Group Decision Making under Multiplicative Hesitant Fuzzy Environment

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Abstract

The multiplicative hesitant fuzzy information proposed by Xia (2012) uses the non-uniform scale to express the preference of the decision makers which has a broad application background. The key difference between multiplicative hesitant fuzzy information and hesitant fuzzy information is the former uses unsymmetrical scale in expressing the preferences about two alternatives but the latter uses the symmetrical scale. In this paper, I investigate the aggregation operators for multiplicative hesitant fuzzy information. I consider the problem of multi-criteria group decision making with the preferences expressed as multiplicative hesitant fuzzy information. I pay attention to the practical application of the proposed group decision making method for solving the talent introduction problems.

Keywords: Multiplicative hesitant fuzzy information, aggregation operator, group decision making.

1. Introduction

Group decision making is a choice of a joint operation carried out by a decision-making group [1]. It refers to the decision-making individuals given their individual preferences for shared decision-making problems, then the individual preferences of each decision-making individuals were integrated into group preference in accordance with certain rules for group preference and the ranking of the alternatives were based on the preferences of the groups [2, 3]. In many areas, there are a lot of multi-attribute group decision making, such as weaponry system development demonstration problem in the military field [4]; mining method selection in the production and management of mining, beneficiation program the preferred issues [5]; colleges assess problems and so on. Scholars have given more and more attention to multi-attribute group decision-making problems recently and make it developed rapidly. Multi-attribute group decision-making problems have become an important research in the decision sciences fields [8-14].

In group decision-making problems, the sort of decision-making program and merit-based process often requires the experts make paired comparison with the object, and given its preference information for decision-making program and construct a judgment matrix. Due to the differences in the interest of experts, preferences, and knowledge structure, the different decision-making experts will give different judgments preferences. For example fuzzy judgment matrix, interval fuzzy judgment matrix, language fuzzy judgment matrix and the fuzzy set and linguistic fuzzy set are used to describe the membership degree of a program better than another program membership. The hesitant fuzzy set (HFS) [15, 16] is a useful extension of the fuzzy set [17, 18]. Hesitate fuzzy theory has more flexibility and practicality than other extended fuzzy set theory in dealing with complex decision making under uncertainty issues [19]. In [20], the overview on HFS was investigated and proposed some future directions of the research about HFS. Based on the hesitant fuzzy linguistic term sets [21], a new group decision making model was proposed [22].

Taken into account in the actual decision-making problems, such as, partner selection in supply chain management, military systems performance assessment, due to the complexity of the objective environment, decision-makers knowledge structure and professional standards, as well as the time and many other factors, there is a lack of hesitation or show a certain degree of knowledge when the decision-makers given their preference information [23-24]. In this case, it is more appropriate to use the hesitant fuzzy information to express preferences of the decision makers and construct hesitate fuzzy judgment matrix [25]. Since the hesitant fuzzy judgment matrix is more comprehensive than the traditional judgment matrix in describing and characterizing the preferences of decision makers, and thus the research on group decision making problem based on the hesitant fuzzy judgment matrix have a high practical value.

An important issue to be recovered is the hesitant fuzzy information is based on the 0.1-0.9 scale which is distributed evenly. However, the information is usually unevenly distributed in reality and it is more suitable to
use Satty’s non-uniform 1-9 scale instead of a uniform 0.1-0.9 scale [26], based on which, the multiplicative hesitation fuzzy set (MHFS) was proposed [27]. Similar to hesitant fuzzy information, the multiplicative hesitant fuzzy information is also a set of possible values indicated the degree that one program is better than another program. But unlike hesitant fuzzy information, the multiplicative hesitant fuzzy information uses the non-uniform 1-9 scale to describe the decision makers’ preferences, the multiplicative hesitant fuzzy information can reflect the preferences of the decision makers more objectively, and the group decision-making based on multiplicative hesitant fuzzy information has broad application prospects.

Multiplicative hesitant fuzzy information concepts have been proposed, but the research is still relatively slow, and the main reason is that its math system is too complex. If we investigate the multiplicative hesitant fuzzy group decision making under the assumption that the relationship between independent expert and indicator are independent or associated respectively, it could establish a more multiplicative hesitant fuzzy group decision-making theory system, further enriched the theory of group decision making. In order to do this successfully, the rest of this paper is organized as follows: Section 2 introduces the basic concepts associated with this article, mainly including hesitant fuzzy set and multiplicative hesitant fuzzy set. Section 3 presents two types of aggregation operators for multiplicative hesitant fuzzy information, such as, independent multiplicative hesitant fuzzy aggregate operator, and associated multiplicative hesitant fuzzy aggregate operator. Based on the proposed aggregation operators, we introduce independent multiplicative hesitant fuzzy group decision making method and associated multiplicative hesitant fuzzy group decision making method in Section 4, a real example about talent introduction is also introduced in this Section. Finally, the conclusions and future works are discussed in Section 5.

2. Preliminaries

As one of the extensions of Fuzzy set (FS), hesitant fuzzy set (HFS) was proposed by Torra [15, 16] and it is characterized by the membership degree of an element to a set illustrated by several possible determined values between 0 and 1. Many researchers have focused on HFS and have received many excellent research results since the theory was put forward. In the following, we first introduce the definition of HFS.

Definition 1 [15, 16]: Let $X$ be a fixed set, a hesitant fuzzy set $HFS$ on $X$ is in terms of a function that when applied to $X$ returns a subset of $[0,1]$, which can be represented as the following mathematical symbol:

$$E = \{ x, f_x(x) | x \in X \}$$

where $f_x(x)$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set $E$. For convenience, Xia and Xu (2011) called $f_x(x)$ a hesitant fuzzy element (HFE).

According to the concept of IFS, Xia [27] take advantages of the 1-9 ratio scale to present the preference information with hesitant fuzzy values and introduced the concept of multiplicative hesitant fuzzy set as follows.

Definition 2: Let $X$ be a fixed set, a multiplicative hesitant fuzzy set (MHFS) is defined as follows:

$$D = \{ (x, \rho_x(x)) | x \in X \}$$

where $a \in [1,9]$, $\rho_x(x)$ is the membership degree of each element $x$, and $\rho_x(x)$ is an subset of $[1/a^a]$. For convenience, Xia [27] defined $\rho = \rho(x)$ as a multiplicative hesitant fuzzy number (MHFN). For comparing any two MHFNs, Xia [27] defined the following comparison laws.

Definition 3: For a MHFN $\rho = \rho(x)$, the score function is defined as

$$s(\rho) = \frac{1}{\sum \eta \in \rho} \eta$$

where $\rho$ is the number of elements of $\rho$. For any two MHFNs $\rho_1$, $\rho_2$, if $s(\rho_1) > s(\rho_2)$, then $\rho_1 > \rho_2$.

3. Aggregation Operators for Multiplicative Hesitant Fuzzy Information

In this Section, we mainly focus on the aggregation operators for MHFNs. We first study the operations of MHFNs and then investigate the two types of aggregation operators, such as, independent multiplicative hesitant fuzzy aggregate operators and associated multiplicative hesitant fuzzy aggregate operators.

3.1 Operations for MHFNs

Based on Einstein operational laws, Xia [27] introduced some operations for MHFNs as follows.

Definition 4 [27]: Let $\rho_1$, $\rho_2$ be two MHFNs, and $\lambda > 0$, then

1) $\rho_1^\lambda = \bigcup_{\eta \in \rho_1} \left\{ \frac{2\eta^\lambda}{2 + \eta^2} \right\}$

2) $\lambda \rho_1 = \bigcup_{\eta \in \rho_1} \left\{ \frac{(1+2\eta)^\lambda - 1}{2} \right\}$

3) $\rho_1 \oplus \rho_2 = \bigcup_{\eta_1, \eta_2 \in \rho_1} \left\{ \frac{(1+2\eta_1)(1+2\eta_2)-1}{2} \right\}$

4) $\rho_1 \otimes \rho_2 = \bigcup_{\eta_1, \eta_2 \in \rho_1} \left\{ \frac{2\eta_1\eta_2}{2 + \eta_1(2+\eta_2)\eta_2} \right\}$
3.2 Independent multiplicative hesitant fuzzy aggregate operators

In the following, we propose the multiplicative hesitant fuzzy weighted averaging operator for aggregating MHFNs under the assumption of the aggregated arguments are independent, than we investigated its desirable properties.

Definition 5: Let $\rho_i, (i=1,2,...,n)$ be a collection of MHFNs, a multiplicative hesitant fuzzy weighted averaging (MHFWA) operator is a mapping $\rho^* \to M$, like this

$$\text{MHFWA}(\rho_1, \rho_2, ..., \rho_n) = w_1\rho_1 \oplus w_2\rho_2 \oplus \cdots \oplus w_n\rho_n \quad (3)$$

In Section 3.1, we have researched the operations, based on which, we can deduced the Theorem 1 as follows.

Theorem 1: Let $\rho_i, (i=1,2,...,n)$ be a collection of MHFNs, and $w_i, (i=1,2,...,n)$ be their weight of $\rho_i$, satisfy $w_i \in [0,1]$ and $\sum_{i=1}^{n}w_i = 1$, then

$$\text{MHFWA}(\rho_1, \rho_2, ..., \rho_n) = w_1\rho_1 \oplus w_2\rho_2 \oplus \cdots \oplus w_n\rho_n = \bigcup_{\eta \in \mathcal{H}} \left( \prod_{i=1}^{n}(1+2\eta)^{-1} \right) \quad (4)$$

Remarks: Xia and Xu [25] introduced the similar aggregation operator. The main difference is the operations. The aggregation operator proposed in reference [25] is based on algebraic operational laws while MHFWA operator proposed above is based on Einstein operational laws. The final aggregation results of Xia and Xu [25] is

$$\text{HFWA}(\rho_1, \rho_2, ..., \rho_n) = \bigcup_{\eta \in \mathcal{H}} \left( \prod_{i=1}^{n}(1+\eta)^{-1} \right) \quad (5)$$

The MHFWA operator has some of the desirable properties, as shown below.

Property 1: Let $\rho_i, (i=1,2,...,n)$ be a collection of MHFNs, if all $\rho_i, (i=1,2,...,n)$ are equal, i.e., for all $i=1,2,...,n$, $\eta = \eta_i$, where $\eta_i$ are elements of multiplicative hesitant fuzzy set $\rho_i$, $\eta$ is the element of multiplicative hesitant fuzzy set $\rho$, then

$$\text{MHFWA}(\rho_1, \rho_2, ..., \rho_n) = w_1\rho_1 \oplus w_2\rho_2 \oplus \cdots \oplus w_n\rho_n = \rho \quad (6)$$

Proof: Since for $i=1,2,...,n$, $\eta = \eta_i$, that is to say, $\rho_i = \rho$

$$\text{MHFWA}(\rho_1, \rho_2, ..., \rho_n) = w_1\rho_1 \oplus w_2\rho_2 \oplus \cdots \oplus w_n\rho_n$$

$$ = w_1\rho_1 \oplus w_2\rho_2 \oplus \cdots \oplus w_n\rho_n = (w_1 + w_2 + \cdots + w_n)\rho = \rho$$

Property 2 (Monotonicity): Let $\rho_i = \bigcup_{\eta \in \mathcal{H}} \{\eta\}$ $(i=1,2,...,n)$ and $\beta = \bigcup_{\eta \in \mathcal{H}} \{\eta\}$ $(i=1,2,...,n)$ be two collection of MHFNs, if all $i, \eta \leq \eta_i$. Then,

$$\text{MHFWA}(\rho_1, \rho_2, ..., \rho_n) \leq \text{MHFWA}(\beta_1, \beta_2, ..., \beta_n) \quad (7)$$

Proof: since

$$\text{MHFWA}(\rho_1, \rho_2, ..., \rho_n) = \bigcup_{\eta \in \mathcal{H}} \left( \prod_{i=1}^{n}(1+2\eta)^{-1} \right)$$

$$\text{MHFWA}(\beta_1, \beta_2, ..., \beta_n) = \bigcup_{\eta \in \mathcal{H}} \left( \prod_{i=1}^{n}(1+2\beta)^{-1} \right)$$

For all $i, \eta \leq \eta_i$, Then, we can easily get the following inequality

$$(2\eta_i + 1)^{-1} \leq (2\eta + 1)^{-1}$$

$$\prod_{i=1}^{n}(1+2\eta)^{-1} \leq \prod_{i=1}^{n}(1+2\beta)^{-1}$$

According to the comparing laws defined in Section 3.1, we know that the property 2 is valid.

Property 3: Let $\rho_i = \bigcup_{\eta \in \mathcal{H}} \{\eta\}$ $(i=1,2,...,n)$ be a collection of MHFNs, $w_i, (i=1,2,...,n)$ be their weight of $\rho_i$, satisfy $w_i \in [0,1]$ and $\sum_{i=1}^{n}w_i = 1$, if $\beta$ is an MHFN, $t$ are elements of multiplicative hesitant fuzzy set $h$, then

$$\text{MHFWA}(\rho_i \otimes \beta, \rho_2 \otimes \beta, ..., \rho_n \otimes \beta) = \text{MHFWA}(\rho_1, \rho_2, ..., \rho_n) \otimes \beta$$

Proof: According to definition 4 about the operations, we can get,

$$\rho_i \otimes \beta = \bigcup_{\eta \in \mathcal{H}_{\rho_i \otimes \beta}} \{ (1+2\eta)(1+2\beta)-1 \}$$

$$\text{MHFWA}(\rho_i \otimes \beta, \rho_2 \otimes \beta, ..., \rho_n \otimes \beta) = \bigcup_{\eta \in \mathcal{H}_{\rho_i \otimes \beta}} \left( \prod_{i=1}^{n}(1+2\eta)^{-1} \right)$$

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\[ \text{MHFWA} \left( \rho_1 \oplus \beta, \rho_2 \oplus \beta, \ldots, \rho_n \oplus \beta \right) \]

**Property 4:** Let \( \rho_i = \bigcup_{n \in \mathbb{N}} \{ \eta_i \} (i = 1, 2, \ldots, n) \) be a collection of MHFNs, \( w_i (i = 1, 2, \ldots, n) \) be the weight of \( \rho_i \), satisfy \( w_i \in [0, 1] \) and \( \sum_{i=1}^{n} w_i = 1 \), and \( r > 0 \), then

\[ \text{MHFWA} (\rho_1, \rho_2, \ldots, \rho_n) = r \text{MHFWA} (\rho_1, \rho_2, \ldots, \rho_n) \]

**Proof:** According to definition 4 about the operations, we can get,

\[ r \rho_i = \bigcup_{n \in \mathbb{N}} \left\{ \left( 1 + 2 \eta_i \right)^{w_i - 1} \right\} \]

Then, \( \text{MHFWA} (\rho_1, \rho_2, \ldots, \rho_n) \)

\[ = \bigcup_{n \in \mathbb{N}} \left( \prod_{i=1}^{n} \left( 1 + 2 \eta_i \right)^{w_i - 1} \right) \]

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Based on the result of property 4, we know

\[ r \text{MHFWA} (\rho_1, \rho_2, \ldots, \rho_n) = \bigcup_{n \in \mathbb{N}} \left( \prod_{i=1}^{n} \left( 1 + 2 \eta_i \right)^{w_i - 1} \right) \]

then

\[ r \text{MHFWA} (\rho_1, \rho_2, \ldots, \rho_n) \oplus \beta \]

\[ = \bigcup_{n \in \mathbb{N}} \left( \prod_{i=1}^{n} \left( 1 + 2 \eta_i \right)^{w_i - 1} \right) \oplus \bigcup_{n \in \mathbb{N}} \{ \beta \} \]
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\[ \phi(\emptyset) = 0, \phi(X) = 1; \]

2) \( B \subseteq C \) implies \( \phi(B) \leq \phi(C) \), for all \( B, C \subseteq X \);
3) \( \phi(B \cup C) = \phi(B) + \phi(C) + \rho(B) \phi(C) \), for all \( B, C \subseteq X \) and \( B \cap C = \emptyset \), where \( \tau > -1 \).

In this Section, we shall apply the Choquet to Multiplicative hesitant fuzzy information aggregation problem and introduce the associated aggregation operator.

**Definition 6:** Let \( \phi \) be a fuzzy measure on \( X \), \( \rho_i = \bigcup_{\eta \in \rho} \{ \eta \} (i = 1, 2, \ldots, n) \) be a set of MHFNs, An associated aggregation operator is a mapping \( H^* \rightarrow H \), and denoted as

\[ \text{CMHFWA}(\rho_1, \rho_2, \ldots, \rho_n) = \bigoplus_{\rho_i} \left[ \phi(B_{\rho(i)} - B_{\rho(i-1)} \rho_{\sigma(i)}) \right] \]

where \( (\sigma(1), \sigma(2), \ldots, \sigma(n)) \) is a permutation of \( (1, 2, \ldots, n) \), such that \( \rho_{\sigma(1)} \geq \rho_{\sigma(2)} \geq \cdots \geq \rho_{\sigma(n)} \),

\[ B_{\rho(i)} = \{ \rho_{\rho(i)} \} \leq k \], for \( k \geq 1 \), and \( B_{\rho(i)} = \emptyset \).

Based on the Definition 4 described in Section 2, we can get the following research results accordingly.

**Theorem 2:** Let \( \phi \) be a fuzzy measure on \( X \), \( \rho_i = \bigcup_{\eta \in \rho} \{ \eta \} (i = 1, 2, \ldots, n) \) be a set of MHFNs, \( \rho_{\sigma(i)} \) be the \( i \) th largest of them, then their aggregated value by using the CMHFWA operator is also an MHFN, and

\[ \text{CMHFWA}(\rho_1, \rho_2, \ldots, \rho_n) \]

\[ = \bigoplus_{\rho_i} \left[ \phi(B_{\rho(i)} - B_{\rho(i-1)} \rho_{\sigma(i)}) \right] \]

The CMHFWA operator has some desirable characteristics similar to the MHFWA operator as follows. In should be noted that the proof of these characteristics are also similar to MHFWA. Therefore, we just list out these properties.

**Property 7:** Let \( \phi \) be a fuzzy measure on \( X \), \( \rho_i = \bigcup_{\eta \in \rho} \{ \eta \} (i = 1, 2, \ldots, n) \) be a collection of MHFNs, \( \rho_{\sigma(j)} \) be the \( j \) th largest of them. If all \( \rho_i \) are equal, i.e., for all \( i = 1, 2, \ldots, n \), \( \eta = \eta_i \), where \( \eta_i \) are elements of multiplicative hesitant fuzzy set \( \rho_i \), \( \eta \) is the element of multiplicative hesitant fuzzy set \( \rho \), then

\[ \text{CMHFWA}(\rho_1, \rho_2, \ldots, \rho_n) \]

\[ = \bigoplus_{\rho_i} \left[ \phi(B_{\rho(i)} - B_{\rho(i-1)} \rho_{\sigma(i)}) \right] \]

**Property 8:** Let \( \phi \) be a fuzzy measure on \( X \), \( \rho_i = \bigcup_{\eta \in \rho} \{ \eta \} (i = 1, 2, \ldots, n) \) be two collection of MHFNs, \( \rho_{\sigma(i)} \) be the \( i \) th largest of \( \rho_i \), then

\[ \text{CMHFWA}(\rho_1, \rho_2, \ldots, \rho_n) \]

\[ = \bigoplus_{\rho_i} \left[ \phi(B_{\rho(i)} - B_{\rho(i-1)} \rho_{\sigma(i)}) \right] \]
\( \beta_i \ (i = 1, 2, ..., n) \), if all \( i \), \( \eta_{(i)} \leq \eta_{(j)} \). Then,
\[
\text{CMHFWA}(\rho_i, \rho_2, \ldots, \rho_n) \leq \text{CMHFWA}(\beta_2, \beta_3, \ldots, \beta_n)
\]

**Property 9:** Let \( \phi \) be a fuzzy measure on \( X \), \( \rho_i = \bigcup_{s \in \eta_i} \{\eta_i\} (i = 1, 2, ..., n) \) be a collection of MHFNs, if \( \beta \) is an MHFN, \( t \) are elements of multiplicative hesitant fuzzy set \( h \), then
\[
\text{CMHFWA}(\rho_i \oplus \beta, \rho_2 \oplus \beta, \ldots, \rho_n \oplus \beta) = \text{CMHFWA}(\rho_i, \rho_2, \ldots, \rho_n) \oplus \beta
\]
easure on \( X \), \( \rho_i = \bigcup_{s \in \eta_i} \{\eta_i\} (i = 1, 2, ..., n) \) be a collection of MHFNs, and \( r > 0 \), then
\[
\text{CMHFWA}(r\rho_1, r\rho_2, \ldots, r\rho_n) = r \text{CMHFWA}(\rho_1, \rho_2, \ldots, \rho_n)
\]

**Property 11:** Let \( \phi \) be a fuzzy measure on \( X \), \( \rho_i = \bigcup_{s \in \eta_i} \{\eta_i\} (i = 1, 2, ..., n) \) be a collection of MHFNs, and \( r > 0 \), if \( \beta \) is an MHFN, \( t \) are elements of multiplicative hesitant fuzzy set \( h \), then
\[
\text{CMHFWA}(r\rho_i \oplus \beta, r\rho_2 \oplus \beta, \ldots, r\rho_n \oplus \beta) = r \text{CMHFWA}(\rho_i, \rho_2, \ldots, \rho_n)
\]

**Property 12:** Let \( \phi \) be a fuzzy measure on \( X \), \( \rho_i = \bigcup_{h \in \beta_i} \{\eta_i\} (i = 1, 2, ..., n) \) and \( \beta_i = \bigcup_{h \in \beta_i} \{\eta_i\} (i = 1, 2, ..., n) \) be two collection of MHFNs, then
\[
\text{CMHFWA}(\rho_i \oplus \beta_2, \rho_2 \oplus \beta_2, \ldots, \rho_n \oplus \beta_n) = \oplus \text{CMHFWA}(\rho_1, \rho_2, \ldots, \rho_n)
\]

**4. Group Decision Making Based on Multiplicative Hesitant Fuzzy Information**

Group decision-making theory is developed with the development of the welfare economics of western countries. The group decision is to give full play to the collective wisdom, by more than one person in participate in decision-making and make decision together [33-37]. In this Section, we intend to apply the proposed aggregation operators to multiplicative hesitant fuzzy group decision making problems.

Suppose a group of objects being evaluated \( (x_1, x_2, ..., x_n) \) is going to be assessed by a panel of experts \((e_1, e_2, ..., e_n)\). In the evaluation process, the experts use the multiplicative hesitant fuzzy number to express their preference information. For example, the multiplicative hesitant fuzzy number \( \rho^i_j = \bigcup_{k \in \eta^i_j} \{\eta^i_k\} (i, j = 1, 2, ..., n; k = 1, 2, ..., p) \) indicated the following meanings: \( \rho^i_j \) on behalf of the degree of the alternative \( x_i \) is priority to the alternative \( x_j \) provided by the expert \( e_k \). The \( \rho^i_j \) is set composed of several real number between \( \frac{1}{9} \) and \( 9 \). When all the experts expressed the multiplicative hesitant fuzzy preference information, the decision maker matrix can be constructed as \( D^{(k)} = \left( \rho^i_j \right)_{i,n} \) \( k = 1, 2, ..., p \). In order to find the best candidate, and the entire candidate object sorting, we propose the following method.

**Step 1:** Utilized the MHFWA or CMHFWA operator to aggregate the \((\rho^i_1, \rho^i_2, ..., \rho^i_n)\) and get the MHFNs \( \rho^i \) which was the comprehensive value of alternative \( x_i \) provided by expert \( e^i \).

**Step 2:** Utilized the MHFWA or CMHFWA operator to aggregate the \((\rho^1, \rho^2, ..., \rho^p)\) and get the aggregated MHFNs \( \rho \) for alternative \( x \).

**Step 3:** In accordance with the rules of the comparison for MHFNs defined in Definition 3, rank the MHFNs \( \rho_i (i = 1, 2, ..., n) \) and rank the alternatives accordingly.

In order to illustrate the effectiveness of this method, we take a real example about research-based talent introduction.

**Example 1:** Management Science and Engineering Institute of the Zhejiang University of Finance and Economics University is the key discipline, rich traditional characteristics of the university, and intend to make close contact with the actual reform and development in Zhejiang Province, local economic construction after several years of development, directly and more prominent the contribution of the service base.

Scientific researchers are the primary resources of the Institute, and the Institute’s leader attaches great importance to the introduction of talent. At present, Management Science and Engineering Institute are going to recruit a scientific researcher who was graduate from key universities home and abroad. There are three alternatives \((x_1, x_2, x_3)\) to apply for this position. The three professors \((e_1, e_2, e_3)\) in Management Science and Engineering Institute are responsible for this evaluation. The three professors express their preference relations for the alternatives and constructed the following multiplicative hesitant fuzzy decision matrices.

\[
D^1 = \begin{pmatrix}
1 & 7/3 & 9/7 \\
1/7 & 1 & 9/7 \\
9/7 & 3/2 & 1/6 & 3/2 & 1/6
\end{pmatrix}
\]

\[
D^2 = \begin{pmatrix}
1 & 7/3 & 9/7 \\
1/7 & 1 & 3/2 & 3/4 \\
9/7 & 3/2 & 1/6 & 3/2 & 1/6
\end{pmatrix}
\]

\[
D^3 = \begin{pmatrix}
1 & 7/3 & 9/7 \\
1/7 & 1 & 3/2 & 3/4 \\
9/7 & 3/2 & 1/6 & 3/2 & 1/6
\end{pmatrix}
\]
First of all, we suppose the three experts are independent, therefore, we use the MHFWA operator to aggregate the \((\rho^i_j, \rho^i_2, \ldots, \rho^i_n)\) and get the MHFNs \(\rho^i\) which was the comprehensive value of alternative \(x_i\) provided by expert \(e^i\).

\[
\rho^i = \{0.5226,0.6757,0.7686\}
\]

\[
\rho^i = \{1.16179,1.6752,1.7371,0.9150,0.9533,0.9947,0.7599,0.7940,0.8308\}
\]

\[
\rho^i = \{1.22131,1.42051,1.4623,1.70471,1.2562,1.2944\}
\]

\[
\rho^i = \{0.4756,0.6371,0.5305,0.5323,0.7032,0.5904\}
\]

\[
\rho^i = \{2.5811,1.6793,2.1207,2.1147,1.3494,1.7240,2.0497,1.3034,1.6687\}
\]

\[
\rho^i = \{0.7599,0.7386,0.6935,0.9071,0.8832,0.8329,0.8925,0.8689,0.8190\}
\]

\[
\rho^i = \{1.5104,1.6363,1.8652,1.0243,1.1198,1.2934,1.0536,1.1510,1.3279\}
\]

\[
\rho^i = \{1.32971,1.1355,0.92881,1.3542,1.1575,0.9480,1.4129,1.2100,0.9938\}
\]

**Step 2**: Use the MHFWA operator to aggregate the \((\rho^1, \rho^2, \ldots, \rho^3)\) and get the aggregated MHFNs \(\rho\) for alternative \(x\). In view of the large quantity of the data, detailed procedure is omitted here.

**Step 3**: According to the score function \(s(\rho) = \sqrt[n]{\prod_{\eta,\eta} \eta} \) defined in the Definition 3, calculate the score of the MHFNs \(\rho^i (i = 1,2,\ldots,n)\).

\[
s(\rho^1) = 0.9673, \quad s(\rho^2) = 1.4387, \quad s(\rho^3) = 1.5424
\]

Since \(s(\rho^1) > s(\rho^2) > s(\rho^3)\), then the ranking of the three alternatives is \(x_1 > x_2 > x_3\) and the best alternative is also \(x_1\).

In the other hand, when the three experts are associate, therefore, we use the CMHFWA operator to aggregate the \((\rho^1_1, \rho^1_2, \ldots, \rho^1_n)\) and get the MHFNs \(\rho^1\) which was the comprehensive value of alternative \(x_i\) provided by expert \(e^i\).

Suppose the fuzzy measures of experts were:

\[
\phi(\emptyset) = 0, \quad \phi(e^1) = 0.38, \quad \phi(e^2) = 0.37, \quad \phi(e^3) = 0.36
\]

\[
\phi(e^1, e^2) = 0.77, \quad \phi(e^1, e^3) = 0.64, \quad \phi(e^2, e^3) = 0.44
\]

\[
\phi(e^1, e^2, e^3) = 1,
\]

We can calculate the weight vector of the three experts is \((0.38,0.39,0.23)^T\).

\[
\rho^1 = \{0.6062,0.7179,0.7836\}
\]

\[
\rho^2 = \{1.8444,1.7071,1.4368\}
\]

\[
\rho^3 = \{1.7048,1.7747,1.8507,0.8922,0.9363,0.9843,0.7196,0.7583,0.8003\}
\]

\[
\rho^1 = \{1.4066,1.5563,1.5870,1.2172,1.3520,1.3797\}
\]

\[
\rho^2 = \{0.5374,0.6530,0.5773,0.6064,0.7297,0.6490\}
\]

\[
\rho^3 = \{2.9581,1.8061,2.3616,2.3679,1.4125,1.8732,2.2868,1.3584,1.8061\}
\]

\[
\rho^1 = \{0.6681,0.6544,0.6252,0.8292,0.8136,0.7805,0.8131,0.7977,0.7649\}
\]

\[
\rho^2 = \{1.7876,1.8855,2.0591,1.1685,1.2400,1.3666,1.2052,1.2782,1.4076\}
\]

\[
\rho^3 = \{1.3973,1.1639,0.9206,1.4263,1.1894,0.9423,1.4960,1.2505,0.9948\}
\]

**Step 2**: Use the CMHFWA operator to aggregate the \((\rho^1_1', \rho^1_2', \ldots, \rho^1_n')\) and get the aggregated MHFNs \(\rho\) for alternative \(x\). Due to a large number of data, we do not put them lists.

**Step 3**: According to the score function \(s(\rho) = \sqrt[n]{\prod_{\rho,\rho} \rho} \) defined in the Definition 3, calculate the score of the MHFNs \(\rho^i (i = 1,2,\ldots,n)\).

\[
s(\rho^1) = 1.0372, \quad s(\rho^2) = 1.1763, \quad s(\rho^3) = 1.3846
\]

Since \(s(\rho^1) > s(\rho^2) > s(\rho^3)\), then the ranking of the three alternatives is \(x_1 > x_2 > x_3\) and the best alternative is also \(x_1\).

It has the same result whichever method we take it. It is also showed the stability of this approach.

### 5. Discuss and Future Works

The independent and associate aggregation operators for MHFNs have been proposed in this paper, based on which, the GDM method under multiplicative hesitant fuzzy environment have been investigated. A real example about the introduction of the researchers in Management Science and Engineering Institute of Zhejiang University of Finance and Economics has been provided to show the effectiveness of the GDM method. In the future research, we intend to extend our method proposed in this paper to accommodate the interval-valued
In practical applications, the multiplicative hesitation fuzzy multi-attribute group decision making method reflects the preferences of the decision makers more objectively, more reasonable and more effective decisions can be made real problem. Multiplicative hesitate fuzzy group decision will facilitate the application and practice of group decision-making theory in the social, economic, energy, military and other fields, to improve the quality of decision-making, resulting in better social and economic benefits. Therefore, this study has important practical significance.

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