \( j = 1, 2, \ldots, n \), then 

   \[ \text{GNNHOWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \text{GNNHOWA}(\tilde{a}_1', \tilde{a}_2', \ldots, \tilde{a}_n'). \]

2. **Theorem 8 (Idempotency):**

   Let \( \tilde{a}_j = \tilde{a}, j = 1, 2, \ldots, n \), then \( \text{GNNHOWA}(\tilde{a}, \tilde{a}, \ldots, \tilde{a}) = \tilde{a} \).

3. **Theorem 9 (Bounded):**

   Let \( \tilde{a}_j = \tilde{a}, j = 1, 2, \ldots, n \), then the GNNHOWA operator lies between the maximum and minimum of \( (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \), i.e., 

   \[ \min(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \text{GNNHOWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \max(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \]

   where \( \min \) and \( \max \) represent the maximum and minimum of \( (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \), respectively.

4. **Theorem 10 (Commutativity):**

   Let \( (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \) and \( (\tilde{a}_1', \tilde{a}_2', \ldots, \tilde{a}_n') \) be two collections of SVNNs, and \( (\tilde{a}_1', \tilde{a}_2', \ldots, \tilde{a}_n') \) is any permutation of \( (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \), then

   \[ \text{GNNHOWA}(\tilde{a}_1', \tilde{a}_2', \ldots, \tilde{a}_n') = \text{GNNHOWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n). \]

Similar to GNNHWA operator, some special cases of the GNNHOWA operator with respect to the parameters \( \lambda \) and \( \gamma \) can be discussed as follows.

1. If \( \lambda = 1 \), then the GNNHOWA operator defined by (37) will be reduced to the neutrosophic number Hamacher ordered weighted averaging (NNHOWA) operator which is defined as follows:

\[ \text{NNHOWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \omega_1 \tilde{a}_1 \Theta_1 \omega_2 \tilde{a}_2 \Theta_2 \cdots \Theta_n \omega_n \tilde{a}_n \]

(42)

According to (38), we can get

\[ \text{NNHOWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left\{ \begin{array}{l}
\prod_{j=1}^{n} (1 + (\gamma - 1)T_{\sigma(j)})^{\omega_j} - \prod_{j=1}^{n} (1 - T_{\sigma(j)})^{\omega_j} \\
\prod_{j=1}^{n} (1 + (\gamma - 1)T_{\sigma(j)})^{\omega_j} + (\gamma - 1)\prod_{j=1}^{n} (1 - T_{\sigma(j)})^{\omega_j} \\
\end{array} \right. \]

\[ \frac{\gamma \prod_{j=1}^{n} I_{\sigma(j)}^{\omega_j}}{\prod_{j=1}^{n} (1 + (\gamma - 1)(1 - I_{\sigma(j)}))^{\omega_j} + (\gamma - 1)\prod_{j=1}^{n} F_{\sigma(j)}^{\omega_j}} \]

\[ \frac{\gamma \prod_{j=1}^{n} F_{\sigma(j)}^{\omega_j}}{\prod_{j=1}^{n} (1 + (\gamma - 1)(1 - F_{\sigma(j)}))^{\omega_j} + (\gamma - 1)\prod_{j=1}^{n} I_{\sigma(j)}^{\omega_j}} \]

Further,
(i) When $\gamma = 1$, the formula (43) will be reduced to the neutrosophic number ordered weighted averaging (NNOWA) operator which is defined as follows:

$$NNOWA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left\{ \frac{1}{n} \sum_{j=1}^{n} \left[ \prod_{j=1}^{n} (1 - T_{\sigma(j)}) \prod_{j=1}^{n} F_{\sigma(j)} \right]^{\frac{1}{\gamma}} \right\}_{\gamma=1}$$

(44)

(ii) When $\gamma = 2$, the formula (43) will be reduced to the neutrosophic number Einstein ordered weighted averaging (NNEOWA) operator which is defined as follows:

$$NNEOWA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left\{ \frac{2}{n} \sum_{j=1}^{n} \left[ \prod_{j=1}^{n} (1 + T_{\sigma(j)}) \prod_{j=1}^{n} F_{\sigma(j)} \right]^{\frac{1}{\gamma}} \right\}_{\gamma=2}$$

(45)

Further,

(2) If $\lambda \to 0$, then the GNNHOWA operator defined by (37) will be reduced to the neutrosophic number ordered weighted geometric (NHNOWG) operator which is defined as follows:

$$\text{NHNOWG}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left\{ \frac{2}{n} \sum_{j=1}^{n} \left[ \prod_{j=1}^{n} (1 + T_{\sigma(j)}) \prod_{j=1}^{n} F_{\sigma(j)} \right]^{\frac{1}{\gamma}} \right\}_{\gamma=2}$$

(46)

According to (38), we can get

$$\text{NHNOWG}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left\{ \frac{2}{n} \sum_{j=1}^{n} \left[ \prod_{j=1}^{n} (1 + T_{\sigma(j)}) \prod_{j=1}^{n} F_{\sigma(j)} \right]^{\frac{1}{\gamma}} \right\}_{\gamma=2}$$

(47)

Further,

(i) When $\gamma = 1$, the formula (47) will be reduced to the neutrosophic number ordered weighted geometric (NNOGW) operator which is defined as follows:

$$\text{NNOGW}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left\{ \prod_{j=1}^{n} T_{\sigma(j)} \prod_{j=1}^{n} F_{\sigma(j)} \right\}_{\gamma=1}$$

(48)

(ii) When $\gamma = 2$, the formula (47) will be reduced to the neutrosophic number Einstein ordered weighted geometric (NNEOWG) operator which is defined as follows:

$$\text{NNEOWG}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left\{ \frac{2}{n} \sum_{j=1}^{n} \left[ \prod_{j=1}^{n} (1 + T_{\sigma(j)}) \prod_{j=1}^{n} F_{\sigma(j)} \right]^{\frac{1}{\gamma}} \right\}_{\gamma=2}$$

(49)

(3) If $\gamma = 1$, then the GNNHOWA operator defined by (37) will be reduced to the generalized neutrosophic number ordered weighted averaging (GNNOWA) operator which is defined as follows:

$$\text{GNNOWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left\{ \frac{2}{n} \sum_{j=1}^{n} \left[ \prod_{j=1}^{n} (1 + T_{\sigma(j)}) \prod_{j=1}^{n} F_{\sigma(j)} \right]^{\frac{1}{\gamma}} \right\}_{\gamma=2}$$

(50)

According to (38), we can get

$$\text{GNNOWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left\{ \frac{2}{n} \sum_{j=1}^{n} \left[ \prod_{j=1}^{n} (1 + T_{\sigma(j)}) \prod_{j=1}^{n} F_{\sigma(j)} \right]^{\frac{1}{\gamma}} \right\}_{\gamma=2}$$

(51)

(4) If $\gamma = 2$, then the GNNHOWA operator defined by (37) will be reduced to the generalized neutrosophic number Einstein ordered weighted averaging (GNNEOWA) operator which is defined as follows:

$$\text{GNNEOWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left\{ \frac{2}{n} \sum_{j=1}^{n} \left[ \prod_{j=1}^{n} (1 + T_{\sigma(j)}) \prod_{j=1}^{n} F_{\sigma(j)} \right]^{\frac{1}{\gamma}} \right\}_{\gamma=2}$$

(52)

According to (38), we can get

$$\text{GNNEOWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left\{ \frac{2}{n} \sum_{j=1}^{n} \left[ \prod_{j=1}^{n} (1 + T_{\sigma(j)}) \prod_{j=1}^{n} F_{\sigma(j)} \right]^{\frac{1}{\gamma}} \right\}_{\gamma=2}$$

(53)

Further, where

$$x_j = (2 - T_j)^{\lambda} + 3T_j^{\lambda} \, , \quad y_j = (2 - T_j)^{\lambda} - T_j^{\lambda} \, ,$$

$$z_j = (1 + I_j)^{\lambda} + 3(1 - I_j)^{\lambda} \, , \quad t_j = (1 + I_j)^{\lambda} - (1 - I_j)^{\lambda} \, ,$$

$$\lambda \to 0$$
\[ u_j = (1 + F_j)^y + 3(1 - F_j)^y, \quad v_j = (1 + F_j)^y - (1 - F_j)^y. \]

As GNNHHWA operator only emphasizes the self-importance of each SVNN, and GNNHOWA operator only emphasizes the ordering position importance of all SVNNs. However, in many practical applications, we need consider these two weights together because they represent different aspects of decision making problems. Obviously, two operators have shortcomings. In order to overcome these defects, a generalized hybrid averaging operator for SVNNs based on Hamacher T-norm and T-conorm is given as follows.

**Definition 11:** Let \( \tilde{a}_j = (T_j, I_j, F_j) \) \( (j = 1, 2, \ldots, n) \) be a collection of the SVNNs, and \( \text{GNNHHWA}: \Omega^3 \rightarrow \Omega \), if

\[
\text{GNNHHWA}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \sum_{i=1}^{n} \hat{a}_i \hat{b}_{n(i)} \tag{54}
\]

where \( \Omega \) is the set of all SVNNs, and \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) is the weighted vector associated with GNNHHWA, such that \( \omega_j \geq 0 \) and \( \sum \omega_j = 1 \).

\( w = (w_1, w_2, \ldots, w_n) \) is the weight vector of \( \tilde{a}_j (j = 1, 2, \ldots, n) \), and \( w_j \in [0, 1], \sum w_j = 1 \). Let \( \tilde{b}_j = n \omega \hat{a}_j = (\tilde{T}_j, \tilde{I}_j, \tilde{F}_j) \), \( n \) is the adjustment factor. Suppose \( (\sigma(1), \sigma(2), \ldots, \sigma(n)) \) is a permutation of \((1, 2, \ldots, n)\), such that \( \hat{b}_{\sigma(i)} \geq \hat{b}_{\sigma(j)} \) for any \( j \), and then function GNNHHWA is called the generalized neutrosophic number Hamacher hybrid weighted averaging (GNNHHWA) operator.

Based on the Hamacher operational rules of the IS-VNNs, we can derive the result aggregated from Definition 11 shown as Theorem 11.

**Theorem 12:** The GNNHHWA and GNNHOWA operators are the special cases of the GNNHHWA operator.

It is easy to prove that when \( w = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right) \), the GNNHHWA operator will reduce to GNNHOWA operator, and when \( \omega = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right) \), the GNNHHWA operator will reduce to GNNHOWA operator.

From definition 11, we can know that the GNNHHWA operator firstly weights the input arguments, and then reordered the weighted values in descending order and weights them. So, the GNNHHWA operator can consider the importance degrees of both the input arguments and their weighted ordered positions.

### 4. The Multiple Attribute Decision Making Methods Based on the Generalized Neutrosophic Number Hamacher Aggregation Operators

In this section, we will use the generalized neutrosophic number Hamacher aggregation operators to the multiple attribute group decision making problems in which the attribute weights take the form of crisp numbers and attribute values take the form of SVNNs.

For a multiple attribute group decision making problem, let \( E = \{e_1, e_2, \ldots, e_s\} \) be the collection of decision makers, \( S = \{S_1, S_2, \ldots, S_s\} \) be the collection of alternatives, and \( C = \{C_1, C_2, \ldots, C_s\} \) be the collection of attributes. Suppose that \( r^i_k = (T^i_k, I^i_k, F^i_k) \) is an attribute value given by the decision maker \( e_k \) for the alternative \( S_i \) with respect to the attribute \( C_j \) which is expressed by a SVNN, \( w = (w_1, w_2, \ldots, w_n) \) is the weight vector of attrib...
ute set $C=\{C_1,C_2,\cdots,C_n\}$, and $w_j \in [0,1], \sum_{j=1}^n w_j = 1$. Let $\xi=(\xi_1,\xi_2,\cdots,\xi_q)$ be the vector of decision makers $\{e_1,e_2,\cdots,e_q\}$, and $\xi_j \in [0,1], \sum_{j=1}^q \xi_j = 1$. Then we use the attribute weights, the decision makers’ weights, and the attribute values to rank the order of the alternatives.

In group decision making, we need aggregate the different attribute values to the comprehensive values and the comprehensive values of different decision makers to collective overall values. As mentioned above, we don’t need to consider the position weight in aggregating the different attribute values to the comprehensive values, so we can select the GNNHWA operator. However, in aggregating the comprehensive values of different decision makers to collective overall values, we can use the GNNHHWA operator so that the weights of decision makers and ordering position of the comprehensive values can be considered together. The steps are shown as follows.

**Step 1:** Utilize the GNNHWA operator
\[
\tilde{r}_i = (\tilde{T}_1, \tilde{T}_2, \tilde{T}_3) = GNNHWA(r_1, r_2, \cdots, r_q) \quad (56)
\]
to derive the comprehensive values $r_i (i=1,2,\cdots,m; k=1,2,\cdots,q)$ of each decision maker.

**Step 2:** Utilize the GNNHHWA operator
\[
r_i = (\tilde{T}_1, \tilde{I}_1, \tilde{F}_1) = GNNHHWA(r_1, r_2, \cdots, r_q) \quad (57)
\]
to derive the collective overall values $r_i (i=1,2,\cdots,m)$.

**Step 3:** Calculate the score function $sc(r_i)$ and accuracy function $ac(r_i)$ ($i=1,2,\cdots,m$) by definition 3.

**Step 4:** Rank all the alternatives $\{S_1,S_2,\cdots,S_n\}$ by definition 4.

**Step 5:** End.

### 5. An Application Example

In order to demonstrate the application of the proposed methods to multi-attribute group decision making problems, we will cite an example about the air quality evaluation (adapted from [38]). To evaluate the air quality of Guangzhou for the 16th Asian Olympic Games, the air quality in Guangzhou for the Novembers of 2006, 2007, 2008 and 2009 were collected in order to find out the air quality in Guangzhou for the Novembers of 2006, 2007, 2008, and 2009 to be the set of alternatives, please give the rank of air quality from 2006 to 2009.

SVNNs (Note: the original data take the form of intuitionistic fuzzy numbers, we can get SVNNs by $I=1-T-F$). Let $(S_1,S_2,S_3) = \{\text{November of 2006, November of 2007, November of 2008, November of 2009}\}$ be the set of alternatives, please give the rank of air quality from 2006 to 2009.

**Table 1. Air quality data from station $e_1$.**

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>(0.265,0.350,0.385)</td>
<td>(0.330,0.390,0.280)</td>
<td>(0.245,0.275,0.480)</td>
</tr>
<tr>
<td>$A_2$</td>
<td>(0.345,0.245,0.410)</td>
<td>(0.430,0.290,0.280)</td>
<td>(0.245,0.375,0.380)</td>
</tr>
<tr>
<td>$A_3$</td>
<td>(0.365,0.300,0.335)</td>
<td>(0.480,0.315,0.205)</td>
<td>(0.340,0.370,0.290)</td>
</tr>
<tr>
<td>$A_4$</td>
<td>(0.430,0.300,0.270)</td>
<td>(0.460,0.245,0.295)</td>
<td>(0.310,0.520,0.170)</td>
</tr>
</tbody>
</table>

**Table 2. Air quality data from station $e_2$.**

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>(0.125,0.470,0.405)</td>
<td>(0.220,0.420,0.360)</td>
<td>(0.345,0.490,0.165)</td>
</tr>
<tr>
<td>$A_2$</td>
<td>(0.355,0.315,0.330)</td>
<td>(0.300,0.370,0.330)</td>
<td>(0.205,0.630,0.165)</td>
</tr>
<tr>
<td>$A_3$</td>
<td>(0.315,0.380,0.305)</td>
<td>(0.330,0.565,0.105)</td>
<td>(0.280,0.520,0.200)</td>
</tr>
<tr>
<td>$A_4$</td>
<td>(0.365,0.365,0.270)</td>
<td>(0.355,0.320,0.325)</td>
<td>(0.425,0.485,0.090)</td>
</tr>
</tbody>
</table>

**Table 3. Air quality data from station $e_3$.**

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>(0.260,0.425,0.315)</td>
<td>(0.220,0.450,0.330)</td>
<td>(0.255,0.500,0.245)</td>
</tr>
<tr>
<td>$A_2$</td>
<td>(0.270,0.370,0.360)</td>
<td>(0.320,0.215,0.465)</td>
<td>(0.135,0.575,0.290)</td>
</tr>
<tr>
<td>$A_3$</td>
<td>(0.245,0.465,0.290)</td>
<td>(0.250,0.570,0.180)</td>
<td>(0.175,0.660,0.165)</td>
</tr>
<tr>
<td>$A_4$</td>
<td>(0.390,0.340,0.270)</td>
<td>(0.305,0.475,0.220)</td>
<td>(0.465,0.485,0.050)</td>
</tr>
</tbody>
</table>

To get the best alternative(s), the following steps are involved:

**Step 1:** Utilize the GNNHWA operator expressed by (57) to derive the comprehensive values $r_i (i=1,2,\cdots,m; k=1,2,\cdots,q)$ of each decision maker (suppose $\lambda=1$, $\gamma=1$), we can get

\[
r_1 = (0.319,0.321,0.350), \quad r_2 = (0.384,0.284,0.332), \quad r_3 = (0.434,0.311,0.256), \quad r_4 = (0.436,0.316,0.228),
\]

\[
r_1^2 = (0.266,0.435,0.277), \quad r_2^2 = (0.322,0.395,0.251), \quad r_3^2 = (0.342,0.465,0.175), \quad r_4^2 = (0.407,0.363,0.192),
\]

\[
r_1^3 = (0.272,0.436,0.282), \quad r_2^3 = (0.291,0.333,0.352), \quad r_3^3 = (0.261,0.534,0.195), \quad r_4^3 = (0.408,0.410,0.138).
\]

**Step 2:** Utilize the GNNHHWA operator expressed by (58) to derive the collective overall values $r_i (i=1,2,\cdots,m)$ (suppose $\lambda=1$, $\gamma=1$, $\omega = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$), we can get

$r_1 = (0.288,0.396,0.304), \quad r_2 = (0.329,0.344,0.310), \quad r_3 = (0.350,0.375,0.275), \quad r_4 = (0.323,0.340,0.337)$. 

Firstly, there are the same ranking results of these methods. Secondly, the aggregation operators proposed in this paper are more general and more flexible according to the different parameter values $\lambda$ and $\gamma$.

6. Conclusions

This paper puts forward a new method to solve MAGDM problems with single valued neutrosophic information. We have defined Hamacher operation rules of single valued neutrosophic numbers by using Hamacher t-conorm and t-norm, and discussed some properties of them. Further, we have developed some new aggregation operators based on Hamacher operations and generalized aggregation operators for single valued neutrosophic information, including the generalized neutrosophic number Hamacher weighted averaging (GNNHWA) operator, generalized neutrosophic number Hamacher ordered weighted averaging (GNNHOWA) operator, and generalized neutrosophic number Hamacher hybrid averaging (GNNHHA) operator, and discussed various properties of these operators and analyzed some special cases of them. Moreover, we applied the developed operators to deal with the MAGDM problems with single valued neutrosophic information, and proposed a new method. This research has showed the proposed operators extended the Algebraic and Einstein aggregation operators, also extended the arithmetic aggregation operators, geometric aggregation operators and quadratic aggregation operators, and the proposed method is more general and more flexible according to the different parameter values $\lambda$ and $\gamma$. In further research, it is necessary and meaningful to apply the proposed operators to real decision making problems, or extend them to the other domains such as pattern recognition, fuzzy cluster analysis and uncertain programming, etc.

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References


