The Fuzzy Jump-Diffusion Model to Pricing European Vulnerable Options

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Abstract

Owing to the fluctuation of financial markets from time to time, some parameters, such as the interest rate, volatility, cannot be precisely described. Under the assumption that the risk-free rate, the volatility, and the average jump intensity are fuzzy numbers, this paper presents the jump-diffusion approach to price vulnerable options in fuzzy environments. We also provide the crisp possibilistic mean jump-diffusion model to price European vulnerable call options and the secant method to obtain the belief degree. Finally, the performance of our model and the algorithm is illustrated with some numerical examples.

Keywords: Fuzzy number, European vulnerable options, jump-diffusion process, secant method.

1. Introduction

Options are popular financial derivatives that play an essential role in global financial markets. Pricing them efficiently and accurately is very important, both in theory and in practice. This theory has been developed and improved on the basis of Black-Scholes log-normal stochastic differential models since the mid-1970s. Despite the advantage of using Black-Scholes model [1] and Merton model [2], these models have the limitation of assuming that the counter-party will not default. For an exchange-traded option the assumption is usually reasonable. But, in the over-the-counter market, the assumption is not feasible. In the real world, many options are traded in the over-the-counter market.

Consequently, it is important to consider the valuation of the vulnerable option pricing. Johnson and Stulz [3] firstly considered the problem of pricing options with default risk under the assumption that the option holder would access to all of the property of the option writer, once the option writer defaults. Hull and White [4] and Jarrow and Turnbull [5] assumed the dependence between the underlying assets and the credit risk, and presented a model for pricing vulnerable option. Klein [6] extended the previous studies and gave an improved model for valuing option when there was default risk. However, under the diffusion process, because a sudden drop in firm values is impossible, firms never default suddenly. Meanwhile, jumps and stochastic volatility are clearly identifiable from stock data, as discussed in A¨ıt-Sahalia et al. [7], Eraker et al. [8], Xu et al. [9, 10]. Amin [11] provided a simple discrete time model to value options when the underlying asset price followed a jump diffusion process. Following Klein [6] and Amin [11], Xu et al. [12] presented a new approach to price vulnerable options under the jump diffusion assumption about the underlying stock prices and firm values.

The above studies value vulnerable options in stochastic environment, which means the input parameters are regarded as the precise real-valued data. However, in the real world, some parameters in the pricing model cannot always be expected in a precise sense. For instance, the risk-free interest rate may occur imprecisely. Fortunately, the fuzzy sets theory proposed by Zadeh [13, 14] is a very useful tool for modeling this kind of imprecise problem. Simonelli [15] provided a methodology evaluating financial instruments using certainty equivalents. Yoshida [16] and Yoshida et al. [17] discussed the valuation of European and American options with uncertainty for both random and fuzziness in the output variables, by introducing fuzzy logic to the stochastic financial model. Wu [18-20] presented the fuzzy pattern of the Black-Scholes formula by introducing a fuzzy arithmetic. Chrysafis and Papadopoulos [21] considered an application of a new method of constructing a fuzzy estimator for volatility in Black-Scholes formula, and analyzed the results to the Greek parameters. Xu et al. [22] presented a fuzzy normal jump-diffusion model for pricing the European options in which the jumps contained the uncertainty of both randomness and fuzziness. Nowak and Romaniuk [23] proposed a method to compute option price for

Xu et al. [12] assumed that the risk-free rate, the volatility and the average jump intensity these market parameters were precise real numbers. However, in the real world, due to imprecise information and the fluctuations of financial market from time to time, it is unreasonable that these market parameters are expected to be constants over time. To reflect this fact, in the paper, fuzziness is introduced to describe the uncertainty as to the risk-free rate, the volatility and the average jump intensity. Based on it, we present the jump-diffusion approach to modeling vulnerable option pricing under fuzzy environments. In the latter part of this section, we provide the crisp possibilistic mean jump-diffusion model to pricing European vulnerable call option. In Section 4, we analyze the results by numerical examples. Section 5 concludes this paper.

The rest of this paper is arranged as follows. Section 2 contains preliminaries. In Section 3, we present the jump-diffusion approach to modeling vulnerable option pricing under fuzzy environments. In the latter part of this section, we provide the crisp possibilistic mean jump-diffusion model to pricing European vulnerable call option. In Section 4, we analyze the results by numerical examples. Section 5 concludes this paper.

2. Fuzzy Set Theory

Definition 2.1 [18]: Let \( \tilde{a} \) be a fuzzy subset of \( \mathbb{R} \). Then \( \tilde{a} \) is called a fuzzy number if the following conditions are satisfied:

(i) \( \tilde{a} \) is a normal and convex fuzzy set;
(ii) its membership function \( \mu_\tilde{a} \) is upper semi-continuous;
(iii) the \( \alpha \)-level set \( \tilde{a}_\alpha \) is bounded.

From Zadeh [13], \( \tilde{A} \) is a convex fuzzy set if and only if each of the \( \alpha \)-level set \( \tilde{A}_\alpha = \{ x | \mu_{\tilde{A}}(x) > \alpha \} \) is a convex set. Therefore, if \( \tilde{a} \) is a fuzzy number, then the \( \alpha \)-level set \( \tilde{a}_\alpha \) is a compact and convex set; that is, \( \tilde{a} \) is a closed interval. Then the \( \alpha \)-level set of \( \tilde{a} \) is denoted by \( \tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U] \).

If the membership function of fuzzy number \( \tilde{a} \) is defined as

\[
\begin{align*}
\mu_\tilde{a}(x) = \begin{cases} 
\frac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2; \\
\frac{x - a_2}{a_3 - a_2}, & a_2 \leq x \leq a_3; \\
0, & \text{otherwise}.
\end{cases}
\end{align*}
\]

then we named fuzzy number \( \tilde{a} \) as a triangular fuzzy number (see Figure 1).

Triangular fuzzy number \( \tilde{a} = (a_1, a_2, a_3) \). A \( \alpha \)-level set of \( \tilde{a} \) is written as

\[
\tilde{a}_\alpha = [a_2 - (a_2 - a_1)(1 - \alpha), a_2 + (a_3 - a_2)(1 - \alpha)],
\]

then

\[
\tilde{a}_\alpha^L = a_2 - (a_2 - a_1)(1 - \alpha), \tilde{a}_\alpha^U = a_2 + (a_3 - a_2)(1 - \alpha).
\]

Because triangular fuzzy number is the simplest form of fuzzy numbers and is easy to be constructed, in this paper, we assume that all the fuzzy numbers are triangular fuzzy numbers.

Proposition 2.2 [14]: Let \( \tilde{A} \) be a fuzzy set with membership function \( \mu_{\tilde{A}} \) and \( \tilde{A}_\alpha = \{ x | \mu_{\tilde{A}}(x) > \alpha \} \).

Then

\[
\mu_{\tilde{A}}(x) = \sup_{a \in [a_0, a_1]} \alpha [1_{\tilde{A}_\alpha}(x)],
\]

where \( 1_{\tilde{A}_\alpha} \) is an indicator function of set \( \tilde{A}_\alpha \), i.e.,

\[
1_{\tilde{A}_\alpha} = 1 \text{ if } x \in \tilde{A}_\alpha \text{ and } 1_{\tilde{A}_\alpha} = 0 \text{ if } x \notin \tilde{A}_\alpha.
\]

Note that the \( \alpha \)-level set of \( \tilde{A} \) is a crisp set.

Moreover, \( \tilde{a} \) is called a crisp number with value \( m \) if its membership function is

\[
\mu_{\tilde{a}}(x) = \begin{cases} 
1, & x = m; \\
0, & \text{otherwise}.
\end{cases}
\]

which is denoted by \( \tilde{a} = 1_m \). It is easy to see that

\[
(1)_\alpha^L = (1)_\alpha^U = \alpha \text{ for all } \alpha \in [0, 1].
\]

We can see that real numbers are regarded as crisp numbers.

In the following, we introduce the arithmetics of any two fuzzy numbers. Let \( \odot \) be a binary operator, \( \odot, \odot, \odot \text{ or } \odot \) correspond to the binary operator \( \circ : +, -, \times \text{ or } \odot \). The membership function of \( \tilde{a} \odot \tilde{b} \) is defined by

Figure 1. The triangular fuzzy number.
\[ \mu_{\alpha \circ b} = \sup_{(x,y) \in [a,b] \times [c,d]} \min \{ \mu_a(x), \mu_b(y) \}. \]

Let \( \circ \) be a binary operator \( \oplus \), \( \ominus \), \( \otimes \), or \( \odot \), between two closed intervals \([a, b]\) and \([c, d]\). Then \( [a,b] \oplus [c,d] = \{ z \in \mathbb{R} \mid z = x + y, \forall x \in [a,b], y \in [c,d] \} \).

**Proposition 2.3** [18]: Let \( a \) and \( b \) be two fuzzy numbers. Then \( a \oplus b, a \ominus b \) and \( a \odot b \) are also fuzzy numbers and their \( \alpha \)-level sets are also fuzzy numbers and their \( \alpha \)-level sets are

\[
\begin{align*}
(a \oplus b)_\alpha &= \bar{a}_\alpha \oplus \bar{b}_\alpha = \left[ a^l_\alpha + b^l_\alpha, a^u_\alpha + b^u_\alpha \right] \\
(a \ominus b)_\alpha &= \bar{a}_\alpha \ominus \bar{b}_\alpha = \left[ a^l_\alpha - b^u_\alpha, a^u_\alpha - b^l_\alpha \right] \\
(a \odot b)_\alpha &= \bar{a}_\alpha \odot \bar{b}_\alpha = \left\{ \begin{array}{ll}
\min \left\{ a^l_\alpha b^l_\alpha, a^l_\alpha b^u_\alpha, a^u_\alpha b^l_\alpha, a^u_\alpha b^u_\alpha \right\}, & \text{if } \alpha \notin [0,1] \\
\max \left\{ a^l_\alpha b^l_\alpha, a^l_\alpha b^u_\alpha, a^u_\alpha b^l_\alpha, a^u_\alpha b^u_\alpha \right\}, & \text{if } \alpha = 0,1
\end{array} \right.
\end{align*}
\]

for all \( \alpha \in [0,1] \). If \( b \) does not contain zero for all \( \alpha \in [0,1] \), then \( a \odot b \) is also a fuzzy number and its \( \alpha \)-level set is

\[
(a \odot b)_\alpha = \bar{a}_\alpha \odot \bar{b}_\alpha = \left\{ \begin{array}{ll}
\min \left\{ a^l_\alpha b^l_\alpha, a^l_\alpha b^u_\alpha, a^u_\alpha b^l_\alpha, a^u_\alpha b^u_\alpha \right\}, & \text{if } \alpha \notin [0,1] \\
\max \left\{ a^l_\alpha b^l_\alpha, a^l_\alpha b^u_\alpha, a^u_\alpha b^l_\alpha, a^u_\alpha b^u_\alpha \right\}, & \text{if } \alpha = 0,1
\end{array} \right.
\]

Let \( \mathcal{F} \) denote the set of all fuzzy subsets of \( \mathbb{R} \). Let \( f(x) \) be a non-fuzzy real-valued function from \( \mathbb{R} \) into \( \mathbb{R} \) and let \( \bar{A} \) be a fuzzy subset of \( \mathbb{R} \). According to the “Extension Principle” in Zadeh [13], the fuzzy-valued function \( \tilde{f}:\mathcal{F} \to \mathcal{F} \) can be induced by the non-fuzzy function \( f(x) \); that is, \( \tilde{f}(\bar{A}) \) is a fuzzy subset of \( \mathbb{R} \). The membership function of \( \tilde{f}(\bar{A}) \) is defined by

\[ \mu_{f(\bar{A})}(r) = \sup_{(x,y) \in f^{-1}(r)} \mu_{\bar{A}}(x) \]

**Proposition 2.4** [18]: Let \( f(x) \) be a real-valued function and let \( \bar{A} \) be a fuzzy subset of \( \mathbb{R} \). The function can induce a fuzzy-valued function \( \tilde{f}:\mathcal{F} \to \mathcal{F} \) via the extension principle. Suppose that the membership function \( \mu_{\bar{A}} \) of \( \bar{A} \) is upper semi-continuous and \( x : r = f(x) \) is a compact set for all \( r \), then the \( \alpha \)-level set of \( \tilde{f}(\bar{A}) \) is

\[
(\tilde{f}(\bar{A}))_\alpha = \{ f(x) : x \in \bar{A}_\alpha \}
\]

**Proposition 2.5** [18]: Let \( f(x_1, \ldots, x_n) \) be a continuous real-valued function defined on \( \mathbb{R}^n \) and \( \bar{a}_1, \ldots, \bar{a}_n \) be \( \alpha \) fuzzy numbers. Let \( \tilde{f}:\mathbb{R}^n \to \mathbb{R} \) be a fuzzy-valued function induced by \( f(x_1, \ldots, x_n) \) via the extension principle. Suppose that each \( \{(x_1, \ldots, x_n) : r = f(x_1, \ldots, x_n)\} \) is a compact subset of \( \mathbb{R}^n \) for \( r \) in the range of \( f \). Then \( \tilde{f}(\bar{a}_1, \ldots, \bar{a}_n) \) is a fuzzy number and its \( \alpha \)-level set is

\[
(\tilde{f}(\bar{a}_1, \ldots, \bar{a}_n))_\alpha = \{ f(x_1, \ldots, x_n) : x_i \in (\bar{a}_i)_\alpha, i = 1, \ldots, n \}
\]

3. The Jump-Diffusion Approach to Modeling Vulnerable Option Pricing Under Fuzzy Environments

Xu et al. [12] modeled the evolutions of the underlying stock price \( S_t \) and the value of the asset of the counterparty \( V_t \) as the following jump diffusion processes, respectively:

\[
dS_t / S_t = (r - \lambda J_t) dt + \sigma J_t dW^s_t + (\xi - 1) dN_t \tag{3}
\]

\[
dV_t / V_t = (r - \lambda J_t) dt + \sigma J_t dW^v_t + (\xi - 1) dN_t \tag{4}
\]

where \( W^s_t \) and \( W^v_t \) are standard Brownian motions and \( Cov(dW^s_t, dW^v_t) = \rho dt \), \( \rho \) denotes the risk-free rate, \( N_t \) is a Poisson process with rate \( \lambda \), \( \xi \) \((i \in \{s,v\})\) is the percentage jump size that is a sequence of log-normally, identically and independently distributed, and

\[
E(\ln \xi) = a, Var(\ln \xi) = \delta^2,
\]

\[
J_i = E(\xi^i - 1) = \exp\{a + \frac{1}{2} \delta^2\} - 1, i \in \{s,v\}
\]

Based on equations (3), (4), and Itô's formula, we have

\[
\ln S_T = \ln S_0 + (r - \frac{1}{2} \sigma^2 - \lambda J_s)(T-t) + \sigma J_t W^s_T + \sum_{i=1}^{N_t} \ln \xi_i^s
\]

\[
\ln V_T = \ln V_0 + (r - \frac{1}{2} \sigma^2 - \lambda J_v)(T-t) + \sigma J_t W^v_T + \sum_{i=1}^{N_t} \ln \xi_i^v
\]

Following the framework of Amin [11], partition the trading interval \([t; T]\) into \( n \) subintervals of length \( h = (T-t) / n \) \((n \) is a fixed positive integer). Suppose that the trading take place at the equidistant time points \( t_i = t + ih, i = 0, 1, \ldots, n, n(\xi = r - \frac{1}{2} \sigma^2 - \lambda J_s) \). The discrete time stochastic process defined on the following state space \((\mathbb{R})\) can be used to simulate the jump diffusion process (5) (see Figure 2). In equation (5), to the diffussion component, the stock price changes are modeled by moving up or down one tick, i.e., the stock
price in state \( j \) at date \( i \) moves to either state \( j+1 \) or state \( j-1 \) at date \( i+1 \). To the jump component, the stock price changes are modeled by multiple ticks on the state space grid at the next date (this does not include adjacent states).

\[
i = 0 \quad 1 \quad 2 \quad \cdots \quad n \quad j =
\]

\[t_0 = t \quad t_1 = t + h \quad t_2 = t + 2h \quad \cdots \quad T = t + nh\]

\[
\ln \frac{S_h}{S_{t+h}} = \zeta h + 2\sigma \sqrt{h} \quad \zeta h + 2\sigma \sqrt{h} \quad \zeta nh + 2n\sigma \sqrt{h} \quad 2n
\]

\[
\ln \frac{S_h}{S_{t+h}} = \zeta h + 2\sigma \sqrt{h} \quad \zeta h + 2\sigma \sqrt{h} \quad \zeta nh + 2n\sigma \sqrt{h} \quad 2n
\]

\[
\ln \frac{S_h}{S_{t+h}} = \zeta h + 1\sigma \sqrt{h} \quad \zeta h + 1\sigma \sqrt{h} \quad \zeta nh + 1n\sigma \sqrt{h} \quad 1n
\]

\[
\ln \frac{S_h}{S_{t+h}} = \zeta h + 2\sigma \sqrt{h} \quad \zeta h + 2\sigma \sqrt{h} \quad \zeta nh + 2n\sigma \sqrt{h} \quad 2n
\]

\[
\ln \frac{S_h}{S_{t+h}} = \zeta h - 1\sigma \sqrt{h} \quad \zeta h - 1\sigma \sqrt{h} \quad \zeta nh - 1n\sigma \sqrt{h} \quad -1n
\]

\[
\ln \frac{S_h}{S_{t+h}} = \zeta h - 2\sigma \sqrt{h} \quad \zeta h - 2\sigma \sqrt{h} \quad \zeta nh - 2n\sigma \sqrt{h} \quad -2n
\]

\[
\ln \frac{S_h}{S_{t+h}} = \zeta h - 2n\sigma \sqrt{h} \quad \zeta h - 2n\sigma \sqrt{h} \quad \zeta nh - 2n\sigma \sqrt{h} \quad -2n
\]

Figure 2. State space (\( \mathbb{R}^n \)) of the discrete approximation of the stock price distribution for a fixed discretisation parameter \( n \).

Define \( q \) as

\[
q = \frac{\exp(rh) - \lambda h E_{\xi^+}(\zeta^+)}{1 - \lambda h} = \frac{\exp(\zeta h - \sigma \sqrt{h}) - \exp(\zeta h - \sigma \sqrt{h})}{1 - \lambda h}
\]

Let \( q \) and \( 1 - q \) be the risk neutral probability of an uptick or downtick conditional on the diffusion component, respectively, and \( \lambda \) be the probability of multiple ticks on the jump component. The option value at date \( i \) can be denoted by

\[
C(i, j) = \exp(-rh) \left[ \lambda h E_{\xi^+}(C_{i+1, j}) + (1 - \lambda h)(qC(i+1, j+1) + (1-q)C(i+1, j-1)) \right]
\]

(7)

where \( E_{\xi^+}(C_{i+1, j}) \) is the expectation operator with respect to the distribution of \( \xi^+ \).

Once the probability measure \( Q \) is defined, according to its value at date \( i+1 \), the European option value at date \( i \) can be written as

\[
C_{i, j} = \exp(-rh)E_{Q_i}[\xi_{i+1, j}].
\]

(8)

Following the framework of Klein [6], the time \( t \) price of a vulnerable option can be written as

\[
C_t = \exp(-r(T-t))E_{Q_i}^T[(S_T - K)^+]
\]

(9)

\[
(1_{(t_1 \leq T)} + 1_{(t_2 \leq T)}) \frac{(1 - \theta)V_T}{D}. \]

By tower property, the pricing formula (8) can be written as

\[
C_t = \exp(-r(T-t))E_{Q_i}^T[(S_T - K)^+]\left[\left(1_{(t_1 \leq T)} + 1_{(t_2 \leq T)}\right) \frac{(1 - \theta)V_T}{D}\right].
\]

(10)

According to Liu and Liu [27], Xu et al. [12] got the value of the vulnerable option at time \( T \) and in state \( j \), denoted by

\[
C_{n, j} = (S_T \exp(\zeta nh + j\sigma \sqrt{h} - K)^+)
\]

\[
\left[\left(1_{(t_1 \leq T)} + 1_{(t_2 \leq T)}\right) \frac{(1 - \theta)V_T}{D}\right].
\]

(11)

where

\[
\phi_{n, j} = \Phi(-f_{n, j}), \psi_{n, j} = g_{n, j} \Phi(f_{n, j} - \sigma_T),
\]

\[
\rho_i = \text{Corr} \{ \sum_{t=1}^{N_i} \ln \xi_t, \sum_{i=1}^{N_i} \ln \xi_t \}
\]

\[
f_{n, i} = (\ln D - \eta_i)/(\sigma_T), g_{n, i} = \exp(\eta_{n, i} + 1/2 \sigma_i^2),
\]

\[
\eta_{n, i} = \eta_i + \frac{\rho \sigma_i \sigma_T \gamma}{(T-t + \lambda T(t-t)} \rho_i \delta_i \delta_T \exp(\ln S_{n, j} - \eta_i),
\]

\[
\sigma_i^2 = \sigma_i^2(T-t) + \lambda T(t-t)(\alpha_i^2 + \delta_T^2),
\]

\[
\lambda_i = \ln S_T + (r - 1/2 \sigma_i^2 - \lambda J_i)(T-t) + \lambda a_i(t-t),
\]

\[
\eta_i = \ln V_T + (r - 1/2 \sigma_i^2 - \lambda J_i)(T-t) + \lambda a_i(t-t),
\]

\[
S_{n, j} = S_T \exp(\zeta nh + j\sigma \sqrt{h}).
\]

As stated above, the risk-free rate, the volatility and the average jump intensity are assumed to be constants. In fact, there are so many factors to affect the financial market that it is hard to estimate these parameters precisely. As a consequence, it is more reasonable to assume that these parameters are fuzzy numbers. In the following, we get the jump-diffusion approach to price vulnerable option pricing under fuzzy environments, denoted as FJD model.

In the FJD model, because the risk-free rate, the volatility and the average jump intensity are fuzzy numbers, the option price is also a fuzzy number. Based on Proposition 2.2, we have

\[
\mu_{\alpha}(x) = \sup_{x \in \tilde{C}_a} \alpha \cdot \mu_{\alpha}(x)
\]

where \( \tilde{C}_a \) is the \( \alpha \)-level set of the fuzzy vulnerable option price \( \tilde{C} \). We display it as \( \tilde{C}_a = [\tilde{C}_a^L, \tilde{C}_a^U] \).

In the following, we derive the jump-diffusion approach to price vulnerable option pricing under fuzzy environments. From Proposition 2.2.2.5, we obtain the following equations:
and Proposition 2.3, we can denote as follows

$$(\eta_n)^V_a = \ln S_i + \left[ r^V_a - \frac{1}{2} (\sigma^V_a)^2 (\sigma^V_a)_{\alpha} - \lambda^V_{\alpha} J_s \right] (T-t)$$

$$(\eta_n)^V_a = \ln S_i + \left[ r^V_a - \frac{1}{2} (\sigma^V_a)^2 (\sigma^V_a)_{\alpha} - \lambda^V_{\alpha} J_s \right] (T-t)$$

$$(\eta_n)^V_a = \ln V_i + \left[ r^V_a - \frac{1}{2} (\sigma^V_a)^2 (\sigma^V_a)_{\alpha} - \lambda^V_{\alpha} J_s \right] (T-t)$$

$$(\eta_n)^V_a = \ln V_i + \left[ r^V_a - \frac{1}{2} (\sigma^V_a)^2 (\sigma^V_a)_{\alpha} - \lambda^V_{\alpha} J_s \right] (T-t)$$

According to Proposition 2.2, we have

$$(\psi_{n,j})^V_a = (g_{n,j})^V_a \Phi \left[ \ln D^* - (\eta^V_{n,j})_{\alpha} \right] = \left( \frac{\ln D^* - (\eta^V_{n,j})_{\alpha}}{(\sigma^V_{\alpha})_{\alpha}} \right)$$

$$(\psi_{n,j})^V_a = (g_{n,j})^V_a \Phi \left[ \ln D^* - (\eta^V_{n,j})_{\alpha} \right] = \left( \frac{\ln D^* - (\eta^V_{n,j})_{\alpha}}{(\sigma^V_{\alpha})_{\alpha}} \right)$$

where

$$(g_{n,j})^V_a = \exp \{ (\eta^V_{n,j})_{\alpha} + \frac{1}{2} (\sigma^V_{\alpha})_{\alpha} \},$$

$$(g_{n,j})^V_a = \exp \{ (\eta^V_{n,j})_{\alpha} + \frac{1}{2} (\sigma^V_{\alpha})_{\alpha} \}.$$

Combining of the foregoing discussion, we have

$$(C(n,j))^V_a = \left[ \frac{S(t) \exp (\xi^V_{n,j} h + j(\sigma^V_{\alpha})_{\alpha} \sqrt{h}) - K}{(\phi_{n,j})_{\alpha} + \frac{1 - \theta}{D} (\psi_{n,j})^V_a} \right]$$

$$(C(n,j))^V_a = \left[ \frac{S(t) \exp (\xi^V_{n,j} h + j(\sigma^V_{\alpha})_{\alpha} \sqrt{h}) - K}{(\phi_{n,j})_{\alpha} + \frac{1 - \theta}{D} (\psi_{n,j})^V_a} \right]$$

According to $\xi^V_{\alpha}$, $\xi^V_{\alpha}$ and Proposition 2.3, we can obtain the following equation

\[
\exp(r^V_{\alpha} h) \frac{\lambda^V_{\alpha} h E_{\xi^V_a} (\xi^V_{\alpha})}{\exp(\xi^V_{\alpha} h - (\sigma^V_{\alpha})_{\alpha} \sqrt{h})} - \exp(\xi^V_{\alpha} h - (\sigma^V_{\alpha})_{\alpha} \sqrt{h})
\]

\[
\exp(r^V_{\alpha} h) \frac{\lambda^V_{\alpha} h E_{\xi^V_a} (\xi^V_{\alpha})}{\exp(\xi^V_{\alpha} h + (\sigma^V_{\alpha})_{\alpha} \sqrt{h})} - \exp(\xi^V_{\alpha} h + (\sigma^V_{\alpha})_{\alpha} \sqrt{h})
\]

\[
\exp(r^V_{\alpha} h) \frac{\lambda^V_{\alpha} h E_{\xi^V_a} (\xi^V_{\alpha})}{\exp(\xi^V_{\alpha} h - (\sigma^V_{\alpha})_{\alpha} \sqrt{h})} - \exp(\xi^V_{\alpha} h - (\sigma^V_{\alpha})_{\alpha} \sqrt{h})
\]

\[
\exp(r^V_{\alpha} h) \frac{\lambda^V_{\alpha} h E_{\xi^V_a} (\xi^V_{\alpha})}{\exp(\xi^V_{\alpha} h + (\sigma^V_{\alpha})_{\alpha} \sqrt{h})} - \exp(\xi^V_{\alpha} h + (\sigma^V_{\alpha})_{\alpha} \sqrt{h})
\]

From the above analysis and equation (7), we have

\[
\left( \hat{C}(i,j) \right)^V_a = \exp(-r^V_{\alpha} h) \left( \lambda^V_{\alpha} h E_{\xi^V_a} \left[ C_{\xi^V_a} (i+1, j) \right] \right)
\]

\[
+ (1 - \lambda^V_{\alpha} h) q^V_{\alpha} \left( C(i+1, j+1) \right)^V_a
\]

\[
+ (1 - \lambda^V_{\alpha} h) q^V_{\alpha} \left( C(i+1, j+1) \right)^V_a
\]

\[
+ (1 - \lambda^V_{\alpha} h) q^V_{\alpha} \left( C(i+1, j-1) \right)^V_a
\]

\[
+ (1 - \lambda^V_{\alpha} h) q^V_{\alpha} \left( C(i+1, j-1) \right)^V_a
\]

Fulcher and Majlender [28] defined the crisp possibilistic mean value of a triangular fuzzy number \( \bar{a} \) with \( \alpha \)-level set \( \bar{a} = [\bar{a}_L, \bar{a}_R, \bar{a}_M] \) as

\[
M(\bar{a}) = \frac{1}{2} \left( \int_0^{1/2} 2a \bar{a}_L \, da + \int_0^{1/2} 2a \bar{a}_R \, da \right)
\]

\[
= \int_0^{1/2} a (\bar{a}_L + \bar{a}_R) \, da.
\]

The crisp possibilistic mean value of triangular fuzzy number \( \bar{a} = (a_L, a_R, a_M) \) can be expressed as:

\[
M(\bar{a}) = \frac{1}{3} (a_L + 4a_M + a_R).
\]
(11), replacing \( r, \sigma, \sigma_v \) and \( \lambda \) with \( M(\tilde{r}), M(\tilde{\sigma}), M(\tilde{\lambda}) \), we can obtain the crisp possibilistic mean jump-diffusion model to pricing European vulnerable call option, denoted as CPM-JD model.

### 4. Computational Method and Numerical Analysis

In the above pricing model, according to a belief degree \( \alpha \), we can obtain the corresponding price interval and the crisp possibilistic mean value of European vulnerable call options. In this section, we discuss how to compute the corresponding belief degree of a given European vulnerable call option price \( c \) of the fuzzy price \( \tilde{C}(i,j) \) at time \( t \). Wu [18] and Zhang [29] have transformed this problem into an optimization problem and have taken a different approach to solve this problem, respectively. In this paper, we consider this issue from the perspective of the equation. Since \( L(\alpha)=\tilde{C}(i,j)_\alpha \) increases with \( \alpha \) and \( U(\alpha)=(\tilde{C}(i,j))_\alpha \) decreases with \( \alpha \), we consider the following situations:

1. If \( L(1) \leq c \leq U(1) \), then the corresponding belief is 1.
2. If \( c \leq L(1) \), since \( c \leq L(1) \leq U(1) \leq U(0) \), then the corresponding belief degree is the solution of the equation \( L(\alpha) = c \).
3. If \( c \geq U(1) \), since \( L(0) \leq L(1) \leq U(1) \leq c \), then the corresponding belief degree is the solution of the equation \( U(\alpha) = c \).

Here, we use the secant method to solve the problem. For the second case, the algorithm is as follows:

**Step 1:** Let \( \varepsilon \) be the tolerance. Set \( x_0 \leftarrow 0 \) and \( x_1 \leftarrow 1 \).

**Step 2:** Find \( L(x_i) \). If \( |L(x_i) - c| < \varepsilon \), then the belief degree is \( x_i \), otherwise go to step 3.

**Step 3:** Set

\[
x_2 = x_1 - \frac{x_1 - x_0}{L(x_1) - L(x_0)} L(x_i), x_0 \leftarrow x_1, x_1 \leftarrow x_2,
\]

and go to step 2.

For the third case, the above algorithm is still applicable.

In the following, we present some numerical results for pricing European vulnerable call options. We use the FJD model with \( n=100 \) and \( n=200 \). For comparison, we use Klein’s vulnerable option pricing model with \( n=\infty \). In the numerical analysis, the calculations of the values of vulnerable options are based on the following basic parameters values:

\[
a_i = a_v = -0.03, \quad \delta_i = \delta_v = 0.2121,
\]

\[
\rho = \rho_J = 0.45, \quad \theta = 0.5, D = 10, D’ = 10
\]

\[
S(t) = 50, \quad V(t) = 10, \quad K = 40, 50, 60,
\]

\[
T-t = 0.25, 1.
\]

Other fuzzy parameters used in the FJD model are:

\[
\tilde{r} = (0.04, 0.045, 0.055), \quad \tilde{\sigma} = (0.2, 0.3, 0.5), \quad \tilde{\lambda} = (3.5, 4.5, 5.5).
\]

### Table 1. The price interval for different \( \alpha \) under the fuzzy model with \( T-t=1, K=40, n=100, \tilde{x} = (3.5, 4.5) \).

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>Interval for fuzzy price</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>[11.4282, 14.1449]</td>
</tr>
<tr>
<td>0.9</td>
<td>[12.0246, 13.3737]</td>
</tr>
<tr>
<td>0.93</td>
<td>[12.2158, 13.1589]</td>
</tr>
<tr>
<td>0.95</td>
<td>[12.3466, 13.0197]</td>
</tr>
<tr>
<td>0.98</td>
<td>[11.5477, 12.8168]</td>
</tr>
</tbody>
</table>

From a practical point of view, it is very important that the \( \alpha \)-level set for fuzzy price like in Table 1 may be seen as the interval of prices which has belief degree \( \alpha \) for the financial analysts. For \( \alpha = 0.93 \), it means that the vulnerable call option price will lie in the closed interval \([12.2158, 13.1589]\) with belief degree 0.93. In another way, if a decision-maker is comfortable with this belief degree 0.93, then he can pick any value from \([12.2158, 13.1589]\) as the vulnerable call option price for the later use.

For the purpose of comparison, we also give the price calculated by Klein’s vulnerable option pricing model, denoted as Klein, in the following tables. From Table 2 and Table 3, we can see that the option prices obtained from the CPM-JD model are a little higher than those obtained from the jump model in Xu et al. [12] under the same conditions. It is a very reasonable conclusion for us, because the uncertainty of the CPM-JD model is more than the jump model. Moreover, we can also see that the option prices obtained from both the CPM-JD model and jump model are clearly different from the prices obtained from the Klein model. It is consistent with the fact that the Klein model is based on the assumptions that both the value of the asset underlying the option and the value of the assets of the counter-party follow the log-normal distribution. From Table 2, we notice that the option prices obtained from both the CPM-JD model and jump model tend to become high for the average jump intensity. It is obvious for us, because the greater the average jump intensity, the more the uncertainty. This result also holds for Table 3. Note also that the option prices obtained from the above three models increase with the maturity \( T-t \).
Table 2. The numerical analysis for European vulnerable call option pricing under CPM-JD model (T = 0.25).

<table>
<thead>
<tr>
<th>K</th>
<th>$\sigma_x$</th>
<th>$\sigma_y$</th>
<th>Klein</th>
<th>n</th>
<th>$\lambda$</th>
<th>Jump model</th>
<th>CPM-JD</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>0.3</td>
<td>0.3</td>
<td>0.4736</td>
<td>100</td>
<td>4.5</td>
<td>1.9101</td>
<td>2.0063</td>
</tr>
<tr>
<td>60</td>
<td>0.3</td>
<td>0.3</td>
<td>0.4736</td>
<td>200</td>
<td>4.5</td>
<td>1.9133</td>
<td>2.0099</td>
</tr>
<tr>
<td>60</td>
<td>0.3</td>
<td>0.3</td>
<td>0.4736</td>
<td>100</td>
<td>4</td>
<td>1.7653</td>
<td>1.8641</td>
</tr>
<tr>
<td>60</td>
<td>0.3</td>
<td>0.3</td>
<td>0.4736</td>
<td>200</td>
<td>4</td>
<td>1.7686</td>
<td>1.8669</td>
</tr>
<tr>
<td>50</td>
<td>0.3</td>
<td>0.3</td>
<td>2.7853</td>
<td>100</td>
<td>4.5</td>
<td>4.4402</td>
<td>4.5569</td>
</tr>
<tr>
<td>50</td>
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<td>0.3</td>
<td>2.7853</td>
<td>200</td>
<td>4.5</td>
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</tr>
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<td>4.2809</td>
<td>4.4022</td>
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<td>0.3</td>
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<td>4</td>
<td>4.2831</td>
<td>4.4049</td>
</tr>
<tr>
<td>40</td>
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<td>0.3</td>
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<td>100</td>
<td>4.5</td>
<td>9.1858</td>
<td>9.2488</td>
</tr>
<tr>
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<td>0.3</td>
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<td>200</td>
<td>4.5</td>
<td>9.1878</td>
<td>9.2513</td>
</tr>
<tr>
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<td>0.3</td>
<td>8.4165</td>
<td>100</td>
<td>4</td>
<td>9.0987</td>
<td>9.1624</td>
</tr>
<tr>
<td>40</td>
<td>0.3</td>
<td>0.3</td>
<td>8.4165</td>
<td>200</td>
<td>4</td>
<td>9.1006</td>
<td>9.1647</td>
</tr>
</tbody>
</table>

Table 3. The numerical analysis for European vulnerable call option pricing under CPM-JD model (T = 1).

<table>
<thead>
<tr>
<th>K</th>
<th>$\sigma_x$</th>
<th>$\sigma_y$</th>
<th>Klein</th>
<th>n</th>
<th>$\lambda$</th>
<th>Jump model</th>
<th>CPM-JD</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>0.3</td>
<td>0.3</td>
<td>2.7853</td>
<td>100</td>
<td>4.5</td>
<td>6.5232</td>
<td>6.7305</td>
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<tr>
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<td>0.3</td>
<td>2.7853</td>
<td>200</td>
<td>4.5</td>
<td>6.5547</td>
<td>6.7653</td>
</tr>
<tr>
<td>60</td>
<td>0.3</td>
<td>0.3</td>
<td>2.7853</td>
<td>100</td>
<td>4</td>
<td>6.2097</td>
<td>6.4255</td>
</tr>
<tr>
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<td>0.3</td>
<td>2.7853</td>
<td>200</td>
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<td>6.2395</td>
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<td>0.3</td>
<td>5.9329</td>
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<td>9.1290</td>
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<tr>
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<td>4.5</td>
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<td>12.8300</td>
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<td>0.3</td>
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<td>4</td>
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<td>12.6108</td>
</tr>
<tr>
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<td>0.3</td>
<td>10.6185</td>
<td>200</td>
<td>4</td>
<td>12.4859</td>
<td>12.6359</td>
</tr>
</tbody>
</table>

5. Conclusions

Option pricing (e.g., the option pricing of European options, American options, and exotic options) is an essential research topic in mathematical finance. The main difficulty of this problem is to price them efficiently and accurately. In the classical sense, the problem of pricing options was considered in terms of a stochastic environment in a virtual world and the input parameters of these pricing models are usually regarded as precise real numbers. However, in the real world, due to the fluctuation of financial market from time to time, it is unreasonable to assume that the risk-free rate, the volatility and the average jump intensity are constants. In this paper, we present the fuzzy pattern of Xu et al. [12] vulnerable option pricing model and obtain the crisp possibilistic mean jump-diffusion model to pricing European vulnerable call options. The fuzzy jump-diffusion model to price vulnerable options and the CPM-JD model are more reasonable, and can instruct financial investors more efficiently in decision-making. However, all fuzzy numbers proposed in this paper are triangular fuzzy numbers. We will consider other forms of fuzzy numbers in the future research.

Table 4. Belief degrees $\alpha$ for European vulnerable call option prices $c$.

<table>
<thead>
<tr>
<th>$c$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.56</td>
<td>0.3971</td>
</tr>
<tr>
<td>10.37</td>
<td>0.5933</td>
</tr>
<tr>
<td>11.58</td>
<td>0.8268</td>
</tr>
<tr>
<td>12.29</td>
<td>0.9406</td>
</tr>
<tr>
<td>12.68</td>
<td>0.9989</td>
</tr>
<tr>
<td>12.82</td>
<td>0.9892</td>
</tr>
<tr>
<td>14.61</td>
<td>0.7457</td>
</tr>
<tr>
<td>15.74</td>
<td>0.6262</td>
</tr>
<tr>
<td>16.53</td>
<td>0.5522</td>
</tr>
</tbody>
</table>

Acknowledgments

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