**Abstract**

This paper studies the $H_\infty$ control problem for a nonlinear large-scale system with unknown disturbance. In the large-scale system, each subsystem “without” nonlinear interconnections and disturbance is transformed into a T-S fuzzy system. Moreover, based on a linear decomposition, the nonlinear interconnection corresponding to each subsystem is decomposed to a linear weighting interconnection which contains a fixed linear term and an uncertain linear term. Based on the above transformation, the total fuzzy rules’ number of the transformed large-scale system is reduced evidently so that the rule explosion problem will be avoided. In this study, some conditions are proposed to guarantee the existence of Parallel Distributed Compensation (PDC) fuzzy controllers and then the controllers using Linear Matrix Inequality (LMI) tool are synthesized such that the $H_\infty$ control performance of the nonlinear large-scale system is achieved. Finally, a numerical example and a balancing double-inverted pendulums example are given to demonstrate the effectiveness of the controller.

**Keywords:** Large-scale systems, fuzzy systems, decomposition, linear matrix inequality (LMI), $H_\infty$ control.

**1. Introduction**

Sometimes, a system is too big or too complicated to do the stability analysis and control design, then the system can be decomposed into several interconnected subsystems. There have been several decomposition methods being used to transform the original system into one large-scale system which is composed of several smaller subsystems with small dimensions. In other words, the original system is divided into several subsystems and the interconnection between any two subsystems is represented by linear or nonlinear functions. Much research (see [1]-[33] and the references therein) has been done on designing controls for a large-scale system. It is possible to split the papers [1]-[33] into two categories based upon the strategies they employed. One category consists of those papers [1]-[21] which do not utilize the fuzzy methods at all; and the other contains those papers [22]-[33] that do use the fuzzy methods. This second category consists of papers [22]-[31] which utilize fuzzy controllers and papers in which the large-scale system is transformed into a T-S fuzzy system [34]. Moreover, papers [32]-[33] design the fuzzy controllers by the neural-network-based approach and DSC technique/high-gain filters, respectively.

Let us review the papers [1]-[21] in the first category. In papers [1]-[9], the authors study the stability and stabilization of a large-scale system which has different types of interconnections. Those interconnections may be generated using $\varepsilon$-decomposition, overlapping decomposition, etc. The output feedback control design problem for the large-scale system is considered in papers [10]-[15]. Some robust controller design methods for a large-scale system with uncertainty are considered in [16]-[18]. Furthermore, papers [19]-[21] focus on the analysis of the relationship between the stability and the interconnections of a large-scale system. For instance, paper [21] gives a method to stabilize the large-scale system without using any controllers. Alternatively, the stability of the large-scale system depends on the parameters adjusting of the interconnections.

As to the second category, papers [22]-[31] study large-scale systems based on their T-S fuzzy models [34]. In general, large-scale systems consist of several nonlinear subsystems; these subsystems are connected by linear or nonlinear interconnections. Next, the methods such as ‘sector nonlinearity’ or ‘local approximation in fuzzy partition spaces’ method [34] can be used to transform each nonlinear subsystem and interconnection into a T-S fuzzy system which consists of a set of fuzzy rules. Though the nonlinear subsystems are transformed into fuzzy rule sets, there is no need to transform the interconnections into fuzzy rule sets if the...
interconnection has a linear form (as in [25], [26], [28], and [29]). However, if the original interconnection has a nonlinear form, then it also must be transformed into a set of fuzzy rules (as in [22]-[24], [27], [30] and [31]). In this paper, we call this type of system as a nonlinear large-scale system. Since each subsystem has several interconnections with the other subsystems, an entire nonlinear large-scale system contains many interconnections. If each interconnection is transformed to a set of fuzzy rules, this may cause the well-known ‘rule-explosion’ problem [35, p. 273]. Therefore, fuzzy control design which uses an LMI tool [36] becomes very complicated because of the large number of fuzzy rules in the computation process. To our knowledge, there are few research papers to deal with this problem.

In this study, we consider nonlinear large-scale systems with nonlinear subsystems, nonlinear interconnections and disturbance. The task is to solve the \( H_\infty \) control performance problem for the systems. In this paper, we just transform each isolated subsystem without interconnection and disturbance into a T-S fuzzy system and use a fixed linear term and an uncertain linear term to replace the nonlinear interconnection. The uncertain linear term covers the nonlinear portion of the original interconnection. Hence, the nonlinear interconnections disappear and the fuzzy rules for approaching the interconnection are not generated. Therefore, the number of fuzzy rules to represent the large-scale system is reduced and then the ‘rule-explosion’ problem in the control design process can be avoided. Thus, with the aids of LMI tool, the PDC fuzzy controller for each subsystem is synthesized to achieve \( H_\infty \) control performance.

The organization of this study is as follows. Section 2 is the system description and problem formulation. In Section 3, the procedure of the replacement of nonlinear interconnections is introduced. In Section 4, the main theorem is proposed to give the existence conditions of the controller and to show the synthesized form of the control gain. The numerical examples are given in Sections 5. Finally, the conclusions are given.

2. System Description and Problem Formulation

Consider a nonlinear large-scale system which consists of \( N \) interconnected subsystems \( \Sigma \) as follows

\[
\dot{x}_l(t) = A_l(x(t))x_l(t) + B_l(x(t))u_t(t) + f_l(x(t)) + E_l v_l(t), \quad l = 1, \ldots, N, \tag{1}
\]

where \( x_l(t) \in \mathbb{R}^{n_l} \) is the current state of the \( l \)-th subsystem, \( u_t(t) \in \mathbb{R}^{m_l} \) is the control input, and \( x(t) \) is the state vector composed of all states \( x_l(t) \).

\( A_l(x(t)) \in \mathbb{R}^{n_l \times n_l} \) and \( B_l(x(t)) \in \mathbb{R}^{n_l \times m_l} \), which may be nonlinear, are system and input matrices, respectively. The term \( f_l(x(t)) \in \mathbb{R}^{n_l} \) represents the nonlinear interconnections from the \( l \)-th subsystem to any other subsystems. \( E_l \in \mathbb{R}^{n_l \times n_l} \) represents the disturbance weights and \( v_l(t) \in \mathbb{R} \) denotes the unknown disturbance.

The objective of this paper is to design a \( H_\infty \) fuzzy control to guarantee two tasks, one is

\[
\int_0^T x_l^T(t)M_l x_l(t) dt < \lambda_l^2 \int_0^T v_l^T(t) v_l(t) dt, \quad l = 1, \ldots, N, \tag{2}
\]

under zero initial conditions with \( v_l(t) \neq 0 \), and the other is to guarantee the stability of the controlled large-scale system in the sense of Lyapunov [37] for \( v_l(t) = 0 \). Here \( T \) is the terminal time of the control, \( \lambda_l \) is a prescribed value which denotes the worst case effect of \( v_l(t) \) on \( x_l(t) \). \( M_l \in \mathbb{R}^{n_l \times n_l} \) is a positive-definite weighting matrix.

3. Decomposition of Nonlinear Interconnections

In the original system (1), it should be noted that there are nonlinear forms in \( A_l(x(t)) \), \( B_l(x(t)) \), and the interconnection \( f_l(x(t)) \). If we like to transform the system (1) into T-S fuzzy system by using the ‘sector nonlinearity’ or ‘local approximation in fuzzy partition spaces’ method [34], there will be a lot of fuzzy rules to be generated because of linearized all nonlinear terms. Therefore, to avoid the large number of fuzzy rules should be considered first. Let the nonlinear \( f_l(x(t)) \) be decomposed to a combination of a linear term

\[
\sum_{n=1}^{N} A_{n}x_n(t) \quad \text{and an uncertain linear term} \quad \sum_{n=1}^{N} g_{n}F_{n}x_n(t) \quad \text{as (3).}
\]

\[f_l(x(t)) = \sum_{n=1}^{N} A_{n}x_n(t) + \sum_{n=1}^{N} g_{n}F_{n}x_n(t), \quad l = 1, \ldots, N, \tag{3}\]

where \( A_{n} \in \mathbb{R}^{n \times n} \) and \( F_{n} \in \mathbb{R}^{n \times n} \) are real constant matrices representing the connection from the \( l \)-th subsystem to the \( m \)-th subsystem. \( g_{n} \) is a scalar within the interval \([-1,1]\) and is used to represent the uncertainty of interconnection, which varies depending on the nonlinear portion of the interconnection. It should be emphasized that (3) does cover the interconnection in (1) since the uncertain linear term covers all possible values from the nonlinear portion of the interconnection.

One simple example is given to show how to use the proposed linear decomposition to decompose a nonlinear interconnection into the form (3).

Example I: For a nonlinear large-scale system with \( N = 2 \), suppose the interconnections are
where \( x_{11} \in [-\pi/2, \pi/2] \), \( x_{21} \in [-2.4] \), \( x_{21} \in [-\pi/2, \pi/2] \), and \( x_{22} \in [-1] \). Now, we replace (4) and (5) by the interconnection form (3) with the following two steps: 

**Step 1:** Separate \( f(x(t)) \) into the linear and nonlinear matrices parts.

**Step 2:** Calculate the maximum bound for the absolute value of each entry of the nonlinear matrices in step 1, then we can obtain (6) and (7) easily such as

\[
\begin{align*}
\max & |0.1 \sin(x_{11})| = 0.1, \quad \max |x_{21}^2| x_{21} = 4, \quad \max |\sin(x_{11})| = 1, \\
& \max |\sin(x_{21})| (x_{21})^2 = 1, \quad \text{and} \quad \max |0.1x_{11}| = 0.1 \times \pi/2 \}.
\end{align*}
\]

Therefore

\[
\begin{align*}
f_{1}(x(t)) &= \left[ \begin{array}{c} 0 \\ 0.1 \sin(x_{11}) \end{array} \right] + \left[ \begin{array}{c} 0 \\ 0 \end{array} \right], \\
\end{align*}
\]

\[
\begin{align*}
f_{2}(x(t)) &= \left[ \begin{array}{c} 0 \\ \sin(x_{21}) \end{array} \right] + \left[ \begin{array}{c} 0 \\ 0 \end{array} \right], \\
\end{align*}
\]

\[
\begin{align*}
+g_{12} \begin{bmatrix} 0.1 & 1 \\ 0 & 0 \end{bmatrix} x_{11} + g_{22} \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} x_{21}, \\
\end{align*}
\]

\[
\begin{align*}
f_{1}(x(t)) &= \left[ \begin{array}{c} 0 \\ 0 \end{array} \right], \\
\end{align*}
\]

\[
\begin{align*}
+g_{12} \begin{bmatrix} 0.1 & 1 \\ 0 & 0 \end{bmatrix} x_{11} + g_{22} \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} x_{21}, \\
\end{align*}
\]

where the interconnection weight is \( g_{12} \in [-1.1] \), \( l = 1, 2 \). Hence, the interconnection \( f(x(t)) \) can be decomposed to the following equation.

\[
f_{1}(x(t)) = \sum_{i=1}^{N} A_{i} x_{i}(t) + \sum_{i=1}^{N} g_{12} F_{12} x_{i}(t), \quad l = 1, 2, \tag{8}
\]

in which,

\[
A_{1} = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \end{bmatrix}, \quad A_{2} = \begin{bmatrix} 0 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad F_{1} = \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \end{bmatrix}, \quad F_{2} = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}, \\
A_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix}, \quad F_{12} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad F_{22} = \begin{bmatrix} 0 & 0 \\ 0.1 \times \pi/2 & 0 \end{bmatrix}.
\]

**Remark 1:** From the example, it is found that a nonlinear interconnection can be decomposed to a linear term \( \sum_{i=1}^{N} A_{i} x_{i}(t) \) and an uncertain linear term \( \sum_{i=1}^{N} g_{12} F_{12} x_{i}(t) \) as (3). This transformation covers most kinds of nonlinear interconnections if the lower and upper bounds of each \( x_{i} \) are known. This assumption is always given in the T-S fuzzy system transformation by the ‘sector nonlinearity’ [34] concept.

It must be noted that we give a \( g_{12} \) in (3) in order to cover the nonlinear terms of the original interconnection such that the nonlinear interconnections disappear. Therefore, when we transform the nonlinear large-scale system (1) into T-S fuzzy system, only each main subsystem without interconnections and disturbance needs to be transformed. Consequently, the fuzzy rules number of the transformed nonlinear large-scale system with (3) will be much less than those of the transformed system which is transformed from the original nonlinear large-scale system containing all nonlinear terms \( A'(x(t)), B'(x(t)), \) and the interconnection \( f(x(t)) \).

The interconnection \( f(x(t)) \) is decomposed to (3), and then the nonlinear form of interconnections disappears. Therefore, let the nonlinear large-scale system (1) be rewritten as (9).

\[
\begin{align*}
x(t) &= A'(x(t)) x(t) + B'(x(t)) u(t) \\
&+ \sum_{i=1}^{N} A_{i} x_{i}(t) + \sum_{i=1}^{N} g_{12} F_{12} x_{i}(t), \quad l = 1, \ldots, N, \tag{9}
\end{align*}
\]

where (9) denotes the \( l \)-th subsystem that has interconnections to other subsystem’s states. It is seen that there is not any nonlinear term in the interconnection portion of (9). Here only \( A'(x(t)) x(t) + B'(x(t)) u(t) \) are still nonlinear; therefore, we only need to transform the individual subsystems (9) without interconnections and disturbance into the T-S fuzzy system using the ‘sector nonlinearity’ or ‘local approximation in fuzzy partition spaces’ method [34].

**Rule 1:** IF \( z_{i}^{j}(t) \) is \( m_{i,j}^{j} \) and... and \( z_{i}^{q}(t) \) is \( m_{i,q}^{q} \), \( \text{THEN} \)

\[
\begin{align*}
x_{i}(t) &= A'_{i} x_{i}(t) + B'u_{i}(t), \quad i = 1, \ldots, q, \tag{10}
\end{align*}
\]

where \( A'_{i} \in \mathbb{R}^{n_{i} \times n_{i}} \) and \( B'_{i} \in \mathbb{R}^{n_{i} \times r_{i}} \) are constant matrices; \( r_{i} \) is the total number of fuzzy rules in the \( l \)-th subsystem; and \( z_{i}^{j}(t) \), \( j = 1, 2, \ldots, q \), are the known premise variables which may be the function of states, and \( m_{i,j}^{j} \) is the fuzzy membership function of \( z_{i}^{j}(t) \). Therefore, we can achieve the large-scale T-S fuzzy system as (11).

\[
\begin{align*}
x(t) &= \sum_{i=1}^{N} \mu_{i}^{j} \left( z_{i}^{j}(t) \right) \left[ A'_{i} x_{i}(t) + B'u_{i}(t) \right] \\
&+ \sum_{i=1}^{N} A_{i} x_{i}(t) + \sum_{i=1}^{N} g_{12} F_{12} x_{i}(t), \quad l = 1, \ldots, N, \tag{11}
\end{align*}
\]

in which \( z_{i}^{j}(t) = \left[ z_{i}^{1}(t), z_{i}^{2}(t), \ldots, z_{i}^{q}(t) \right] \), \( \mu_{i}^{j} \left( z_{i}^{j}(t) \right) = \frac{n_{i}^{j} \left( z_{i}^{j}(t) \right)}{\sum_{j=1}^{q} n_{i}^{j} \left( z_{i}^{j}(t) \right)} \), \( \mu_{i}^{j} \left( z_{i}^{j}(t) \right) \geq 0 \), and \( \sum_{j=1}^{q} \mu_{i}^{j} \left( z_{i}^{j}(t) \right) = 1 \).

4. Controller Synthesis

This section likes to synthesize the \( H_{\infty} \) fuzzy controller for the large-scale T-S fuzzy system (11). Here, the PDC [34] fuzzy controller is adopted as the controller form.
Rule i: IF $z^i(t)$ is $m^i_{m1}$ and AND $z^j(t)$ is $m^j_{m2}$ THEN $u(t) = -K^i x(t)$, $i = 1,..., r_m$, (12)

where $K^i \in \mathcal{R}^{m \times m}$ is the control gain. The overall fuzzy controller is

$$u(t) = \sum_{i = 1}^{r_m} \mu^i(z^i(t)) K^i x(t).$$

Therefore, by combining (13) and (11), we can obtain the large-scale T-S fuzzy system with a PDC controller:

$$\dot{x}_i(t) + 2 \sum_{i \in j \in j} \mu^i(z^i(t)) G^i x(t) + \nabla \nu_i(t) = 0,$$

in which $G^i = A^i - B^i K^i$. Next, we have the following theorem which provides the existence conditions of the PDC controller and the synthesis method of the controller.

Theorem 1: The large-scale T-S fuzzy system (11) with the PDC fuzzy controller (13), where $K^0 = H^T P x(t)$, is stable in the sense of Lyapunov when $v_m(t) = 0$ and the $H_m$ control performance in (2) is guaranteed for the given prescribed values $\lambda_m, m = 1,..., N$, when $v_m(t) \neq 0$, if there exist positive definite matrices $Q_m = Q_m^T > 0, \tilde{M}_m = \tilde{M}_m^T > 0$, positive semi-definite matrices $\tilde{A}_m = \tilde{A}_m^T \geq 0, \tilde{D}_m = \tilde{D}_m^T \geq 0$, symmetric matrices $F_m^i (i = 1,..., r_m)$, matrices $\bar{F}_m^i = \bar{F}_0^T (i < j \leq r_m)$, and $H_m^m$ such that

$$\begin{bmatrix}
\Phi_0^m Q_m A_m \cdots Q_m A_m \sqrt{\bar{V}^T F_m^1} \cdots \sqrt{\bar{V}^T F_m^N} \\
* -I \cdots 0 \cdots 0 \\
* * \cdots -I \cdots 0 \\
* * * \cdots -I \cdots 0 \\
* * * * \cdots -I \cdots 0 \\
\end{bmatrix} < 0,$$

$$\bar{V}_m < 0, \quad m = 1,..., N,$$

and

$$\bar{V}_m < 0, \quad l, m = 1,..., N,$$

and

$$\Phi_0^m Q_m A_m \cdots Q_m A_m \sqrt{\bar{V}^T F_m^1} \cdots \sqrt{\bar{V}^T F_m^N} + 2N \cdot I + \bar{M}_m < 0, \quad i = j = 1,..., r_m,$$

if $i = j$, then $i = j$, (18)

$$\Phi_0^m = \frac{1}{2} \left( \Phi_0^m Q_m A_m \cdots Q_m A_m + A_m^T Q_m^T A_m \cdots A_m^T Q_m^T \right) - H_m^T B_m^T - H_m^T B_m^T - \bar{F}_m^i - \bar{F}_m^i \sqrt{\bar{V}^T F_m^1} \cdots \sqrt{\bar{V}^T F_m^N} + 2N \cdot I + \bar{M}_m,$$

$$\nabla \nu_i(t) = \sum_{i = 1}^{r_m} \lambda_m(t) E_m \left[ \frac{1}{2} \lambda_m(t) \right] E_m^T,$$

$$V(x(t)) = \sum_{i = 1}^{N} V_i(x_i(t)) = \sum_{i = 1}^{N} \left[ x_i^T(t) (G_i^T P + P G_i) x_i(t) \right] + 2N \cdot I + \bar{M}_m + \sum_{i = 1}^{r_m} \left[ \lambda_m(t) E_m \left[ \frac{1}{2} \lambda_m(t) \right] E_m^T \right] + E_m \left[ \frac{1}{2} \lambda_m(t) \right] E_m^T,$$

if $i = j \leq r_m$, (19)

$$\bar{V}_m = \begin{bmatrix}
\bar{V}_m^T \\
\bar{V}_m^T \\
\bar{V}_m^T \\
\bar{V}_m^T \\
\bar{V}_m^T \\
\end{bmatrix},$$

(20)

$$\bar{V}_m = \begin{bmatrix}
-\tilde{A}_m + \tilde{D}_m & 0 & 0 \\
* -\tilde{A}_m & \sqrt{\bar{V}^T F_m^1} \cdots \sqrt{\bar{V}^T F_m^N} - I \\
* * -\tilde{A}_m & \sqrt{\bar{V}^T F_m^1} \cdots \sqrt{\bar{V}^T F_m^N} - I \\
* * * -\tilde{A}_m & \sqrt{\bar{V}^T F_m^1} \cdots \sqrt{\bar{V}^T F_m^N} - I \\
* * * * -\tilde{A}_m & \sqrt{\bar{V}^T F_m^1} \cdots \sqrt{\bar{V}^T F_m^N} - I \\
\end{bmatrix},$$

(21)

$l, m = 1,..., N$, and the asterisk symbol (*) denotes the transposed elements (matrices) for symmetric positions.

Proof: Let the Lyapunov function [37] be

$$V(x(t)) = \sum_{i = 1}^{N} V_i(x_i(t)) = \sum_{i = 1}^{N} x_i^T(t) P x_i(t),$$

(22)

where $P = P^T$ is a positive definite matrix. The derivative of the $l$-th Lyapunov function $V_i(x_i(t))$ along the trajectory of the $l$-th subsystem is

$$V_i(x_i(t)) = \sum_{i = 1}^{N} \mu_i(z^i(t)) \left[ x_i^T(t) (G_i P + PG_i) x_i(t) \right] + 2 \sum_{i \in j} \mu_i(z^i(t)) \mu_j(z^i(t)) \times \left[ x_i^T(t) \left( \frac{G_i + G_j}{2} \right) P + P \left( \frac{G_i + G_j}{2} \right) x_i(t) \right] + 2 \sum_{i = 1}^{N} x_i^T(t) A_i^T P x_i(t) + 2 \sum_{i = 1}^{N} x_i^T(t) F_i^T P x_i(t) + 2 \sum_{i = 1}^{N} x_i^T(t) P E_i x_i(t).$$

(23)

Using $X^T Y + Y^T X \leq X^T X + (1/\lambda) Y^T Y$ to the three last terms of (23). Hence,

$$V_i(x_i(t)) \leq \sum_{i = 1}^{N} \mu_i(z^i(t)) \left[ x_i^T(t) (G_i P + PG_i) x_i(t) \right] + 2 \sum_{i \in j} \mu_i(z^i(t)) \mu_j(z^i(t)) \times \left[ x_i^T(t) \left( \frac{G_i + G_j}{2} \right) P + P \left( \frac{G_i + G_j}{2} \right) x_i(t) \right] + \sum_{i = 1}^{N} x_i^T(t) A_i^T P x_i(t) + N \cdot x_i^T(t) P x_i(t) + \sum_{i = 1}^{N} x_i^T(t) F_i^T P x_i(t) + \sum_{i = 1}^{N} x_i^T(t) E_i x_i(t),$$

(24)

Therefore,

$$V(x(t)) = \sum_{i = 1}^{N} V_i(x_i(t)) \leq \sum_{i = 1}^{N} \left[ \sum_{i = 1}^{N} \mu_i(z^i(t)) \left[ x_i^T(t) (G_i P + PG_i) x_i(t) \right] + 2 \sum_{i \in j} \mu_i(z^i(t)) \mu_j(z^i(t)) \times \left[ x_i^T(t) \left( \frac{G_i + G_j}{2} \right) P + P \left( \frac{G_i + G_j}{2} \right) x_i(t) \right] + \sum_{i = 1}^{N} x_i^T(t) A_i^T P x_i(t) + N \cdot x_i^T(t) P x_i(t) + \sum_{i = 1}^{N} x_i^T(t) F_i^T P x_i(t) + \sum_{i = 1}^{N} x_i^T(t) E_i x_i(t) \right].$$

(25)
Therefore, the following inequality (26) is necessary.

\[
\begin{align*}
+\sum_{n=1}^{N} \left[ 2 \sum_{i,j=1}^{\infty} \mu_i^n \left( z^n(t) \right) \mu_j^n \left( z^n(t) \right) \right] \\
\times x^n(t) \left( \frac{G_{nn}^n + G_{nn}^n}{2} \right)^{\frac{1}{2}} P_{n} + P_{n} \left( \frac{G_{nn}^n + G_{nn}^n}{2} \right) x_n(t) \right] \\
+\sum_{n=1}^{N} \left[ \sum_{i,j=1}^{\infty} \mu_i^n \left( z^n(t) \right) \mu_j^n \left( z^n(t) \right) \nu^n(t) \right] A_{i}^n x_n(t) \right] \\
+\sum_{n=1}^{N} \left[ \sum_{i,j=1}^{\infty} \mu_i^n \left( z^n(t) \right) \mu_j^n \left( z^n(t) \right) \right] x^n(t) \left( N \cdot P_{n} P_{n} \right) x_n(t) \right] \\
+\sum_{n=1}^{N} \left[ \sum_{i,j=1}^{\infty} \mu_i^n \left( z^n(t) \right) \mu_j^n \left( z^n(t) \right) \nu^n(t) \right] A_{i}^n x_n(t) \right] \\
\leq \sum_{n=1}^{N} \left[ \frac{1}{2} F_{nn}^n \left( A_{n} + \Omega_{m} \right) + (g_{m}^n) \left( \frac{1}{2} F_{nn}^n - A_{n} \right) \right] \\
+\left( 1 - (g_{m}^n) \right) \left( \frac{1}{2} F_{nn}^n - \Omega_{m} \right) \right] x_n(t),
\end{align*}
\]

where \( A_{n} = A_{n}^n \geq 0 \) and \( \Omega_{m} = \Omega_{m} \geq 0 \). The proof of (26) can be found in the Appendix.

Hence,

\[
\begin{align*}
V(x(t)) &\leq \sum_{n=1}^{N} \left[ \frac{1}{2} F_{nn}^n \left( A_{n} + \Omega_{m} \right) + (g_{m}^n) \left( \frac{1}{2} F_{nn}^n - A_{n} \right) \right] \\
&\times x^n(t) \left( \frac{G_{nn}^n + G_{nn}^n}{2} \right)^{\frac{1}{2}} P_{n} + P_{n} \left( \frac{G_{nn}^n + G_{nn}^n}{2} \right) x_n(t) \right] \\
&\times \left( N \cdot P_{n} P_{n} \right) x_n(t) \\
&+ \sum_{n=1}^{N} \left[ \sum_{i,j=1}^{\infty} \mu_i^n \left( z^n(t) \right) \mu_j^n \left( z^n(t) \right) \nu^n(t) \right] A_{i}^n x_n(t) \right] \\
&\leq \sum_{n=1}^{N} \left[ \frac{1}{2} F_{nn}^n \left( A_{n} + \Omega_{m} \right) + (g_{m}^n) \left( \frac{1}{2} F_{nn}^n - A_{n} \right) \right] \\
&\times \left( 1 - (g_{m}^n) \right) \left( \frac{1}{2} F_{nn}^n - \Omega_{m} \right) \right] x_n(t)
\end{align*}
\]

(27)

If the following inequality (28) holds

\[
\begin{align*}
\sum_{n=1}^{N} \left[ \frac{1}{2} F_{nn}^n \left( A_{n} + \Omega_{m} \right) + (g_{m}^n) \left( \frac{1}{2} F_{nn}^n - A_{n} \right) \right] \\
&+ \sum_{n=1}^{N} \left[ \sum_{i,j=1}^{\infty} \mu_i^n \left( z^n(t) \right) \mu_j^n \left( z^n(t) \right) \nu^n(t) \right] A_{i}^n x_n(t) \right] \\
&\leq \sum_{n=1}^{N} \left[ \frac{1}{2} F_{nn}^n \left( A_{n} + \Omega_{m} \right) + (g_{m}^n) \left( \frac{1}{2} F_{nn}^n - A_{n} \right) \right] \\
&\times \left( 1 - (g_{m}^n) \right) \left( \frac{1}{2} F_{nn}^n - \Omega_{m} \right) \right] x_n(t) \right]
\end{align*}
\]

(28)

then (27) becomes

\[
\begin{align*}
\dot{V}(x(t)) &\leq \sum_{n=1}^{N} \left[ \frac{1}{2} F_{nn}^n \left( A_{n} + \Omega_{m} \right) + (g_{m}^n) \left( \frac{1}{2} F_{nn}^n - A_{n} \right) \right] \\
&\times \left( 1 - (g_{m}^n) \right) \left( \frac{1}{2} F_{nn}^n - \Omega_{m} \right) \right] x_n(t)
\end{align*}
\]

(29)

The above inequality implies that

\[
\begin{align*}
\dot{V}_{m}(x_{m}(t)) &<-x_{m}^{\nu}(t) M_{m} x_{m}(t) + \nu_{m}^{\nu}(t) \left( \frac{1}{2} \Omega_{m} \right) v_{m}(t)
\end{align*}
\]

(30)

Integrating from \( t = 0 \) to \( t = t_f \), yields

\[
\begin{align*}
\dot{V}_{m}(x_{m}(t_f)) &<-\int_{0}^{t_f} x_{m}^{\nu}(t) M_{m} x_{m}(t) dt + \nu_{m}^{\nu}(t) \left( \frac{1}{2} \Omega_{m} \right) v_{m}(t) dt
\end{align*}
\]

(31)
Under zero-initial conditions and $\nu_n(x_n(t)) \geq 0$, we have

$$\int_0^t x_n^2(t) M_n x_n(t) dt < \lambda_n^2 \int_0^t \nu_n^2(t) v_n(t) dt.$$  \hspace{1cm} (32)

It can be seen that $H_n$ control performance (2) is achieved with a prescribed $\lambda_n$. Actually, if the disturbance $\nu_n(t)$ is zero, we have

$$V_n(x_n(t)) = -x_n^T(t) M_n x_n(t).$$  \hspace{1cm} (33)

Therefore, the system is stable in the sense of Lyapunov too.

Furthermore, the stability condition (28) can be relaxed as follows. If there exist symmetric matrices $Y^m_n$ and matrices $Y^m_n = Y^m_{n^T}$ such that the following two inequalities hold,

$$G_n^m P_n + P_n G_n^m + \sum_{i=1}^{N} A_{n}^T A_{n} + N \cdot P_n P_n$$

$$+ \sum_{i=1}^{N} \left[ \frac{1}{2} E_n^T F_n + (A_{n} + \Omega_m) \right]$$

$$+ N \cdot P_n P_n + \sum_{i=1}^{N} \left[ \frac{1}{2} E_n^T F_n + (A_{n} + \Omega_m) \right]$$

and

$$\left( \frac{G_n^m + G_n^m}{2} \right) P_n + P_n \left( \frac{G_n^m + G_n^m}{2} \right) + \sum_{i=1}^{N} A_{n}^T A_{n}$$

$$+ \sum_{i=1}^{N} \left[ \frac{1}{2} E_n^T F_n + (A_{n} + \Omega_m) \right]$$

$$+ N \cdot P_n P_n + \sum_{i=1}^{N} \left[ \frac{1}{2} E_n^T F_n + (A_{n} + \Omega_m) \right]$$

and

$$\left( \frac{G_n^m + G_n^m}{2} \right) P_n + P_n \left( \frac{G_n^m + G_n^m}{2} \right) + \sum_{i=1}^{N} A_{n}^T A_{n}$$

$$+ \sum_{i=1}^{N} \left[ \frac{1}{2} E_n^T F_n + (A_{n} + \Omega_m) \right]$$

where

$$G_n^m =$$

$$2 Y_{11}^m \cdots 2 Y_{1N}^m$$

$$2 Y_{21}^m \cdots 2 Y_{2N}^m$$

$$\vdots$$

$$2 Y_{N1}^m \cdots 2 Y_{NN}^m$$

$$I$$

$$0$$

$$\vdots$$

$$0$$

$$\vdots$$

$$0$$

$$A_n + \Omega_m$$

$$0$$

$$\vdots$$

$$0$$

$$\vdots$$

$$0$$

$$\psi_m =$$

$$\frac{1}{2} E_n^T F_n - \Omega_m$$

Hence, if (34), (35), $\Gamma_n < 0$, and $\psi_m \leq 0$ are satisfied, then (28) holds. Next, we need to transform the stabilization conditions into the LMI forms [36]. Let

$$Q_m = P_m^{-1}, \quad H_n = K_n Q_{m}, \quad F_n^m = Q_{m} Y_{n}^m Q_{m}, \quad F_n^m = Q_{m} Y_{n}^m Q_{m}, \quad A_n = Q_{m} A_{n} Q_{m} \quad \tilde{A}_n = Q_{m} A_{n} Q_{m}, \quad \text{and} \quad \tilde{A}_n = Q_{m} A_{n} Q_{m}.$$

Pre and post multiplication of $Q_m$ in (34) and (35), then (15) and (16) are obtained, respectively. Matrix (20) is obtained by pre and post multiplication of $\text{diag}(Q_m, \ldots, Q_m)$ in (37). Moreover, (21) is obtained by pre and post multiplication of $\text{diag}(Q_m, \ldots, Q_m)$ in (38) and by Schur-complements [36]. Thus, the proof is completed.

In Theorem 1, it is seen that the existence conditions of a $H_n$ fuzzy controller for the large-scale system (14) are proposed and the controller is synthesized.

The proposed fuzzy controller synthesis procedure can be summarized as follows.

Step 1: Let $f_i(x(t))$ be separated into a fixed linear term and an uncertain linear term as (3); and transform each subsystem without interconnections and disturbances into a T-S fuzzy system.

Step 2: Based on Theorem 1, find the matrices $Q_m = Q_m^T > 0$, $\tilde{A}_n = \tilde{A}_n^T > 0$, positive semi-definite matrices $\Lambda_m = \Lambda_m^T > 0, \tilde{A}_n = \tilde{A}_n^T > 0$, symmetric matrices $Y_n^m$ ($i=1, \ldots, r_m$), matrices $F_n^m = F_n^m T$ ($i < j < r_m$), and matrices $H_n^m$ to satisfy (15)-(17).

Step 3: The control gain $K_n^m$ of the controller is obtained from $K_n^m = H_n^m Q_m^{-1}$.

5. Examples

Two examples are given in the following to show the effectiveness of the proposed control synthesis in the paper. Here, the numerical software MATLAB LMI-toolbox [39] is used to get the control gain and do the simulation.

A. A Numerical example

A numerical example is given here to illustrate the $H_n$ fuzzy controller synthesis for the nonlinear large-scale system (1). Consider a nonlinear large-scale system

$$\dot{x}_i(t) = A_i^i (x(t)) x_i(t) + B_i (x(t)) u_i(t) + f_i (x(t)) + E_{ni}(t), \quad i = 1, 2, 3, \ldots$$  \hspace{1cm} (39)

where $x(t) = [x_1, x_2, x_3]^T$,

$$x_1 = [x_{11}, x_{12}]^T, \quad x_2 = [x_{21}, x_{22}]^T, \quad x_3 = [x_{31}, x_{32}]^T,$$

$$A_i^i (x(t)) = \begin{bmatrix} 0 & 3 \\ 4 & -10 + (x_{11})^2 \end{bmatrix},$$

$$A_i^2 (x(t)) = \begin{bmatrix} 0 & 1.5 + \sin(x_{21}) \\ 8 & -10 \end{bmatrix}. $$
\[
A(t) = \begin{bmatrix} 0 & 6 \\ 10 & 0.1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 3 \\ 4 & 10 \end{bmatrix}, \quad B_1 = B_2 = 1, \\
\infty
\]

\[
\mu_i(z_i) = \begin{cases} 
1, & z_i \leq \pi/2 \\mu_i'(z_i) = 1 - \mu_i(z_i), \\
0, & \text{others}
\end{cases}
\]

and \( \mu_i(z_i) = 1 - \mu_i'(z_i) \).

Step 2: Based on Theorem 1 and following the procedure in Section 4, all matrix parameters are obtained as follows.

\[
Q_1 = \begin{bmatrix} 0.0627 & -0.3355 \\ -0.3355 & 1.3151 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.0401 & -0.3859 \\ -0.3859 & 1.8830 \end{bmatrix}, \\
B_1 = \begin{bmatrix} 0.8297 & -0.5702 \\ -0.5702 & 0.4789 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.12244 & 0.0000 \\ 0.0000 & 0.9825 \end{bmatrix}, \\
M_1 = \begin{bmatrix} 52.4157 & 15.6957 \\ 15.6957 & 4.9126 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix}, \\
M_3 = \begin{bmatrix} 52.4157 & 15.6957 \\ 15.6957 & 4.9126 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 52.4157 & 15.6957 \\ 15.6957 & 4.9126 \end{bmatrix}, \\
F_1 = \begin{bmatrix} -9.7206 & -0.0000 \\ -0.0000 & -3.1161 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -9.7206 & -0.0000 \\ -0.0000 & -3.1161 \end{bmatrix}, \\
F_3 = \begin{bmatrix} -9.7206 & 0.0000 \\ -0.0000 & 0.0000 \end{bmatrix}, \quad F_4 = \begin{bmatrix} -9.7206 & -0.0000 \\ -0.0000 & -3.1161 \end{bmatrix}, \\
F_5 = \begin{bmatrix} -9.7206 & -0.0000 \\ -0.0000 & -3.1161 \end{bmatrix}, \quad F_6 = \begin{bmatrix} -9.7206 & 0.0000 \\ -0.0000 & 0.0000 \end{bmatrix}, \\
F_7 = \begin{bmatrix} -9.7206 & -0.0000 \\ -0.0000 & -3.1161 \end{bmatrix}, \quad F_8 = \begin{bmatrix} -9.7206 & 0.0000 \\ -0.0000 & 0.0000 \end{bmatrix}, \\
F_9 = \begin{bmatrix} -9.7206 & -0.0000 \\ -0.0000 & -3.1161 \end{bmatrix}, \quad F_10 = \begin{bmatrix} -9.7206 & 0.0000 \\ -0.0000 & 0.0000 \end{bmatrix}. 
\]
Step 3: Thus, the fuzzy control gains $\mathbf{K}_l$ are

$$
\mathbf{K}_1 = [135.2321, 43.3631], \quad \mathbf{K}_2 = [135.7647, 44.2531],
$$

$$
\mathbf{K}_3 = [188.5928, 41.9877], \quad \mathbf{K}_4 = [188.5928, 39.9877],
$$

$$
\mathbf{K}_5 = [450.8608, 543.6006], \quad \text{and} \quad \mathbf{K}_6 = [464.7078, 563.7475].
$$

The simulation results with initial conditions $(x_1(0), x_2(0)) = (1.4)$, $(x_3(0), x_4(0)) = (0.0)$, and $(x_5(0), x_6(0)) = (0.0)$, and the values of $\lambda_i = \sqrt{0.0674 \times 10^{-3}}$, $\lambda_2 = \sqrt{0.0069 \times 0.0082}$, and $\lambda_3 = \sqrt{0.5 \times 10^{-3}} = 0.0224$, all of which are less than the prescribed value 0.55. The $H_{\infty}$ control performance is achieved.

There is some discussion about this example in the following. Since each nonlinear term in the plant and interconnection needs at least two fuzzy rules to construct the T-S fuzzy model. Therefore, if we linearize all nonlinear terms to be fuzzy rules by using 'sector nonlinearity' method [34], it needs $2^{1+1} + 2^{1+3} + 2^{1+3}$ rules totally to construct the large-scale T-S fuzzy system (see Table 1).

However, the number of rules is reduced to $2^4 + 2^2 + 2^2$ if the proposed method is used, because the nonlinear terms of interconnections are not transformed to fuzzy rules. It is obvious that the number of rules is reduced. Table 1 shows the rule number comparison between the former method and the proposed method.
B. Balancing double-inverted pendulums

Here, we consider a practical example which is the problem of balancing double-inverted pendulums [28] connected by a torsional spring. Each pendulum is positioned by a torque input \( u_i \) applied by a servomotor at its base. \( \theta_i \) and \( \dot{\theta}_i \) are angular position and its rate.

We assume both \( \theta_i \) and \( \dot{\theta}_i \) are measureable and available to the \( l \)-th controller for \( l = 1, 2 \). Furthermore, \( v_1(t) = 5 \sin(\pi t/2) \) and \( v_2(t) = 5 \sin(\pi t/4) \) are the torque disturbances. The motion equations of the pendulums are defined as

\[
\begin{align*}
\dot{x}_{11}(t) &= x_{12}(t), \\
\dot{x}_{12}(t) &= \frac{m_g(r_i + R_i(t))}{J_1}\sin(x_{11}(t)) \cdot \frac{k}{J_1}x_{11}(t) + \frac{1}{J_1}u_i(t) \\
&\quad + \frac{k}{J_1}x_{11}(t) + \frac{1}{J_1}v_1(t),
\end{align*}
\]

where \( x_{11} = \theta_i \in [-\pi/2, \pi/2] \) and \( x_{12} = \dot{\theta}_i \in [-\pi/2, \pi/2] \) are the angular displacements of the pendulums from the vertical reference, the end masses of the pendulums are \( m_1 = 1 \text{kg} \) and \( m_2 = 2.5 \text{kg} \), the moments of inertia are \( J_1 = 2 \text{kg m}^2 \) and \( J_2 = 2.5 \text{kg m}^2 \), the constant of the connecting torsional spring is \( k = 1 \text{N m rad}^{-1} \), the gravitational acceleration is \( g = 9.81 \text{m/s}^2 \), the two pendulum’s lengths are \( r_i(t) = R_i(t) \), \( R_i(t) = r_i + R_i(t) \) respectively. \( R_i(t) \in [0, R] \), and \( R_i(t) \in [0, R] \), where \( r_i \) and \( R_i(t) \) are the constant value and variable value used to represent the unchanged part and changeable part of the pendulum length, respectively. The torsional spring is relaxed when both pendulums are in the upright position. Therefore, the origin \( x_{11} = x_{12} = x_{31} = x_{32} = 0 \) is the equilibrium point of the system. Here, the positions of the masses of the pendulums depend on the values of \( \theta_i \) and \( \dot{\theta}_i \), i.e., \( R_i(t) \) is a function of \( \theta_i \) and \( \dot{\theta}_i \). Therefore, the additional terms \( R_i(t) \) belong to the interconnections of these two subsystems. This assumption makes the interconnection between two pendulums be highly connected and nonlinear.

Table 1. Comparison of rule numbers between the proposed and the original T-S fuzzy large-scale system.

<table>
<thead>
<tr>
<th>( S^1 )</th>
<th>Nonlinear terms in the plant ( A'(x(t)) ), ( B'(x(t)) ), and interconnection ( f_i(x(t)) )</th>
<th>Rule numbers of the proposed T-S fuzzy large-scale system</th>
<th>Rule numbers of the original T-S fuzzy large-scale system</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plant: ( x_{11}^2 )</td>
<td>2 ( i )</td>
<td>2 ( i )</td>
<td></td>
</tr>
<tr>
<td>Interconnection: ( 0.3 \sin(x_{11}), 0.1x_{22}, x_{31} )</td>
<td>0</td>
<td>2 ( i )</td>
<td></td>
</tr>
<tr>
<td>Total rule numbers</td>
<td>2 ( i )</td>
<td>2 ( i )</td>
<td></td>
</tr>
<tr>
<td>( S^2 )</td>
<td>Plant: ( \sin(x_{21}) )</td>
<td>2 ( i )</td>
<td>2 ( i )</td>
</tr>
<tr>
<td>Interconnection: ( x_{11}, 0.2x_{12}, 0.1x_{21}, (x_{31})^2, \sin(x_{32}) )</td>
<td>0</td>
<td>2 ( i )</td>
<td></td>
</tr>
<tr>
<td>Total rule numbers</td>
<td>2 ( i )</td>
<td>2 ( i )</td>
<td></td>
</tr>
<tr>
<td>( S^3 )</td>
<td>Plant: ( \cos(x_{31})x_{32} )</td>
<td>2 ( i )</td>
<td>2 ( i )</td>
</tr>
<tr>
<td>Interconnection: ( \sin(x_{11})(x_{31})^2, x_{21}, 0.2x_{32} )</td>
<td>0</td>
<td>2 ( i )</td>
<td></td>
</tr>
<tr>
<td>Total rule numbers</td>
<td>2 ( i )</td>
<td>2 ( i )</td>
<td></td>
</tr>
</tbody>
</table>
Next, let (45) be rewritten as the following form.
\[ x_1(t) = U^1(x_1(t), u_1(t)) + f_1(x(t)) + E_1v_1(t), \quad l = 1, 2 , \] (46)
where \( x_1 = [x_{11}, x_{12}]^T \), \( x_2 = [x_{21}, x_{22}]^T \),

\[ U^1 = \left[ \begin{array}{c} x_{11}(t) \\ x_{12}(t) \end{array} \right] = \left[ \begin{array}{c} \frac{m_g R_2}{J_1} \sin(x_{11}(t)) + \frac{1}{J_1} u_1(t) \\ \frac{m_g R_2}{J_2} \sin(x_{12}(t)) + \frac{1}{J_2} u_2(t) \end{array} \right], \]
\[ U^2 = \left[ \begin{array}{c} x_{21}(t) \\ x_{22}(t) \end{array} \right] = \left[ \begin{array}{c} \frac{m_g R_1}{J_1} \sin(x_{21}(t)) + \frac{1}{J_1} u_1(t) \\ \frac{m_g R_1}{J_2} \sin(x_{22}(t)) + \frac{1}{J_2} u_2(t) \end{array} \right], \]
and the fuzzy membership functions are depicted as Fig. 3. Moreover, we decompose the nonlinear interconnection terms \( f_i(x(t)) \) to a fixed linear term and an uncertain linear term.

\[ f_i(x(t)) = g_{i1}(t) \left( \frac{m_g R_i}{J_i} \right) \sin(x_i(t)) + \frac{k}{J_i} x_i(t) + \frac{k}{J_i} \dot{x}_i(t) \]
\[ = \sum_{k=1}^2 A_{i, k} x_k(t) + \sum_{k=1}^2 g_{i, k} F_{i, k} x_k(t), \]
\[ f_{i1}(x(t)) = 0, \]
where \( A_{i, k} = \left[ \begin{array}{cc} 0 & 1 \\ 0 & \frac{m_g R_i}{J_i} \end{array} \right], \]
\( B_{i, k} = \left[ \begin{array}{c} \frac{1}{J_1} \\ \frac{1}{J_2} \end{array} \right], \)
\( A_{i, 1} = \left[ \begin{array}{cc} 0 & 1 \\ 0 & \frac{m_g R_i}{J_i} \end{array} \right], \]
\( B_{i, 1} = \left[ \begin{array}{c} \frac{1}{J_1} \\ \frac{1}{J_2} \end{array} \right], \)
\( A_{i, 2} = \left[ \begin{array}{cc} 0 & 1 \\ 0 & \frac{m_g R_i}{J_i} \end{array} \right], \]
\( B_{i, 2} = \left[ \begin{array}{c} \frac{1}{J_1} \\ \frac{1}{J_2} \end{array} \right], \)

and \( E_1 = \left[ \begin{array}{c} 0 \\ \frac{1}{J_1} \end{array} \right], \) (47)
The individual subsystems (46) excluding the nonlinear interconnection \( f_i(x(t)) \) and disturbance \( v_1(t) \) are transformed into the T-S fuzzy system by the method of ‘local approximation in fuzzy partition spaces’ [34].

Subsystem 1:
Rule 1: IF \( x_{11}(t) \) is about 0 ,
\[ \text{THEN } \dot{x}_{11}(t) = A_{1, 1} x_{11}(t) + B_{1, 1} u_1(t), \] (48)
Rule 2: IF \( x_{11}(t) \) is about \( \pm \pi/2 \),
\[ \text{THEN } \dot{x}_{11}(t) = A_{1, 2} x_{11}(t) + B_{1, 2} u_1(t), \] (49)

Subsystem 2:
Rule 1: IF \( x_{21}(t) \) is about 0 ,
\[ \text{THEN } \dot{x}_{21}(t) = A_{2, 1} x_{21}(t) + B_{2, 1} u_1(t), \] (50)
Rule 2: IF \( x_{21}(t) \) is about \( \pm \pi/2 \),
\[ \text{THEN } \dot{x}_{21}(t) = A_{2, 2} x_{21}(t) + B_{2, 2} u_1(t), \] (51)
where
\[ A_{1, 1} = \left[ \begin{array}{cc} 0 & 1 \\ 0 & \frac{m_g R_1}{J_1} \end{array} \right], \]
\[ B_{1, 1} = \left[ \begin{array}{c} \frac{1}{J_1} \\ \frac{1}{J_2} \end{array} \right], \]
\[ A_{1, 2} = \left[ \begin{array}{cc} 0 & 1 \\ 0 & \frac{m_g R_1}{J_1} \end{array} \right], \]
\[ B_{1, 2} = \left[ \begin{array}{c} \frac{1}{J_1} \\ \frac{1}{J_2} \end{array} \right], \]
\[ A_{2, 1} = \left[ \begin{array}{cc} 0 & 1 \\ 0 & \frac{m_g R_1}{J_2} \end{array} \right], \]
\[ B_{2, 1} = \left[ \begin{array}{c} \frac{1}{J_1} \\ \frac{1}{J_2} \end{array} \right], \]
\[ A_{2, 2} = \left[ \begin{array}{cc} 0 & 1 \\ 0 & \frac{m_g R_1}{J_2} \end{array} \right], \]
\[ B_{2, 2} = \left[ \begin{array}{c} \frac{1}{J_1} \\ \frac{1}{J_2} \end{array} \right]. \]
Theorem 1 and follows the procedure in Section 4, all matrix parameters are obtained as follows.

\[ Q_1 = 10^3 \times \begin{bmatrix} 0.0002 & -0.0083 \\ -0.0083 & 1.6169 \end{bmatrix}, Q_2 = 10^3 \times \begin{bmatrix} 0.0002 & -0.0083 \\ -0.0083 & 1.6169 \end{bmatrix}, \]

\[ \bar{M}_1 = 10^3 \times \begin{bmatrix} 0.0012 & 0.0000 \\ 0.0000 & 1.0109 \end{bmatrix}, \bar{M}_2 = 10^3 \times \begin{bmatrix} 0.0012 & 0.0000 \\ 0.0000 & 1.0109 \end{bmatrix}, \]

\[ F_{11} = F_{22} = \begin{bmatrix} -2.8079 & 0.2577 \\ 0.2577 & -870.2926 \end{bmatrix}, \]

\[ F_{12} = \begin{bmatrix} 0.2883 & -0.1289 \\ -0.1289 & -70.2959 \end{bmatrix}, F_{21} = F_{22} = \begin{bmatrix} -2.8079 & 0.2577 \\ 0.2577 & -870.2926 \end{bmatrix}, \]

\[ A_{11} = 10^3 \times \begin{bmatrix} 0.0035 & -0.0209 \\ -0.0209 & 2.5994 \end{bmatrix}, A_{21} = 10^3 \times \begin{bmatrix} 0.0035 & -0.0209 \\ -0.0209 & 2.5994 \end{bmatrix}, \]

\[ \bar{A}_{11} = 10^3 \times \begin{bmatrix} 0.0018 & -0.0234 \\ -0.0234 & 1.9615 \end{bmatrix}, \bar{A}_{21} = 10^3 \times \begin{bmatrix} 0.0018 & -0.0234 \\ -0.0234 & 1.9615 \end{bmatrix}, \]

\[ (54) \]

Thus, the fuzzy control gains \( K_l \) are

\[ K_1 = 10^4 \times \begin{bmatrix} 1.9587 & 0.0106 \\ 0.0106 & 2.4477 \end{bmatrix}, K_2 = 10^4 \times \begin{bmatrix} 3.0421 & 0.0133 \\ 0.0133 & 2.4473 \end{bmatrix}, \]

\[ K_2 = 10^4 \times \begin{bmatrix} 3.0421 & 0.0133 \\ 0.0133 & 2.4473 \end{bmatrix}. \]

\[ (55) \]

The simulation results with initial conditions \((x_{11}(0), x_{12}(0)) = (\pi/2, 0)\) and \((x_{21}(0), x_{22}(0)) = (-\pi/2, 0)\) are shown in Fig. 4.

![Figure 4](image-url)

**Figure 4. System responses of subsystems \( S^1 \) and \( S^2 \).**

![Figure 5](image-url)

**Figure 5. Position variation of pendulums for subsystems \( S^1 \) and \( S^2 \).**

It is seen that the controlled double-inverted pendulums (45) with the fuzzy controller are stable in the sense of Lyapunov. Next, under zero initial conditions, i.e. \((x_{11}(0), x_{12}(0)) = (0, 0)\) and \((x_{21}(0), x_{22}(0)) = (0, 0)\), the values of \( \lambda_l \), \( l = 1, 2 \), are obtained.

\[ \lambda_1 = \sqrt{1.3518 \times 10^{-4}} = 0.0116 \quad \text{and} \quad \lambda_2 = \sqrt{3.0421 \times 10^{-4}} = 0.0174, \]

respectively, which is less than the prescribed value 0.5. Figure 5 shows the position variation of pendulums for subsystems \( S^1 \) and \( S^2 \) when the system is controlled. Two simulation results show that the fuzzy controller proposed in this paper can stabilize the nonlinear large-scale systems where the T-S fuzzy model is with small number of fuzzy rules.

### 6. Conclusion

In this paper, the synthesis of a \( H_{\infty} \) fuzzy controller for nonlinear large-scale systems has been studied. The system can contain nonlinear subsystems and nonlinear interconnections. Since the nonlinear interconnections are decomposed to a combination of a fixed linear term and an uncertain linear term according to the proposed linear decomposition, there is no need to transform the nonlinear interconnections into a set of fuzzy rules. Therefore, the transformed large-scale T-S fuzzy system does not have the ‘rule-explosion’ problem. Then, the synthesis of the \( H_{\infty} \) control with the aid of an LMI tool can achieved successfully. Furthermore, in this paper, we make \( g_{\infty} \) a scalar within the interval \([-1,1]\). However, the actually value of \( g_{\infty} \) may affect the solution of stability conditions. Therefore, in the future research direction, we will pay attention on it.
Appendix

Due to $0 \leq (g_{in})^2 \leq 1$, $A_{in} = A_{in}' \geq 0$ and $\Omega_{in} = \Omega_{in}' \geq 0$, we have

$$
(g_{in})^2 \left( F_{in}' F_{in} - (g_{in})^2 F_{in}' F_{in} + \frac{1}{2} F_{in}' F_{in} - A_{in} \right) + \left(1 - (g_{in})^2 \right) \left( \frac{1}{2} F_{in}' F_{in} - \Omega_{in} \right) = \frac{1}{2} F_{in}' F_{in} + \left(1 - (g_{in})^2 \right) \frac{1}{2} F_{in}' F_{in} - A_{in} + \Omega_{in} \frac{1}{2} F_{in}' F_{in} - \Omega_{in}.
$$

Due to $-(g_{in})^2 \Omega_{in} \leq 0$ and $(g_{in})^2 A_{in} \leq A_{in}$, we have

$$
(56) \leq \frac{1}{2} F_{in}' F_{in} + (g_{in})^2 \left( \frac{1}{2} F_{in}' F_{in} - A_{in} \right) + \left(1 - (g_{in})^2 \right) \left( \frac{1}{2} F_{in}' F_{in} - \Omega_{in} \right) + A_{in} + \Omega_{in}.
$$

Acknowledgment

This work was supported by the National Science Council (NSC) of Taiwan under the contracts NSC 101-2218-E-008-015 and NSC 99-2221-E-008-093-MY3. The authors like to appreciate the support from NSC.

References


Wen-June Wang was born in Hsin-Chu, Taiwan in 1957. He received the B.S. degree in Control Engineering from National Chiao-Tung University, Hsin-Chu, Taiwan, R.O.C., in 1980, and the M.S. degree in Electrical Engineering from Tatung University, Taipei, Taiwan, R.O.C., in 1984. He received the Ph.D. degree in the Institute of Electronics from National Chiao-Tung University of Taiwan in 1987. Dr. Wang is presently a chair professor in the Department of Electrical Engineering, and serves as the Dean of College of Electrical Engineering and Computer Science, National Central University, Chung-Li, Taiwan. He has published more than 140 journal papers and 150 conference papers. He received the Distinguished Research Award from the National Science Council of Taiwan in 1999, 2001 and 2003, respectively. He is an IEEE fellow and serves as a member of the editorial board of numerous journals, including the International Journal of Electrical Engineers, IEEE Trans. on Fuzzy Systems, and IEEE Trans. on Cybernetics. He is also the Editor in Chief of the International Journal of Fuzzy Systems. His research interests include the areas of fuzzy theory and systems, robust control, neural networks, and pattern recognition.

Nai-Jen Li was born in Taipei, Taiwan, in 1984. He received the B.S. and M.S. degree in electrical engineering from Tamkang University, Tamsui, Taiwan, in 2007 and 2009, respectively. He is currently working toward his Ph.D. degree in electrical engineering at National Central University, Jhongli, Taiwan. His currently research interests are in the areas of evolutionary algorithms, neural networks, image processing and fuzzy control.

Hao-Gong Chou was born in Taipei, Taiwan, in 1984. He received the B.S. and M.S. degree from Chien Hsin University of Science and Technology, Jhongli, Taiwan, in 2007 and 2010, respectively. He is currently working toward the Ph.D. degree with the Department of Electrical Engineering, National Central University, Jhongli, Taiwan. His current research interests include fuzzy control, and field-programmable gate array chip design.

Jun-Wei Chang was born in Taipei, Taiwan, in 1985. He received the B.S. and M.S. degree from National Formosa University, Yunlin, Taiwan, in 2007 and 2009, respectively. He is currently working toward the Ph.D. degree with the Department of Electrical Engineering, National Central University, Jhongli, Taiwan. His current research interests include fuzzy control, image processing, and intelligent robots.