Generalized Interval-Valued Atanassov’s Intuitionistic Fuzzy Power Operators and Their Application to Group Decision Making

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Abstract

We generalize the power averaging operators to interval-valued Atanassov’s intuitionistic fuzzy environments, and develop a series of generalized interval-valued intuitionistic fuzzy power aggregation operators. The main advantages of these operators are that they not only accommodate situations in which the input arguments are interval-valued intuitionistic fuzzy numbers (IVIFNs), but also consider information about the relationship between the IVIFNs being fused. The properties of these operators are investigated and the relationships among these operators are discussed. Moreover, approaches to multiple attributes group decision making based on the proposed operators are given and two examples are illustrated to show the feasibility and validity of the new approaches to the application of multiple attributes group decision making.

Keywords: Interval-valued Atanassov’s intuitionistic fuzzy sets; power aggregation operators; generalized interval-valued intuitionistic fuzzy power aggregation operators; multiple attribute group decision making.

1. Introduction

As a generalization of fuzzy set (FS) [24], Atanassov [1] presented intuitionistic fuzzy set (IFS), which considers the influence of the degree of membership function, the degree of non-membership and the degree of hesitate at the same time. Thus, it is more flexibility and practicability in dealing with vagueness and fuzzy information.

Information aggregation is a pervasive activity in our daily life, many operators had been proposed on this issue [5-10, 15, 21, 23]. Traditional aggregation operator is the weighted average operator [13] and the weighted geometric averaging operator [8]. Yager [21] proposed the ordered weighted average operator. Xu [11] presented some intuitionistic fuzzy aggregation operators, including the intuitionistic fuzzy weighted averaging operator, the intuitionistic fuzzy ordered weighted averaging (IFOWA) operator and the intuitionistic fuzzy hybrid averaging (IFHA) operator.

After these pioneering works, intuitionistic fuzzy multi-criteria decision making problems have received much more attention [3, 4, 7, 9, 12, 16, 18-20, 25-28, 31, 32]. Considering that all the above aggregation operators do not consider information about the relationship between the values being fused in the process of multi-attribute group decision making, Yager [22] originally introduced the power average operator, which allow the evaluated values being aggregated to support and reinforce each other. Xu and Yager [17] proposed the power geometric operator. Zhou and Chen [27] developed a generalized power average operator. Much more attentions [13, 26, 29, 30] have been focused on intuitionistic fuzzy power aggregation operators from researchers in the later works to accommodate situations in which the input arguments are intuitionistic fuzzy numbers. Xu [13] developed some Atanassov’s intuitionistic fuzzy power geometric aggregation operators. Zhang [29] presented a series of generalized Atanassov’s intuitionistic fuzzy power geometric aggregation operators.

However, because of the uncertainty of the environment and the hesitancy of the decision makers, it is more appropriate to present the evaluated values by interval-valued Atanassov’s intuitionistic fuzzy numbers (IVIFNs) in some cases. To accommodate these situations, this paper aims at extending the PA operators to interval-valued Atanassov’s intuitionistic fuzzy environments, and presenting a series of interval-valued Atanassov’s intuitionistic fuzzy power aggregation operators, including the interval-valued Atanassov’s intuitionistic fuzzy power averaging (IVIFPA) operator, the interval-valued Atanassov’s intuitionistic fuzzy power ordered weighted averaging (IVIFPOWA) operator and the corresponding generalized aggregation operators. The key advantages of these operators are that...
they not only accommodate situations in which the input arguments are IVIFNs, but also consider information about the relationship between the IVIFNs being fused by allowing the values being aggregated to support and reinforce each other.

The rest of the paper is organized as follows. Section 2 briefly reviews some basic concepts. Section 3 and 4 develop some interval-valued Atanassov’s intuitionistic fuzzy power averaging operators. Section 5 presents approaches to multiple attribute group decision making based on the proposed operators. Section 6 shows the feasibility and validity of the approaches to the application of multi-attribute decision making by numerical examples. Section 7 provides the concluding remarks.

2. Preliminaries

Definition 1 [2]: Let X be an ordinary finite non-empty set. An interval-valued Atanassov’s intuitionistic fuzzy set (IVIFS)  is in X is defined as where  and  are the membership degree range and the non-membership degree range, respectively, satisfying 0 ≤ u(x) ≤ 1 and 0 ≤ v(x) ≤ 1.

Each two-tuples  is called an interval-valued intuitionistic fuzzy number (IVIFN) in [14], where  and  are interval numbers, satisfying 0 ≤ u(x) + v(x) ≤ 1.

For convenience, we denote all interval-valued intuitionistic fuzzy numbers as IVIFNs(X).

If  and  then IVIFS is reduced to IFS.

If  and  then IVIFS is reduced to Zadeh’s FS.

Xu and Chen [14] defined the score function and accuracy degree of IVIFS  as follows:

\[ S(\tilde{A}) = u_A^i - v_A^i + u_A^i - v_A^i / 2. \]

\[ H(\tilde{A}) = u_A^i + v_A^i + u_A^i + v_A^i / 2. \]

Definition 2 [14]: Let  be any two IVIFN s, then  if and only if (i)  (ii)  and  (iii)  .

Definition 3 [14]: Let  be any two IVIFN s, then

\[ A \oplus B = \{ [u_A^i + u_B^i - u_A^i u_B^i, u_A^i - u_A^i u_B^i], [v_A^i + v_B^i - v_A^i v_B^i, v_A^i - v_A^i v_B^i] \} \]

\[ A \otimes B = \{ [u_A^i u_B^i, u_A^i u_B^i], [v_A^i v_B^i, v_A^i v_B^i] \} \]

\[ \lambda \tilde{A} = \{ [(1 - u_A^i)^i, 1 - (1 - u_A^i)^i], [(1 - v_A^i)^i, (1 - v_A^i)^i] \} \]

4. \[ \tilde{A} = \{ [(u_A^i)^i, (u_A^i)^i], [(1 - v_A^i)^i, 1 - (1 - v_A^i)^i] \} \]

In order to provide a tool to aid and provide more versatility in the data aggregation process, Yager [22] introduced the PA operator and the power ordered weighted average (POWA) operator.

Definition 4 [22]: Let  be a collection of data. The PA operator is the mapping defined as

\[ PA(a_1, \ldots, a_n) = \frac{\sum_{i=1}^{n} (1 + T(a_i)) a_i}{\sum_{i=1}^{n} (1 + T(a_i))}, \]

where \( T(a_i) = \sum_{j \in \text{index}(i)} \text{Supp}(a_i, a_j), \)

\[ \text{POWA}(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} u_{a_{\text{index}(i)}} \]

where \( u_i = g(R_{ij}/TV) \) and \( R_i = \sum_{j \in \text{index}(i)} V_{i,j}, TV = \sum_{j \in \text{index}(i)} V_{i,j}, V_{i,j} = 1 + T(a_{\text{index}(j)}) \)

\[ T(a_{\text{index}(j)}) = \sum_{i \in \text{index}(j)} \text{Supp}(a_i, a_{\text{index}(j)}), \]

\[ a_{\text{index}(j)} \] is the support of \( a_j \), and \( a_j \) . The detailed properties were discussed in [22].

Definition 5 [22]: Let  be a collection of data. The POWA operator is the mapping defined as

\[ \text{POWA}(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} u_{a_{\text{index}(i)}} \]

\[ \text{POWA}(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} u_{a_{\text{index}(i)}} \]

where \( u_i = g(R_{ij}/TV) \) and \( R_i = \sum_{j \in \text{index}(i)} V_{i,j}, TV = \sum_{j \in \text{index}(i)} V_{i,j}, V_{i,j} = 1 + T(a_{\text{index}(j)}) \)

\[ T(a_{\text{index}(j)}) = \sum_{i \in \text{index}(j)} \text{Supp}(a_i, a_{\text{index}(j)}), \]

\[ a_{\text{index}(j)} \] is the support of \( a_j \), and \( a_j \). The detailed properties were discussed in [22].

Definition 5 [22]: Let  be a collection of data. The POWA operator is the mapping defined as

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\[ \text{POWA}(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} u_{a_{\text{index}(i)}} \]

where \( u_i = g(R_{ij}/TV) \) and \( R_i = \sum_{j \in \text{index}(i)} V_{i,j}, TV = \sum_{j \in \text{index}(i)} V_{i,j}, V_{i,j} = 1 + T(a_{\text{index}(j)}) \)

\[ T(a_{\text{index}(j)}) = \sum_{i \in \text{index}(j)} \text{Supp}(a_i, a_{\text{index}(j)}), \]

\[ a_{\text{index}(j)} \] is the support of \( a_j \), and \( a_j \). The detailed properties were discussed in [22].

3. Some Interval-Valued Atanassov’s Intuitionistic Fuzzy Power Averaging Operators

A. The IVIFPA operator

Definition 6: Let  be any two IVIFN s, then the IVIFPA operator is defined as

\[ \text{IVIFPA}(\tilde{A}_1, \ldots, \tilde{A}_n) = \sum_{i=1}^{n} \text{Supp}(\tilde{A}_i, \tilde{A}_j), \]

\[ \text{Supp}(\tilde{A}_i, \tilde{A}_j) \] is the support for \( \tilde{A}_i \) and \( \tilde{A}_j \), satisfying:

(a) \( \text{Supp}(\tilde{A}_i, \tilde{A}_j) \in [0,1] \),

(b) \( \text{Supp}(\tilde{A}_i, \tilde{A}_j) = \text{Supp}(\tilde{A}_j, \tilde{A}_i) \),

(c) \( \text{Supp}(\tilde{A}_i, \tilde{A}_j) \geq \text{Supp}(\tilde{A}_i, \tilde{A}_k) \), If \( d(\tilde{A}_i, \tilde{A}_j) < d(\tilde{A}_i, \tilde{A}_k) \),

\[ d(\tilde{A}_i, \tilde{A}_j) \] is the distance between \( \tilde{A}_i \) and \( \tilde{A}_j \).

If  and  the IVIFPA operator becomes the intuitionistic fuzzy power averaging (IFPA) operator.
[27].

Theorem 1: Let \( \tilde{A} = \{\{u_i, a_i^+, [v_i, v_i^+]\}\} \in \text{IVIFNs}(X) (i = 1, \ldots, n) \), then \( \text{IVIFPA}(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) \in \text{IVIFNs}(X) \). Moreover,

\[
\text{IVIFPA}(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) = \left\{ \left[ \frac{1}{\prod_{j=i}^n (1-\alpha_{i,j})(\sum_{j=1}^n (1-\alpha_{i,j}))} \right], \left[ \frac{1}{\prod_{j=i}^n (1-\alpha_{i,j})(\sum_{j=1}^n (1-\alpha_{i,j}))} \right] \right\},
\]

for \( i = 1, \ldots, n \).

Proof: Xu [11] proved the following equation by mathematical induction on \( n \).

\[
\text{IFWA}(A_1, A_2, \ldots, A_n) = \left( 1 + \frac{1}{\prod_{j=i}^n (1-\alpha_{i,j})(\sum_{j=1}^n (1-\alpha_{i,j}))} \right) \frac{1}{\prod_{j=i}^n (1-\alpha_{i,j})(\sum_{j=1}^n (1-\alpha_{i,j}))}.
\]

By replacing \( w_j (i = 1, 2, \ldots, n) \) with \( (1 + T(\tilde{A}_i))(\sum_{j=1}^n (1 + T(\tilde{A}_j))) \),

\[
(i = 1, 2, \ldots, n), \quad A_i = \{u_i, a_i^+, [v_i, v_i^+]\} \text{ with } \tilde{A}_i = \left[ \frac{1}{\prod_{j=i}^n (1-\alpha_{i,j})(\sum_{j=1}^n (1-\alpha_{i,j}))} \right], \left[ \frac{1}{\prod_{j=i}^n (1-\alpha_{i,j})(\sum_{j=1}^n (1-\alpha_{i,j}))} \right], \left[ \frac{1}{\prod_{j=i}^n (1-\alpha_{i,j})(\sum_{j=1}^n (1-\alpha_{i,j}))} \right].
\]

Then by Definition 1, we have that the aggregated value by using the \text{IVIFPA} operator is also an \text{IVIFN}.

Theorem 2: \( \tilde{A} = \{[u_i, a_i^+, [v_i, v_i^+]]\} \in \text{IVIFNs}(X) (i = 1, \ldots, n) \).

(1) Idempotency: If all \( \tilde{A} = \tilde{A}_i \), then \( \text{IVIFPA}(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3) = \tilde{A}_i \).

(2) Boundedness: \( \{[\min [u_i, a_i^+, [v_i, v_i^+]], [\max [v_i, v_i^+]]] \} \leq \text{IVIFPA}(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3) \leq \{[\max [u_i, a_i^+, [v_i, v_i^+]], [\min [v_i, v_i^+]]] \} \leq \text{IVIFPA}(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3) \).

C. The \text{IVIFPOWA} operator

Definition 8: \( \tilde{A} = \{[u_i, a_i^+, [v_i, v_i^+]]\} \in \text{IVIFNs}(X) (i = 1, \ldots, n) \).

Then the \text{IVIFPOWA} operator is defined as

\[
\text{IVIFPOWA}(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) = \sum_{j=i}^n \tilde{A}_i,
\]

where \( u_i = g(D_i) - g(D_{i-1}) \).

4. Some Generalized Interval-Valued Atanassov’s Intuitionistic Fuzzy Power Operators

A. The \text{GIVIFPA} operator

Definition 9: \( \tilde{A} = \{[u_i, a_i^+, [v_i, v_i^+]]\} \in \text{IVIFNs}(X) (i = 1, \ldots, n) \).

Then the generalized interval-valued intuitionistic fuzzy power averaging (\text{GIVIFPA}) operator is defined as

\[
\text{GIVIFPA}(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) = \left( \frac{1}{\prod_{j=i}^n (1-\alpha_{i,j})(\sum_{j=1}^n (1-\alpha_{i,j}))} \right) \frac{1}{\prod_{j=i}^n (1-\alpha_{i,j})(\sum_{j=1}^n (1-\alpha_{i,j}))}.
\]

Proof: The proof is similar to Theorem 1, so omitted.
\[ \text{GIVIFPA}(\tilde{\alpha}, \ldots, \tilde{\alpha}) = \left( \varphi + T(\tilde{A}) \right) / \sum_{j=1}^{n} (1 + T(\tilde{A})) \right)^{\lambda}, \quad (24) \]

where \( T(\tilde{A}) \) satisfies Eq. (17).

**Theorem 7:** Let \( \tilde{\alpha} \in \text{IVIFNs}(X) \) and \( \text{GIVIFPA}(\tilde{\alpha}, \ldots, \tilde{\alpha}) \in \text{IVIFNs}(X) \). Moreover, \( \text{GIVIFPA}(\tilde{\alpha}, \ldots, \tilde{\alpha}) \in \text{IVIFNS}(X) \).

**Proof:** Zhang [29] proved that \( \text{WGIFPA}_{\lambda}(\tilde{\alpha}, \ldots, \tilde{\alpha}) \)

\[ \left( \left[ 1 - \prod_{i=1}^{n} (1 - u_{\tilde{\alpha}})^{\lambda} / \sum_{i=1}^{n} (1 - u_{\tilde{\alpha}})^{\lambda} \right]^{\lambda} \right) \]

2) **Boundedness:** \( \lambda \rightarrow 0 \), then \( \lim_{\lambda \rightarrow 0} \text{GIVIFPA}(\tilde{\alpha}, \ldots, \tilde{\alpha}) \)

\[ \left( \left[ 1 - \prod_{i=1}^{n} (1 - u_{\tilde{\alpha}})^{\lambda} / \sum_{i=1}^{n} (1 - u_{\tilde{\alpha}})^{\lambda} \right]^{\lambda} \right) \]

3) **Commutativity:** \( \text{GIVIFPA}(\tilde{\alpha}, \ldots, \tilde{\alpha}) = \text{GIVIFPA}(\tilde{\alpha}', \ldots, \tilde{\alpha}') \),

where \( (\tilde{\alpha}', \ldots, \tilde{\alpha}') \) is any permutation of \( (\tilde{\alpha}, \ldots, \tilde{\alpha}) \).

**Theorem 9:** Let \( \tilde{\alpha} = \left( \left[ u_{\tilde{\alpha}}, u_{\tilde{\alpha}} \right], \left[ v_{\tilde{\alpha}}, v_{\tilde{\alpha}} \right] \right) \in \text{IVIFNs}(X) \).

1) If \( \lambda \rightarrow 0 \), then \( \lim_{\lambda \rightarrow 0} \text{GIVIFPA}(\tilde{\alpha}, \ldots, \tilde{\alpha}) \)

\[ \left( \left[ 1 - \prod_{i=1}^{n} (1 - u_{\tilde{\alpha}})^{\lambda} / \sum_{i=1}^{n} (1 - u_{\tilde{\alpha}})^{\lambda} \right]^{\lambda} \right) \]

2) For the given arguments \( \tilde{\alpha}(i = 1, \ldots, n) \), the \( \text{GIVIFPA} \) is monotone increasing with respect to the parameter \( \lambda \).

**Proof:** 1) Zhang [29] proved the following equation.

\[ \lim_{\lambda \rightarrow 0} \left( 1 - \prod_{i=1}^{n} (1 - u_{\tilde{\alpha}})^{\lambda} / \sum_{i=1}^{n} (1 - u_{\tilde{\alpha}})^{\lambda} \right) \]

And \( \lim_{\lambda \rightarrow 0} \left( 1 - \prod_{i=1}^{n} (1 - u_{\tilde{\alpha}})^{\lambda} / \sum_{i=1}^{n} (1 - u_{\tilde{\alpha}})^{\lambda} \right) \)

\[ 1 - e^{-\left( \prod_{i=1}^{n} (1 - u_{\tilde{\alpha}})^{\lambda} / \sum_{i=1}^{n} (1 - u_{\tilde{\alpha}})^{\lambda} \right)} \]

Similarly, we have

\[ \lim_{\lambda \rightarrow 0} \left( 1 - \prod_{i=1}^{n} (1 - u_{\tilde{\alpha}})^{\lambda} / \sum_{i=1}^{n} (1 - u_{\tilde{\alpha}})^{\lambda} \right) \]

Thus, 1) in Theorem 9 is established.

2) Zhang [29] proved that

\[ 1 - \prod_{i=1}^{n} (1 - u_{\tilde{\alpha}})^{\lambda} / \sum_{i=1}^{n} (1 - u_{\tilde{\alpha}})^{\lambda} \]

is monotonically decreasing with respect to the parameter \( \lambda \), and

\[ 1 - \prod_{i=1}^{n} (1 - v_{\tilde{\alpha}})^{\lambda} / \sum_{i=1}^{n} (1 - v_{\tilde{\alpha}})^{\lambda} \]

is monotonically increasing with respect to the parameter \( \lambda \).
is reduced to the $WGIFPA$ operator, i.e., Eq. (13).

If $u_i = u_j = u_k$, $v_j = v_j'$ and $\lambda = 1$, and $\text{Supp}(A, \bar{A}) = k$ for all $i$ and $j$, then the $GIVIFPA$ operator is reduced to the $IFA$ operator[13, 27], i.e., Eq. (26)

Theorem 11: Let $\bar{A} = \{[u_i, a_i], [v_j, b_j]\} \in IVIFns(X)(i = 1, \ldots, n)$

1) Idempotency: If $\bar{A} = \bar{A}_0$ (i = 1, 2, \ldots, n), then

$WGIVIFA(\bar{A}, \bar{A}_0, \ldots, \bar{A}_0) = \bar{A}$.

2) Boundedness:

$$\langle \min[u_i], \min[u_i'] \rangle, \langle \max[v_j], \max[v_j'] \rangle \leq GIVIFPOWA(\bar{A}, \ldots, \bar{A})$$

Theorem 12: Let $\bar{A} \in IVIFns(X)(i = 1, \ldots, n)$, and $\lambda > 0$.

The $GIVIFPA$ is monotone increasing with respect to the parameter $\lambda$.

Proof: The proof is similar to Theorem 9, so omitted.

C. The $GIVIFPOWA$ operator

Definition 11: Let $\bar{A} \in IVIFns(X)(i = 1, \ldots, n)$, then the $GIVIFPOWA$ operator is defined as

$$GIVIFPOWA(\bar{A}, \bar{A}_0, \ldots, \bar{A}_0) = \left( \sum_{i=1}^{n} u_i \hat{A}_i \right)^{\lambda}, \quad (29)$$

where $u_i$ satisfies Eq. (22).

Theorem 13: Let $\bar{A} \in IVIFns(X)(i = 1, \ldots, n)$.

Then $GIVIFPOWA(\bar{A}, \bar{A}_0, \ldots, \bar{A}_0) \in IVIFns(X)$.

Moreover,

$$GIVIFPOWA(\bar{A}, \bar{A}_0, \ldots, \bar{A}_0) = \left[ \left[ -\prod_{i=1}^{n} (1-\bar{u}_i^{\lambda-1}) \sum_{i=1}^{n} \bar{v}_i^{\lambda-1} \right]^{\lambda} \right], \quad (30)$$

Proof: The proof is similar to Theorem 7, so it is omitted here.

If $\lambda = 1$, then the $GIVIFPOWA$ operator is reduced to the $IVIFPOWA$ operator, i.e., Eq. (23).

If $u_i = u_j$ and $v_j = v_j'$, then the $GIVIFPOWA$ operator is reduced to the $IFA$ operator, i.e., Eq. (14).

If $u_i = u_j$ and $v_j = v_j'$ and $\lambda = 1$, then the $GIVIFPOWA$ operator is reduced to the $IFPOWA$ operator.[27]

If $u_i = u_j$ and $v_j = v_j'$, $\lambda = 1$ and $\text{Supp}(\bar{A}, \bar{A}_0) = k$ for all $i$ and $j$, then the $GIVIFPA$ operator is reduced to the $IFA$ operator[13, 27], i.e., Eq. (26)

Theorem 14: Let $\bar{A} \in IVIFns(X)(i = 1, \ldots, n)$.

1) Idempotency: If $\bar{A} = \bar{A}_0$ (i = 1, 2, \ldots, n), then

$GIVIFPOWA(\bar{A}, \bar{A}_0, \ldots, \bar{A}_0) = \bar{A}$.

2) Boundedness:

$$\langle \min[u_i], \min[u_i'] \rangle, \langle \max[v_j], \max[v_j'] \rangle \leq GIVIFPOWA(\bar{A}, \ldots, \bar{A})$$
\[ \begin{align*} &\leq \left\{ \max\{u_i^1\}, \max\{u_i^2\}, \min\{v_i^1\}, \min\{v_i^2\} \right\}. \\
3) \text{ Commutativity:} & \quad \text{GIVIFPOWA}(\tilde{A}, \ldots, \tilde{A}) = \text{GIVIFPOWA}(\tilde{A}', \ldots, \tilde{A}') \ , \quad \text{where} \\
& \quad (\tilde{A}', \ldots, \tilde{A}') \text{ is any permutation of } (\tilde{A}, \ldots, \tilde{A}). \\
\text{Theorem 15:} & \quad \text{Let } \tilde{A} \in \text{IVFNS}(X)(i=1, \ldots, n), \text{ and } \lambda > 0. \text{ The GIVIFPOWA is monotone increasing with respect to the parameter } \lambda. \\
\text{Proof:} & \quad \text{The proof is similar to Theorem 7, so omitted.} \\
\end{align*} \]

5. Approaches to MAGDM with Interval-Valued Atanassov’s Intuitionistic Fuzzy Information

For a multiple attributes decision making problem, let \( X = \{x_1, \ldots, x_m\} \) be a set of candidate alternatives, and \( G = \{g_1, \ldots, g_n\} \) be a set of attributes with the associated weighting vector \( w = (w_1, \ldots, w_p) \) satisfying \( w_j \in [0,1] \) and \( \sum_{j=1}^{p} w_j = 1 \). \( D = \{d_1, \ldots, d_l\} \) be the decision makers with the weighting vector \( \omega = (\omega_1, \ldots, \omega_l) \), satisfying \( \omega_j \in [0,1] \) and \( \sum_{j=1}^{l} \omega_j = 1 \). Assume that the characteristics of the alternatives \( x_i(i=1, \ldots, m) \) under attribute \( g_j(j=1, \ldots, n) \) by decision maker \( d_l(l=1, \ldots, k) \) are represented by interval-valued Atanassov’s intuitionistic fuzzy sets \( \tilde{X}^{ij} = \left[ u^{ij}_w, u^{ij}_v \right], \left[ v^{ij}_w, v^{ij}_v \right] \) \((i=1, \ldots, m; j=1, \ldots, n; l=1, \ldots, k)\), where \([u^{ij}_w, u^{ij}_v],[v^{ij}_w, v^{ij}_v]\) indicates the degree that the alternative \( x_i \) \((i=1, \ldots, m)\) satisfies the attribute \( g_j \) \((j=1, \ldots, n)\), \([v^{ij}_w, v^{ij}_v]\) indicates the degree that the alternative \( x_i \) \((i=1, \ldots, m)\) doesn’t satisfy the attribute \( g_j \) \((j=1, \ldots, n)\). If the attributes are benefit attributes, the bigger attribute values, the better. If the attributes are cost attributes, the smaller attribute values, the better.

Before the decision making, we need to obtain a normalized interval-valued Atanassov’s intuitionistic fuzzy decision matrix \( \tilde{X}^{ij} = \left( \tilde{x}^{ij}_{l} \right)_{m \times n} \) by transforming the cost-type attribute values into benefit-type attribute values similar to the method in [10] as follows:

\[ \tilde{x}^{ij}_{l} = \begin{cases} A^{ij}_{l}, & \text{for benefit attribute } c_j \\ \left( A^{ij}_{l} \right)^{\prime}, & \text{for cost attribute } c_j \end{cases}, \quad (31) \]

where \( A^{ij}_{l} = \left[ \left( v^{ij}_w, v^{ij}_v \right), \left[ u^{ij}_w, u^{ij}_v \right] \right] \). According to the weighting vectors of the decision makers and the attributes is known or not, we divide the MAGDM into two cases. Then we have the following decision making methods.

Case 1: The weighting vectors of the decision makers and the attributes are known, we develop a method based on the WGIIVIFPA operator.

Method 1

Step 1: Get the normalized decision matrix by Eq. (31).

Step 2: Calculate the supports:

\[ d(\tilde{x}^{ij}_{l}, \tilde{x}^{ij}_{l'}) = \frac{1}{4} \left[ |v^{ij}_w - u^{ij}_w| + |v^{ij}_v - u^{ij}_v| + |v^{ij}_w - v^{ij}_v| + |u^{ij}_w - v^{ij}_v| \right], \quad (33) \]

\[ d(\tilde{x}^{ij}_{l}, \tilde{x}^{ij}_{l'}) \text{ is the normalized Hamming distance between } \tilde{x}^{ij}_{l} \text{ and } \tilde{x}^{ij}_{l'}, \text{ which was defined in [12].} \]

Step 3: Calculate the weights associated with the IVIFN \( \tilde{r}^{ij}_{l} \) \((l=1,2,\ldots,k)\) with the formula

\[ \theta^{ij}_{l} = \omega_k \left( 1 + T(\tilde{r}^{ij}_{l}) \right) \sum_{l'=1}^{k} \omega_k \left( 1 + T(\tilde{r}^{ij}_{l'}) \right), l = 1,2,\ldots,k, \quad (34) \]

where \( T(\tilde{r}^{ij}_{l}) \) is the weighted support of IVIFN \( \tilde{r}^{ij}_{l} \) by the other IVIFN \( \tilde{r}^{ij}_{l'} \) \((l, l' = 1,2,\ldots,k; l \neq l')\), and

\[ T(\tilde{r}^{ij}_{l}) = \sum_{l'=1, l' \neq l}^{k} \omega_k \text{Supp}(\tilde{r}^{ij}_{l}, \tilde{r}^{ij}_{l'}). \quad (35) \]

Step 4: Aggregate all individual decision matrices \( \tilde{R}^{ij} = \left( \tilde{r}^{ij}_{l} \right)_{m \times n} \) into the collective decision matrix \( \tilde{R}^{ij} = \left( \tilde{r}^{ij}_{l} \right)_{m \times n} \) by the WGIIVIFPA operator (Eq. (28)), i.e.,

\[ \tilde{r}^{ij}_{l} = \left[ \left( 1 - \prod_{i=1}^{m} \left( 1 - u^{ij}_{l} \right)^{\omega_k} \right)^{\frac{1}{\omega_k}}, \left( 1 - \prod_{i=1}^{m} \left( 1 - v^{ij}_{l} \right)^{\omega_k} \right)^{\frac{1}{\omega_k}} \right] \quad (36) \]

Step 5: Calculate the supports:

\[ \text{Supp}(\tilde{r}^{ij}_{l}, \tilde{r}^{ij}_{l'}) = 1 - d(\tilde{r}^{ij}_{l}, \tilde{r}^{ij}_{l'}), l = 1,2,\ldots,m; j,t = 1,2,\ldots,n; t \neq j \quad (37) \]

Step 6: Calculate the weights associated with the IVIFN \( \tilde{r}^{ij}_{l} \) with the formula

\[ \gamma^{ij}_{l} = w_k \frac{1 + T(\tilde{r}^{ij}_{l})}{\sum_{l'=1}^{k} w_k \left( 1 + T(\tilde{r}^{ij}_{l'}) \right)}, j = 1,2,\ldots,n, \quad (38) \]

where

\[ T(\tilde{r}^{ij}_{l}) = \sum_{l'=1, l' \neq l}^{k} w_k \text{Supp}(\tilde{r}^{ij}_{l}, \tilde{r}^{ij}_{l'}). \quad (39) \]

Step 7: Aggregate all the preference number \( \tilde{r}^{ij}_{l} \) \((j=1,\ldots,n)\) into the collective number \( \tilde{r}^{ij}_{l} \) \((i=1,\ldots,m)\) by the WGIIVIFPA operator (Eq. (28)), i.e.,

\[ \tilde{r}^{ij}_{l} = \left[ \left( 1 - \prod_{i=1}^{m} \left( 1 - v^{ij}_{l} \right)^{\omega_k} \right)^{\frac{1}{\omega_k}}, \left( 1 - \prod_{i=1}^{m} \left( 1 - u^{ij}_{l} \right)^{\omega_k} \right)^{\frac{1}{\omega_k}} \right] \]  

\[ \left[ 1 - \left( 1 - \prod_{i=1}^{m} \left( 1 - v^{ij}_{l} \right)^{\omega_k} \right)^{\omega_k}, 1 - \left( 1 - \prod_{i=1}^{m} \left( 1 - u^{ij}_{l} \right)^{\omega_k} \right)^{\omega_k} \right]^{\frac{1}{\omega_k}} \quad (40) \]

Step 8: Calculate the score of \( \tilde{r}^{ij}_{l} \) \((i=1,2,\ldots,m)\).

Step 9: Rank \( \tilde{r}^{ij}_{l} \) \((i=1,2,\ldots,m)\) in descending order by
the result in Step 8 and Definition 2.

**Step 10:** Rank all the candidate alternatives.

**Step 11:** Select the best one(s).

**Case 2:** Suppose that the weighting vectors of the decision makers and the attributes are unknown, we develop a method based on the **GIVIFPOWA** operator.

**Method 2**

Similar to Method 1, except that the aggregation operator used in step 4 and 7 is replaced by Eq. (30). So the detailed processes are omitted here.

### 6. Illustrative Example

**Example 1:** (revised from [9, 29]). Wang and Lee [9] wanted to select a software, which is a MAGDM problem. The alternatives are the software packages, and the evaluated values are fuzzy numbers. Considering that the decision information may be given by the forms of intuitionistic fuzzy numbers as the changing of the environment, Zhang [29] reconsider this problem under intuitionistic fuzzy environment, he assumed that a computer center desires to select a new information system for the improving of the work productivity. After preliminary screening, four alternatives \( x_i (i=1,2,3,4) \) were taken into consideration. The considered attributes are as follows.

- \( g_1 \): The costs of the hardware and software investment.
- \( g_2 \): The reliability of outsourcing software development.
- \( g_3 \): The contribution to organization performance.
- \( g_4 \): The effort to transition from the current systems.

where \( g_1 \) is of cost type and the others are of benefit type, and the weight vector of the attributes is \( w = (0.5, 0.3, 0.1, 0.1)^T \).

Three experts \( d_i (i=1,2,3) \) have been invited to act as decision makers from a committee, whose weight vector is \( \omega = (0.2, 0.5, 0.3) \).

Now, we reconsidered this problem in interval-valued Atanassov’s intuitionistic fuzzy environment, and the evaluated values of the alternatives under attributes are constructed by interval-valued Atanassov’s intuitionistic fuzzy numbers by the invited experts (see Tables 1-3).

<table>
<thead>
<tr>
<th>Table 1. The interval-valued Atanassov’s intuitionistic fuzzy decision matrix ( \tilde{A}^{(1)} ) by ( d_1 ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_1 )</td>
</tr>
<tr>
<td>( x_1 )</td>
</tr>
<tr>
<td>( x_2 )</td>
</tr>
<tr>
<td>( x_3 )</td>
</tr>
<tr>
<td>( x_4 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2. The interval-valued Atanassov’s intuitionistic fuzzy decision matrix ( \tilde{A}^{(2)} ) by ( d_2 ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_1 )</td>
</tr>
<tr>
<td>( x_1 )</td>
</tr>
<tr>
<td>( x_2 )</td>
</tr>
<tr>
<td>( x_3 )</td>
</tr>
<tr>
<td>( x_4 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 3. The interval-valued Atanassov’s intuitionistic fuzzy decision matrix ( \tilde{A}^{(3)} ) by ( d_3 ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_1 )</td>
</tr>
<tr>
<td>( x_1 )</td>
</tr>
<tr>
<td>( x_2 )</td>
</tr>
<tr>
<td>( x_3 )</td>
</tr>
<tr>
<td>( x_4 )</td>
</tr>
</tbody>
</table>

Since the weights of the decision makers and the attributes are known, we select Method 1 as follows.

**Step 1-Step2:** Transform matrix \( \tilde{A}^{(i)} \) into the normalized decision matrix \( \tilde{R}^{(i)} (i=1,2,3) \) by Eq. (31), and obtain the supports \( \text{Supp}(\tilde{r}_y^{(1)},\tilde{r}_y^{(2)}), (l_1,l_2=1,2,3, l_1 \neq l_2) \) as follows.
Step 3: By Eq. (35), we get the weighted support of IVIFN $\tilde{r}_i^0$ ($i=1,2,3$) by other IVIFN $\tilde{r}_j$ as follows.

\[
T(\tilde{r}_i^0) = \begin{bmatrix}
0.6400 & 0.6900 & 0.7850 & 0.5700 \\
0.7150 & 0.6700 & 0.4850 & 0.5600 \\
0.5050 & 0.6100 & 0.6550 & 0.7600 \\
0.3650 & 0.4400 & 0.5000 & 0.6900 \\
0.4150 & 0.3900 & 0.4850 & 0.3300 \\
0.4150 & 0.3700 & 0.3200 & 0.4400 \\
0.2950 & 0.3400 & 0.3850 & 0.4600 \\
0.3350 & 0.2600 & 0.3500 & 0.4200 \\
0.5350 & 0.5100 & 0.6650 & 0.5300 \\
0.5350 & 0.5300 & 0.5900 & 0.6400 \\
0.4950 & 0.5400 & 0.4450 & 0.6400 \\
0.5350 & 0.4600 & 0.5500 & 0.5600 
\end{bmatrix}.
\]

Then by Eq. (34), we obtain the weights associated with the IVIFN $\tilde{r}_j$ ($i=1,2,3$) as follows.

\[
\theta_{ij} = \begin{bmatrix}
0.2193 & 0.2275 & 0.2233 & 0.2184 \\
0.2270 & 0.2260 & 0.2071 & 0.2047 \\
0.2155 & 0.2215 & 0.2272 & 0.2236 \\
0.1949 & 0.2124 & 0.2083 & 0.2230 \\
0.4729 & 0.4677 & 0.4644 & 0.4624 \\
0.4682 & 0.4635 & 0.4603 & 0.4724 \\
0.4635 & 0.4608 & 0.4753 & 0.4638 \\
0.4764 & 0.4646 & 0.4688 & 0.4683 \\
0.3078 & 0.3048 & 0.3124 & 0.3192 \\
0.3048 & 0.3106 & 0.3326 & 0.3228 \\
0.3210 & 0.3177 & 0.2975 & 0.3126 \\
0.3287 & 0.3230 & 0.3229 & 0.3087 
\end{bmatrix}.
\]

Step 4: By Eq. (36), and let $\lambda=0.5$, we aggregate all the individual decision matrices $\tilde{R}^i=\tilde{r}_j^i$ ($i=1,2,3$) into the collective decision matrix $\tilde{R}=(\tilde{r}_j)$ as Tables 4.

Step 5-Step 6: Similar to Step 2-3, by Eqs. (32), (33), (37), (38) and (39), we get the weights associated with the IVIFN $\tilde{r}_j$ as $\gamma_j$, and

\[
(\gamma_j)_{4x4} = \begin{bmatrix}
0.3919 & 0.3271 & 0.1409 & 0.1401 \\
0.3927 & 0.3200 & 0.1428 & 0.1446 \\
0.4037 & 0.3289 & 0.1520 & 0.1155 \\
0.3853 & 0.3296 & 0.1512 & 0.1338 
\end{bmatrix}.
\]

Step 7: By Eq. (40), and let $\lambda=0.5$, we aggregate Table 4 as $\left(\tilde{r}_1^0, \tilde{r}_2^0, \tilde{r}_3^0, \tilde{r}_4^0\right)^T$, and

\[
\begin{bmatrix}
\tilde{r}_1 \\
\tilde{r}_2 \\
\tilde{r}_3 \\
\tilde{r}_4 
\end{bmatrix} = \begin{bmatrix}
[0.3094,0.4144] & [0.3262,0.4417] \\
[0.3833,0.4919] & [0.3394,0.4530] \\
[0.3678,0.4843] & [0.2713,0.3960] \\
[0.4173,0.5364] & [0.2328,0.3605] 
\end{bmatrix}.
\]

Step 8-9: By Eq. (1), we have $S(\tilde{r}_1)=0.1802 > S(\tilde{r}_2)=0.0924 > S(\tilde{r}_3)=0.0414 > S(\tilde{r}_4)=-0.0221$.

Step 10-11: Obviously, $x_1 \succ x_2 \succ x_3 \succ x_4$. $x_4$ is the best alternative.

It should be noted that different results will be obtained when the decision makers change the parameter $\lambda$, which can be seen clearly by Tables 5.

In order to analyze how the different attitudinal character plays a role in the aggregation results, we consider different value of $\lambda$ : 0.001, 0.002, ..., 30, which are provided by the decision maker. The score functions $S(\tilde{r}_i)(i=1,2,3,4)$ of $\tilde{r}_i$ are shown in Fig. 1.

![Figure 1. Variation of score functions with respect to $\lambda$.](image-url)

From Figure 1, we find that the overall trend of $S(\tilde{r}_i)(i=1,2,3,4)$ is increasing as $\lambda$ increases on $(0, 30)$, and the ranking is always $x_1 \succ x_2 \succ x_3 \succ x_4$.

**Example 2:** Let us reconsider Example 5.1 in the cases that the weights of the decision makers and the attributes are unknown for the increasing uncertainty of decision systems. Then we use Method 2 to solve the multiple attribute group decision making problems above.

By method 2, we have $x_1 \succ x_2 \succ x_3 \succ x_4$. $x_4$ is the best alternative. Since the calculation process is similar to method 1, so it is omitted here.

Obviously, the result get by method 2 is different from the result obtained by method 1, which reflects that the weights of decision makers and attributes play important roles in practical multiple attribute decision making problems. Moreover, the decision makers and attributes in method 2 are treated equally, which indicates that the weight of decision makers and attributes are determined.
subjectively to some degree. Therefore, we should use the method 1 to solve the problems in real life as far as possible. However, in the cases that the weighting vectors of the decision makers and the attributes are unknown, we can handle multiple attribute group decision making with the method 2.

Table 4. The interval-valued Atanassov’s intuitionistic fuzzy decision matrix.

<table>
<thead>
<tr>
<th></th>
<th>$g_1$</th>
<th>$g_2$</th>
<th>$g_3$</th>
<th>$g_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>(0.344, 0.448), (0.342, 0.466)</td>
<td>(0.237, 0.341), (0.3333, 0.444)</td>
<td>(0.400, 0.500), (0.329, 0.429)</td>
<td>(0.285, 0.396), (0.268, 0.385)</td>
</tr>
<tr>
<td>$x_2$</td>
<td>(0.424, 0.529), (0.326, 0.430)</td>
<td>(0.321, 0.426), (0.419, 0.526)</td>
<td>(0.420, 0.545), (0.259, 0.401)</td>
<td>(0.369, 0.474), (0.305, 0.419)</td>
</tr>
<tr>
<td>$x_3$</td>
<td>(0.324, 0.433), (0.264, 0.399)</td>
<td>(0.496, 0.614), (0.182, 0.293)</td>
<td>(0.388, 0.498), (0.341, 0.449)</td>
<td>(0.100, 0.200), (0.612, 0.714)</td>
</tr>
<tr>
<td>$x_4$</td>
<td>(0.516, 0.634), (0.151, 0.270)</td>
<td>(0.345, 0.467), (0.324, 0.456)</td>
<td>(0.465, 0.572), (0.212, 0.332)</td>
<td>(0.223, 0.326), (0.382, 0.490)</td>
</tr>
</tbody>
</table>

Table 5. Score values obtained by the $WGIVIFPA$ operator and the rankings of alternatives.

<table>
<thead>
<tr>
<th></th>
<th>$WGIVIFPA_{0.3}$</th>
<th>$WGIVIFPA_{0.8}$</th>
<th>$WGIVIFPA_{1}$</th>
<th>$WGIVIFPA_{2}$</th>
<th>$WGIVIFPA_{5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>-0.0270</td>
<td>-0.0145</td>
<td>-0.0093</td>
<td>0.0167</td>
<td>0.0849</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0.0348</td>
<td>0.0516</td>
<td>0.0584</td>
<td>0.0931</td>
<td>0.1877</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0.0802</td>
<td>0.1108</td>
<td>0.1230</td>
<td>0.1808</td>
<td>0.3027</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0.1674</td>
<td>0.1991</td>
<td>0.2115</td>
<td>0.2676</td>
<td>0.3767</td>
</tr>
</tbody>
</table>

Ranking: $x_1 > x_2 > x_3 > x_4$

<table>
<thead>
<tr>
<th></th>
<th>$WGIVIFPA_{10}$</th>
<th>$WGIVIFPA_{15}$</th>
<th>$WGIVIFPA_{20}$</th>
<th>$WGIVIFPA_{25}$</th>
<th>$WGIVIFPA_{30}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>0.1614</td>
<td>0.2078</td>
<td>0.2388</td>
<td>0.2616</td>
<td>0.2794</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0.2938</td>
<td>0.3583</td>
<td>0.4018</td>
<td>0.4333</td>
<td>0.4570</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0.4058</td>
<td>0.4594</td>
<td>0.4914</td>
<td>0.5123</td>
<td>0.5263</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0.4597</td>
<td>0.4988</td>
<td>0.5213</td>
<td>0.5358</td>
<td>0.5460</td>
</tr>
</tbody>
</table>

Ranking: $x_4 > x_3 > x_2 > x_1$

7 Conclusions

In this paper, we present a series of interval-valued Atanassov’s intuitionistic fuzzy power operators to accommodate situations in which the input arguments are interval-valued intuitionistic fuzzy numbers. The properties of these operators are investigated and the relationships among these operators are also discussed. Approaches to multi-criteria decision making based on the proposed operators is given under interval-valued intuitionistic fuzzy environment. Two examples are illustrated to show the feasibility and validity of the new approaches. In the succeeding work, we plan to extend our research work to fuzzy number intuitionistic fuzzy environment and apply some new aggregation operators to multi-attribute decision making, pattern recognition.

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