Adaptive Fuzzy Stabilization for a Class of Pure-Feedback Systems with Unknown Dead-Zones

Jianjiang Yu

Abstract

This paper presents an adaptive fuzzy controller for a class of completely pure-feedback nonlinear systems. Takagi-Sugeno type fuzzy logic systems are used to approximate the unknown nonlinear functions. The controller synthesis is developed by the backstepping technique and the small gain approach. The closed-loop control system is proved to be semi-globally uniformly ultimately bounded (SGUUB) and the tracking error converges to a residual set. The simulation results show the effectiveness of the control scheme.

Keywords: Pure-feedback nonlinear system, fuzzy control, backstepping, small gain theorem, input-to-state stability (ISS).

1. Introduction

Based on universal function approximators, such as neural networks (NNs) or fuzzy logic systems, adaptive control for uncertain nonlinear systems has received much attention in recent years [1-7]. As a breakthrough in nonlinear control area, adaptive backstepping approach was presented to obtain global stability and asymptotic tracking for a large class of nonlinear system, mostly the strict-feedback system [8-14]. Nevertheless, relatively fewer results have been obtained for the pure-feedback systems. The pure-feedback system has a more representative form than the strict-feedback system, and there are many systems falling into this category. Several special cases of pure-feedback systems were studied in [15] and [16], but they were still affine in control $u$. In [17], the external disturbances or unmoulded dynamics in a class of pure-feedback systems is considered. The completely pure-feedback system was investigated in [18] via small gain theory. Nonsmooth nonlinear characteristics such as dead-zone, backlash, hysteresis are common in actuator and sensors such as mechanical connections, hydraulic actuators and electric servomotors. Dead-zone is one of the most important nonsmooth nonlinearities in many industrial processes. Its presence severely limits system performance, and its study has been attracting much interest for a long time [19, 20]. In [21], an adaptive neural controller was developed for a class of uncertain multi-input multi-output nonlinear state time-varying delay systems in triangular control structure with unknown nonlinear dead-zones and gain signs.

In this paper, the problem of adaptive fuzzy stabilization for the completely pure-feedback system with both uncertain parameters and disturbances is considered. The synthesis is developed by use of the input-to-state stability, the backstepping technique, and small gain approach. At last, a simulation is carried out to show the effectiveness of the control scheme.

This paper is organized as follows. The preliminaries and problem formulation are presented in Section 2 and section 3, respectively. In Section 4, a systematic procedure for the synthesis of the adaptive fuzzy tracking controller is developed. In Section 5, a simulation example is used to demonstrate the effectiveness of the proposed scheme. Finally, the conclusion is given in Section 6.

2. Preliminaries

2.1 ISS and Small Gain Theorem

We give some notions about ISS and small-gain theory in this section.

Definition 1 [22]: For the system $\dot{x} = f(x,u)$, it is said to be input-to-state practically stable (ISpS) if there exist a function $\gamma$ of class $K$, called the nonlinear $L_{\infty}$ gain, and a function $\beta$ of class $KL$ such that, for any initial condition $x(0)$, each measurable essentially bounded control $u(t)$ defined for all $t > 0$ and nonnegative constant $d$, the associated solutions $x(t)$ are defined on $[0, \infty)$ and satisfy:

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|u\|_\infty) + d$$

when $d = 0$ in (1), the ISpS property reduces to the input-to-state stability (ISS) property.

Definition 2 [22]: A $C_1$ function $V$ is said to be an
ISpS-Lyapunov function for the system \( \dot{x} = f(x,u) \) if there exist functions \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) of class \( K \) and a constant \( d > 0 \) such that

\[
\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \forall x \in \mathbb{R}^n
\]

\[
\frac{\partial V(x)}{\partial x} f(x,u) \leq -\alpha_3(|x|) + \alpha_4(|u|) + d
\]

When (3) holds with \( d = 0 \), \( V \) is referred to as an ISS-Lyapunov function.

**Lemma 1:** The system \( \dot{x} = f(x,u) \) is ISpS if and only if there exists an ISpS-Lyapunov function.

**Small-gain theorem:** Consider a system in composite feedback form

\[
\sum_{i=1}^{m} \dot{x}_i = f_i(x_i, \omega)
\]

\[
\dot{z} = H(x)
\]

\[
\sum_{i=1}^{m} \dot{y}_i = g_i(y_i, z)
\]

\[
\omega = K(Y, z, \bar{z})
\]

of two ISpS systems. In particular, there exist two constants \( d_i > 0, d > 0 \), and let \( \beta_\omega \) and \( \beta_\xi \) of class \( KL \), and \( \gamma_\omega \) and \( \gamma_\xi \) of class \( K \) be such that, for each \( \omega \) in the \( L_\infty \) supremum norm, each \( x \in \mathbb{R}^n \) and each \( y \in \mathbb{R}^n \), all the solutions \( X(\omega;\alpha_\omega,t) \) and \( Y(\gamma;\bar{z},t) \) are defined on \([0, \infty)\) and satisfy, for almost all \( t \geq 0 \):

\[
\|H(x;\omega,\alpha_\omega,t)\| \leq \beta_\omega (\|x\|_\omega) + \gamma_\omega (\alpha_\omega) + d_i
\]

\[
\|K(Y;\bar{z},\gamma_\omega,t)\| \leq \beta_\xi (\|\bar{z}\|_\omega) + \gamma_\xi (\gamma) + d_2
\]

under these conditions, if

\[
\gamma_\omega (\gamma_\omega(s)) < s \quad \text{(resp. } \gamma_\xi (\gamma_\xi(s)) < s, \forall s > 0)\]

then the solution of the composite system is ISpS [18].

### 2.2 T-S fuzzy systems

We briefly describe the structure of fuzzy systems in this section. Let \( R \) denote the real numbers, \( \mathbb{R}^n \) the real \( n \)-vectors, \( \mathbb{R}^{n \times m} \) the real \( n \times m \) matrices. Let \( S \) be a compact simply connected set in \( \mathbb{R}^p \). With map \( f : S \rightarrow \mathbb{R}^m \), define \( C^m(S) \) to be the function space such that \( f \) is continuous. A fuzzy system can be employed to approximate the function \( f(x) \) in order to design the adaptive fuzzy robust control law, thus the configuration of T-S type fuzzy logic system called T-S fuzzy system for short.

Consider a T-S fuzzy system to uniformly approximate a continuous multidimensional function \( y = f(x) \) that has a complicated formulation, where \( x \) is input vector with \( n \) independent \( x = (x_1, x_2, \cdots, x_n)^T \). The domain of \( x \) is \( \Theta = \theta_1 \times \theta_2 \times \cdots \theta_n = [a_i, b_i] \times [a_j, b_j] \times \cdots \times [a_n, b_n] \).

In order to construct a fuzzy system, the interval \([a_i, b_i] \) is divided into \( N_i \) subintervals

\[
a_i = C_{w_i} < C_i < \cdots < C_N_{w_i} = b_i, 1 \leq i \leq n.
\]

On each interval \( \theta_i (1 \leq i \leq n) \), \( N_i + 1 (N_i > 0) \) continuous input fuzzy sets, denoted by \( A_{ji}(0 \leq j \leq N_i) \), are defined to fuzzify \( x_i \). The membership function of \( A_{ji} \) is denoted by \( \mu_i^j(x_i) \), which can be represented by triangular, trapezoid, generalized bell or Gaussian type and so on.

Generally, T-S fuzzy system can be constructed by the following \( M(M > 1) \) fuzzy rules:

\[
R_i : \text{If } x_i \text{ is } A_{hi} \text{ AND } \cdots \text{ AND } x_n \text{ is } A_{hb},
\]

\[
\text{Then } y_i = a_{0i} + a_{1i} x_i + \cdots + a_{ni} x_n
\]

where \( A_{ji}(j = 0, 1, \cdots, n, i = 1, 2, \cdots, M \) are the unknown constants. The product fuzzy inference is employed to evaluate the ANDs in the fuzzy rules. After being defuzzified by a typical center average defuzzifier, the output of the fuzzy system is

\[
F(x) = \frac{\sum_{i=1}^{M} y_i \prod_{j=1}^{n} \mu_i^j(x_j)}{\sum_{i=1}^{M} \prod_{j=1}^{n} \mu_i^j(x_j)}
\]

which is called a fuzzy base function. So, restructuring (9) as follows:

\[
F(x) = \sum_{i=1}^{M} y_i \xi_i(x).
\]

Let \( \xi_i(x) = [\xi_i^1(x), \cdots, \xi_i^M(x)]^T \), \( Z = [1, x_1, x_2, \cdots, x_n]^T \),

\[
A_z = \begin{bmatrix}
    a_{01} & a_{11} & \cdots & a_{n1} \\
    a_{02} & a_{12} & \cdots & a_{n2} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{0M} & a_{1M} & \cdots & a_{nM}
\end{bmatrix}
\]

then (9) can be rewritten as

\[
F(x) = \xi(x) A_z = \xi(x) A_z^T + \xi(x) A_x,
\]

where \( x = [x_1, x_2, \cdots, x_n]^T \), \( A_z^T = [a_{01}, a_{12}, \cdots, a_{nM}]^T \), and

\[
A_x = \begin{bmatrix}
    a_{01} & a_{11} & \cdots & a_{n1} \\
    a_{02} & a_{12} & \cdots & a_{n2} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{0M} & a_{1M} & \cdots & a_{nM}
\end{bmatrix}
\]
Lemma 2 [18]: Suppose that the input universal of discourse $U$ is a compact set in $R^r$. Then for any given real continuous function $f(x)$ on $U$ and $\forall \in > 0$, there exists a fuzzy system $F(x)$ in the form of (12) express such that

$$\sup_{x \in U} \| f(x) - F(x) \| = \sup_{x \in U} \| f(x) - \xi(x) A_z \| \leq \varepsilon. \quad (13)$$

3. Problem

Consider the following pure-feedback nonlinear system:

Plant:

$$\begin{align*}
\dot{x}_i &= f_i(\overline{x}, x_{i+1}, \xi) + d_i(t) \quad 1 \leq i \leq n-1 \\
\dot{\bar{x}}_n &= f_n(x, u, \xi) + d_n(t) \\
y &= x_i
\end{align*} \quad (14)$$

Dead-zone:

$$u = D(v) = \begin{cases} g_i(v) & \text{if } v \geq b_i, \\
0 & \text{if } b_i < v < b_i, \\
g_i(v) & \text{if } v \leq b_i, \end{cases} \quad (15)$$

where $\overline{x} = (x_1, \cdots, x_T)^T, i=1, \cdots, n$, $x = \overline{x} \in R^n$, $y \in R$ are state variables and output, respectively, $\xi \in W \subset R^q$ is an $q$-dimension of parameter uncertain vector, $W$ is a compact set. $v(t) \in R$ is the input to the dead-zone, $b_i$ and $b_i$ are the unknown parameters of the dead-zone. $f_i(\overline{x}, \xi), f_{n+1}(x, u, \xi)$ with $f_i(0, \xi) = 0, i = 1, \cdots, n$ are unknown smooth functions. $d_i(t)$ is continuous disturbance which belongs to $L_2[0, T]$.

As in [20, 21], the unknown dead-zone input is satisfying the following assumptions:

Assumption 1: The dead-zone output, $u$, is not available. The dead-zone parameters, $b_i$ and $b_i$, are unknown bounded constants, but their signs are known, i.e., $b_i > 0$ and $b_i < 0$.

Assumption 2: The functions, $g_i(v)$ and $g_i(v)$, are smooth, and there exist unknown positive constants, $k_{10}, k_{11}, k_{10}$, and $k_{11}$ such that

$$\begin{align*}
0 \leq k_{10} \leq g_i(v) & \leq k_{11}, \quad \forall v \in (-\infty, b_i] \quad (16) \\
0 \leq k_{10} \leq g_i(v) & \leq k_{11}, \quad \forall v \in [b_i, +\infty) \quad (17)
\end{align*}$$

and $b_i \leq \min \{k_{10}, k_{11}\}$ is a known positive constant, where $g_i(v) = \frac{dg_i(\xi)}{dz} \big|_{z=\xi}$. $g_i(v) = \frac{dg_i(\xi)}{dz} \big|_{z=\xi}$.

Based on Assumption 2, the dead-zone (15) can be rewritten as follows as shown in [15]:

$$u = D(v) = K^T(t) \Phi(t) v + d(v), \quad (18)$$

where $|d(v)| \leq p^*_v$, $p^*_v$ is an unknown positive constant with $p^*_v = (k_{11} + k_{11}) \max \{b_i, -b_i\}$, and

$$\Phi(t) = \begin{bmatrix} \varphi_1(t) \\ \varphi_1(t) \end{bmatrix}, \quad (19)$$

$$K(t) = \begin{bmatrix} g_1(\xi(v)) & g_2(\xi(v)) \\ g_2(\xi(v)) & g_3(\xi(v)) \end{bmatrix}^T, \quad (20)$$

$$\varphi_i(t) = \begin{cases} 1 & \text{if } v(t) > b_i, \\
0 & \text{if } v(t) \leq b_i, \\
1 & \text{if } v(t) < b_i, \\
0 & \text{if } v(t) \geq b_i, \end{cases} \quad (21)$$

$$d(v) = \begin{cases} -g_1(\xi(v)) b_i & \text{if } v \geq b_i, \\
-g_2(\xi(v)) b_i & \text{if } v < b_i, \end{cases} \quad (23)$$

$$dg z = g_1(\xi(v)) \xi(v), \quad (24)$$

The control objective is to design an adaptive fuzzy controller for the system (14) such that: all the signals in the closed-loop remain semi-globally uniformly ultimately bounded; the output $y$ can be rendered small enough.

Define $g_i(\overline{x}, x_{i+1}, \xi) = \frac{\partial f_i(\overline{x}, x_{i+1}, \xi)}{\partial x_{i+1}}$, $g_n(x, u, \xi) = \frac{\partial f_n(x, u, \xi)}{\partial u}$, $i = 1, \cdots, n-1$.

Assumption 3: There exist positive constants $g_{\alpha}$ such that $0 < g_{\alpha} \leq g_i(\cdot)$.

4. The Design and Analysis of the Robust Control

Use backstepping to design an adaptive fuzzy controller.

Step 1: Define $z_i = x_1, z_2 = x_2 - \alpha_i$. Using the mean value theorem, the derivation of $z_i$ is

$$\begin{align*}
\dot{z}_i &= f_i(x_1, x_2, \xi) + d_i(t) \\
&= f_i(x_1, 0, \xi) + g_2(x_2, d_i(t)) \\
&= f_i(x_1, 0, \xi) + g_2 x_2 + d_i(t)
\end{align*} \quad (24)$$

where $g_i(x_1, x_2, \xi) = \lambda_i x_2, 0 \leq \lambda_i \leq 1$ defined as $g_i$ for brevity. By employing T-S fuzzy system to approximate $f_i(x_1, \xi)$. Then $f_i(x_1, \xi)$ can be expressed as

$$f_i(x_1, \xi) = \varpi_1 x_1 + e_i = C_{\varpi} \varpi_1 x_1 + e_i$$

where $\varpi_1 = A_{\varpi} x_1 = \| A_{\varpi} \|, A_{\varpi} = C_{\varpi}^T A_1, \| A_{\varpi} \| \leq 1$.

Then

$$\begin{align*}
\dot{z}_i &= g_2 x_2 + C_{\varpi} \varpi_1 x_1 + v_i \\
&= e_i + d_i(t)
\end{align*} \quad (25)$$

where $v_i = e_i + d_i(t)$. Consider the following Lyapunov function candidate [9]

$$V_i = \frac{\dot{z}_i^2}{2} + \frac{1}{2} g_{\alpha} \gamma_i \dot{\theta}^2 \quad (26)$$

where $g_{\alpha} = \min \{g_{\alpha}, g_{\alpha}, \cdots, g_{\alpha} \}$, $\dot{\theta} = \theta - \dot{\theta}$. $\dot{\theta}$ is the
estimate of \( \theta = g_0^{-1}C_{\gamma}^2 \), \( C_{\gamma} = \max(C_{\alpha1}, C_{\alpha2}, \ldots, C_{\alpha_n}) \), and \( C_{\gamma_i}, i = 2, \ldots, n \) will be given in the following. The time derivative of \( V_1 \) is

\[
\dot{V}_1 = z_1(z_1 - g_0\Gamma^{-1}\hat{\theta}) \\
= z_1(g_4(z_1 + \alpha_1) + C_{\alpha1}(x_i)\alpha_1 + \xi_1) - g_0\Gamma^{-1}\hat{\theta}
\]

Let \( \gamma > 0, \rho > 0 \), we have

\[
C_{\alpha1}(x_i)\alpha_1z_1 = C_0z_1^2 + \gamma_1^2\omega_1^2\alpha_1 + \gamma_1^2\omega_1^2\alpha_1 \\
\leq \frac{C_0z_1^2}{4\rho^2}z_1 + \gamma_1^2\omega_1^2\alpha_1 \\
\leq \frac{g_0}{4\rho^2}z_1^2 + \gamma_1^2\omega_1^2\alpha_1 + \frac{g_0}{4\rho^2}z_1^2
\]

\[
\dot{V}_1 = z_1(z_1 - g_0\Gamma^{-1}\hat{\theta}) = z_1(g_4(z_1 + \alpha_1) + C_{\alpha1}(x_i)\alpha_1 + \xi_1) - g_0\Gamma^{-1}\hat{\theta}
\]

Then

\[
\dot{V}_1 = z_1(g_4(z_1 + \alpha_1) + \frac{g_0}{4\rho^2}z_1 + \frac{g_0}{2\rho^2}z_1)
\]

Define virtual control \( \alpha_1 \) as

\[
\alpha_1 = -k_1z_1 - \frac{\hat{\theta}}{4\rho^2}z_1 + \frac{1}{2\rho^2}z_1
\]

with design constant \( k_1 > 0 \), it is easy to get

\[
g_4 \alpha_1 z_1 = g_4(-k_1z_1 - \frac{\hat{\theta}}{4\rho^2}z_1 + \frac{1}{2\rho^2}z_1) \\
\leq g_0(-k_1z_1 - \frac{\hat{\theta}}{4\rho^2}z_1 + \frac{1}{2\rho^2}z_1)
\]

Then

\[
\dot{V}_1 \leq g_4\alpha_1 z_1 z_1 - g_0k_1z_1^2 + \gamma_1^2\omega_1^2\alpha_1 + \frac{g_0}{2\rho^2}z_1 + \frac{g_0}{2\rho^2}z_1
\]

\[
\dot{V}_1 \leq g_4\alpha_1 z_1 z_1 - g_0k_1z_1^2 + \gamma_1^2\omega_1^2\alpha_1 + \frac{g_0}{2\rho^2}z_1 + \frac{g_0}{2\rho^2}z_1
\]

Step 2: The derivation of \( \dot{z}_2 \) is \( \dot{z}_2 = f_2(x_3, \xi) + d_1(t) \), where \( \alpha_1 \) is

\[
\alpha_1 = \frac{\partial \alpha_1}{\partial x_1}x_1 + \frac{\partial \alpha_1}{\partial \theta} \dot{\theta}
\]

Define \( z_3 = x_3 - \alpha_1 \), then the derivation of \( \dot{z}_2 \) is

\[
\dot{z}_2 = f_2(x_1, x_2, x_3, \xi) + d_1(t) + \frac{\partial \alpha_1}{\partial x_1}d_1(t)
\]

\[
\dot{V}_1 = \frac{\partial \alpha_1}{\partial x_1}f_1(x_1, x_2, \xi) + g_4 z_1 + \frac{\partial \alpha_1}{\partial \theta} \dot{\theta}
\]

\[
+ \frac{g_0}{4\rho^2}z_1^2 z_1 - g_4 z_1 + \frac{g_0}{4\rho^2}z_1^2 \\
- \frac{g_0}{4\rho^2}z_1^2 z_1 - \frac{g_0}{4\rho^2}z_1^2 \\
\]

\[
\dot{V}_2 = \frac{-\partial \alpha_1}{\partial x_1}f_1(x_1, x_2, \xi) + \frac{g_0}{4\rho^2}z_1^2 z_1 - g_4 z_1 + \frac{g_0}{4\rho^2}z_1^2 \\
+ \frac{g_0}{4\rho^2}z_1^2 z_1 - \frac{g_0}{4\rho^2}z_1^2
\]

Define \( \bar{f}_2(z_1, z_2) = f_2(x_1, x_2, z_1, \xi) - \frac{\partial \alpha_1}{\partial x_1}f_1(x_1, x_2, \xi) + g_4 z_1 \\
+ \frac{g_0}{4\rho^2}z_1^2 z_1 - \frac{g_0}{4\rho^2}z_1^2 \)

Then

\[
\dot{V}_2 = \frac{g_0}{4\rho^2}z_1^2 z_1 - g_4 z_1 + \frac{g_0}{4\rho^2}z_1^2 \\
+ \frac{g_0}{4\rho^2}z_1^2 z_1 - \frac{g_0}{4\rho^2}z_1^2
\]

where \( \tau_2 \) will be given in the following, but we know that \( \tau_2 \) is a continuous function of variables \( z_1 \) and \( z_2 \). We also use T-S fuzzy systems to approximate the unknown function \( \bar{f}_2(z_1, z_2) \) as

\[
\bar{f}_2(z_1, z_2) = \xi_2(z_1, z_2)A_2^Tz_2 + \epsilon_2 = C_{\alpha1}z_1 + \epsilon_2,
\]

where \( \epsilon_2 = A_2^Tz_2 \) and \( C_{\alpha2} = A_2^T \), and \( A_2'' \leq 1 \). Then

\[
\dot{z}_2 = g_4z_1 + g_4z_1 + C_{\alpha1}z_2 + \epsilon_2
\]

\[
\dot{z}_2 = g_4z_1 + g_4z_1 + C_{\alpha1}z_2 + \epsilon_2
\]

Consider the following Lyapunov function candidate

\[
V_2 = V_1 + \frac{1}{2}z_1^2
\]

The time derivative of \( V_2 \) is

\[
\dot{V}_2 \leq g_4z_1^2z_1 - g_0k_1z_1^2 + \gamma_1^2\omega_1^2\alpha_1 + \frac{g_0}{2\rho^2}z_1 + \frac{g_0}{2\rho^2}z_1
\]

Then

\[
\dot{V}_2 \leq g_4z_1^2z_1 - g_0k_1z_1^2 + \gamma_1^2\omega_1^2\alpha_1 + \frac{g_0}{2\rho^2}z_1 + \frac{g_0}{2\rho^2}z_1
\]

It is also easy to get

\[
C_{\alpha2}z_2 z_2 \leq g_0\frac{\partial \alpha_1}{\partial x_1}z_1 + \frac{g_0}{4\rho^2}z_1^2 z_1 + \frac{g_0}{4\rho^2}z_1^2 z_1
\]

\[
\dot{V}_2 \leq g_0\frac{\partial \alpha_1}{\partial x_1}z_1 + \frac{g_0}{4\rho^2}z_1^2 z_1 + \frac{g_0}{4\rho^2}z_1^2 z_1
\]

\[
\dot{V}_2 \leq g_0\frac{\partial \alpha_1}{\partial x_1}z_1 + \frac{g_0}{4\rho^2}z_1^2 z_1 + \frac{g_0}{4\rho^2}z_1^2 z_1
\]

So,

\[
\dot{V}_2 \leq g_0\frac{\partial \alpha_1}{\partial x_1}z_1 + \frac{g_0}{4\rho^2}z_1^2 z_1 + \frac{g_0}{4\rho^2}z_1^2 z_1
\]

Then

\[
\dot{V}_2 \leq g_0\frac{\partial \alpha_1}{\partial x_1}z_1 + \frac{g_0}{4\rho^2}z_1^2 z_1 + \frac{g_0}{4\rho^2}z_1^2 z_1
\]

Define \( z_3 = x_3 - \alpha_2 \), then the derivation of \( \dot{z}_2 \) is

\[
\dot{z}_2 = f_2(x_1, x_2, x_3, \xi) + d_1(t) + \frac{\partial \alpha_1}{\partial x_1}d_1(t)
\]
where \( \psi_2 = \frac{1}{4\gamma^2} e_j^T \xi_j z_j \), \( r_2 = r_1 + \gamma \psi_2 z_2 \), \( \delta_2 = \delta_1 + \frac{g_0^1 \rho^2 (e_i^2 + d_i^1 + d_i^2)}{2} \).

Select virtual control \( \alpha_2 \) as
\[
\alpha_2 = -k_2 z_2 - \frac{\dot{\theta}}{4\gamma^2} e_j^T \xi_j z_j - \frac{1}{2\rho^2} z_2^2 ,
\]
then,
\[
\dot{V}_2 \leq g_{\lambda} z_2 z_2 - g_0 \sum_{j=1}^N k_j z_j^2 + \sum_{j=1}^N \gamma^2 \omega_j^T \omega_j + \frac{g_0}{4\rho^2} \frac{\partial \alpha_2}{\partial \theta} \frac{\partial \alpha_2}{\partial \theta} + \delta_2 \] (37)
Step \( k \) \( (3 \leq k \leq n - 1) \): A similar procedure is employed recursively for each step \( k \). Define \( z_k = x_k - \alpha_{k-1} \), its derivation is \( \dot{z}_k = f_k (\tau_{x_{k-1}}, z_{k-1}) + d_k (t) - \alpha_{k-1} \). Where \( \alpha_{k-1} \) is
\[
\dot{\alpha}_{k-1} = \frac{\hat{\gamma}}{\alpha_{j-1}} \frac{\partial \alpha_{k-1}}{\partial \theta} (f_k (\tau_{x_{k-1}}, z_{k-1}) + d_k (t)) + \sum_{j=1}^N \frac{\partial \alpha_{k-1}}{\partial \theta} \omega_j
\]
then,
\[
\dot{z}_k = f_k (\tau_{x_{k-1}}, z_{k-1}) + d_k (t) - \alpha_{k-1}
\]
\[
= f_k (\tau_{x_{k-1}}, z_{k-1}) - \frac{\partial \alpha_{k-1}}{\partial \theta} \tau_k - g_0 \sum_{j=1}^N k_j z_j^2 - \frac{g_0}{4\rho^2} \frac{\partial \alpha_{k-1}}{\partial \theta} \frac{\partial \alpha_{k-1}}{\partial \theta} z_k
\]
\[
- \sum_{j=1}^N \frac{\partial \alpha_{k-1}}{\partial \theta} \Gamma \psi_j + \sum_{j=1}^N \frac{g_0}{4\rho^2} \frac{\partial \alpha_{k-1}}{\partial \theta} \frac{\partial \alpha_{k-1}}{\partial \theta} \psi_j
\]
\[
+ \sum_{j=1}^N \frac{\partial \alpha_{k-1}}{\partial \theta} \Gamma \psi_k + \sum_{j=1}^N \frac{\partial \alpha_{k-1}}{\partial \theta} \psi_k + \frac{\partial \alpha_{k-1}}{\partial \theta} d_k (t) + g_0 \frac{\partial \alpha_{k-1}}{\partial \theta} \frac{\partial \alpha_{k-1}}{\partial \theta} \psi_k
\]
where \( \tau_k, \psi_k \) will be given in the following, but we know that \( \tau_k, \psi_k \) is a continuous function of variables \( \tau_k \). We also use T-S fuzzy systems to approximate the unknown function \( f_k (\tau_{x_{k-1}}) \)
\[
\tilde{f}_k (\tau_{x_{k-1}}) = \xi_k \Gamma_k + \epsilon_k = C_{\phi_k} \xi_k \omega_k + \epsilon_k
\]
where \( \omega_k = A_\omega^T \xi_k + C_{\phi_k} = \| A_k \|, \ A_\omega^T = C_{\phi_k} A_k \), and \( \| A_\omega \| \leq 1 \). Then
\[
\tilde{f}_k (\tau_{x_{k-1}}) = \xi_k \Gamma_k + \epsilon_k = C_{\phi_k} \xi_k \omega_k + \epsilon_k
\]
where \( \omega_k = A_\omega^T \xi_k + C_{\phi_k} = \| A_k \|, \ A_\omega^T = C_{\phi_k} A_k \), and \( \| A_\omega \| \leq 1 \). Then
\[
\dot{z}_k = g_{\lambda} (z_k + \alpha_k) - g_{\lambda} z_k - \sum_{j=1}^N \frac{g_0}{4\rho^2} \frac{\partial \alpha_{k-1}}{\partial \theta} \frac{\partial \alpha_{k-1}}{\partial \theta} z_k
\]
\[
+ \frac{\partial \alpha_{k-1}}{\partial \theta} (\tau_k - \dot{\theta}) + \sum_{j=1}^N \frac{\partial \alpha_{k-1}}{\partial \theta} \Gamma \psi_k + \epsilon_k
\]
where \( \psi_k = \epsilon_k + d_k (t) - \sum_{j=1}^N \frac{\partial \alpha_{k-1}}{\partial \theta} d_j (t) \).

We choose the Lyapunov candidate functions and the virtual control laws as follows,
\[
V_k = V_{k-1} + \frac{1}{2} z_k^2
\]
\[
\alpha_k = -k_1 z_k - \frac{\dot{\theta}}{4\gamma^2} e_j^T \xi_j z_j - \frac{1}{2\rho^2} z_k^2
\]
and we get
\[
\dot{V}_k \leq g_{\lambda} z_k z_k - \sum_{j=1}^N k_j z_j^2 + \sum_{j=1}^N \gamma \omega_j^T \omega_j
\]
\[
+ (g_0 \Gamma \psi_k) \frac{\partial \alpha_{k-1}}{\partial \theta} \frac{\partial \alpha_{k-1}}{\partial \theta} \psi_k - \frac{\partial \epsilon_k}{\partial \theta} \frac{\partial \epsilon_k}{\partial \theta} \psi_k + \sum_{j=1}^N \frac{\partial \alpha_{k-1}}{\partial \theta} \psi_k + \sum_{j=1}^N \frac{\partial \alpha_{k-1}}{\partial \theta} \psi_k + \sum_{j=1}^N \frac{\partial \alpha_{k-1}}{\partial \theta} \psi_k
\]
where \( \psi_k = \frac{1}{4\gamma^2} e_j^T \xi_j z_j \), \( \tau_k = \tau_{k-1} + \Gamma \psi_k z_k \), \( \delta_k = \delta_{k-1} + g_{\lambda} \rho^2 (e_i^2 + \sum_{j=1}^N d_j^2) \).

Step \( n \) : The same as above steps, at the last step, picking he control law \( u = \alpha_n \) and the adaptive law \( \dot{\theta} = \tau_n \), we get
\[
\dot{V}_n \leq -g_0 \sum_{j=1}^N k_j z_j^2 + \sum_{j=1}^N \gamma \omega_j^T \omega_j + \delta_n
\]
\[
\leq -g_0 \sum_{j=1}^N k_j z_j^2 + \gamma \| \omega \|^2 + \delta_n
\]
where \( \omega = [\omega_1, \omega_2, \ldots, \omega_n]^T \), \( \delta_n = \delta_{n-1} + g_{\lambda} \rho^2 (e_i^2 + \sum_{j=1}^N d_j^2) \).

Define \( \omega^T (t) = [\epsilon_1, \ldots, \epsilon_n, d_1 (t), \ldots, d_n (t)] \). It is easy to get \( \delta_n \leq n g_{\lambda} \rho^2 \| \omega \|^2 \).

The main result on the asymptotic stability of the closed-loop system is summarized in the following theorem.

**Theorem:** Consider the pure-feedback nonlinear system (14) with the control law defined by (45)
\[
u = -k_1 z_n - \frac{\dot{\theta}}{4\gamma^2} e_j^T \xi_j z_n - \frac{1}{2\rho^2} z_n
\]
and the adaptive laws defined by (46)
\[
\dot{\theta} = \tau_n = \frac{1}{4\gamma^2} e_j^T \xi_j z_n
\]
If we pick \( \gamma < 1 \) and \( k_1 > \frac{1}{g_0} \), then it can make all the solutions \( z(t), \tau(t) \) of the derived
closed-loop system uniformly ultimately bounded.

**Proof:** In order to use small-gain theorem, we construct a system in composite feedback form with $\Sigma_{st0}$-subsystem and $\Sigma_{st1}$-subsystem.

**$\Sigma_{st0}$-subsystem:**

$$
\begin{align*}
\dot{z}_1 &= g_{st0}(\alpha, z_1) + C_{st01} \xi_1 \omega + v_1 \\
\dot{z}_2 &= g_{st0}(\alpha, z_1) - g_{st01} + C_{st02} \xi_2 \omega_2 \\
&\quad - \frac{g_0}{4\rho^2} \left( \frac{\partial \alpha}{\partial z} \right)^2 z_2 + \frac{\partial \alpha}{\partial \hat{\theta}} (\tau_2 - \hat{\theta}) + v_2 \\
\vdots \\
\dot{z}_n &= g_{st0} u_n - g_{st01} z_{n-1} + C_{st02} \xi_2 \omega - \\
&\quad - \frac{g_0}{4\rho^2} \sum_{j=1}^{n-1} \left( \frac{\partial \alpha}{\partial z} \right)^2 z_n + \frac{\partial \alpha}{\partial \hat{\theta}} (\tau_n - \hat{\theta}) \\
\dot{z} &= H(z) = z
\end{align*}
$$

where $\omega = [\omega_1, \omega_2, \ldots, \omega_n]^T$ is considered as the virtual input and $\dot{z}$ as the output. Then

$$
\dot{V}_n \leq -z^T + \gamma^2 \| \omega \|^2 + \delta_n.
$$

By definition, we propose the robust adaptive fuzzy tracking controller such that the requirement of ISpS for system $\Sigma_{st0}$ can be satisfied with the functions $\alpha_i(s) = s^2$ and $\alpha_i = \gamma^2 s^2$ of class $K\infty$, then we get a gain function $\gamma_i(s)$ of $\Sigma_{st1}$-subsystem

$$
\gamma_i(s) = \alpha_i^{-1} \alpha_i^0 \alpha_i^{-1} \alpha_i,
$$

where $\alpha_i(z) \leq V_i(z) \leq \alpha_i^0(z)$.

For $\Sigma_{st1}$-subsystem

$$
\begin{align*}
\omega_1 &= A^{\omega}_{n1} z_1 \\
\omega_2 &= A^{\omega}_{n2} [z_1, z_2]^T = A^{\omega}_{n3} z_2 \\
\vdots \\
\omega_n &= A^{\omega}_{n3} [z_1, z_2, \ldots, z_n]^T = A^{\omega}_{n4} z_n
\end{align*}
$$

It is also be re-written as

$$
\omega = A z
$$

and obtain

$$
\| \omega \| \leq \| A \| \| z \| = \gamma' \| z \|.
$$

Then the gain function $\gamma_i$ for system $\Sigma_{st1}$ is $\gamma_i(s) = \gamma' s$. Since $\gamma' = \| A \| \leq 1$, by choosing $\gamma < 1$, we can get $\gamma_i(\gamma_i(s)) < s$, it also means the composite closed-loop system has bounded solutions over $[0, \infty)$. More precisely, there exists a class $KL$-function $\beta$ and a positive constant $\delta^*$ such that

$$
\| z(t), \hat{\theta}(t) \| \leq \beta(\| z(0), \hat{\theta}(0) \|, t) + \delta^*
$$

where

$$
\hat{\theta} = [\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m]^T.
$$

This, in turn, implies that the tracking error vector $z(t)$ is bounded over $[0, \infty)$. So, there exists an ISpS-Lyapunov function for the composite closed-loop system, and the ISpS-Lyapunov function is satisfied as follows

$$
\dot{V}_n \leq -z^T Q_n z + \gamma^2 \| z \|^2 + \delta^*_n
$$

$$
\leq -z^T Q_n z + \gamma^2 \| z \|^2
$$

where $Q_n = \text{diag}([g_0 k_1 -1, g_0 k_2 -1, \ldots, g_0 k_n -1]$). From (53), we obtain $H_n$ performance,

$$
\int_0^t \| z(t) \|^2 \leq n\lambda_n^{-1}(Q_n) \int_0^t \| \omega(t) \|^2 dt + n\lambda_n^{-1}(Q_n) V(0)
$$

This completes the proof.

5. Computer Simulation

Consider the following perturbed pure-feedback nonlinear systems:

$$
\begin{align*}
\dot{x}_1 &= \sin(x_1) + x_2 + \frac{x_1^3}{5} + 6 \sin(t) \\
\dot{x}_2 &= x_1 x_2 + u + \frac{u^3}{7} - 0.5 \sin^3(t) \\
y &= x_1
\end{align*}
$$

We choose the control law

$$
\begin{align*}
u &= -k_1 z_1 - \frac{\hat{\theta}^T}{4\gamma'} [g_1 \hat{g}_1 z_1, \hat{g}_2, \ldots]^T - \frac{1}{2\rho^2} z_1 \\
u &= -k_2 z_2 - \frac{\hat{\theta}^T}{4\gamma'} [g_2 \hat{g}_2 z_2, \hat{g}_2, \ldots]^T - \frac{1}{2\rho^2} z_2
\end{align*}
$$

where $z_1 = y - y_d - x_1 - \gamma', z_2 = x_2 - \alpha_i, \gamma = 0.5, \rho = 0.6$.

The adaptive law

$$
\dot{\hat{\theta}} = r_2 = \Gamma \sum_{i=1}^3 [g_i \hat{g}_i z_i^2]
$$

where $\Gamma = 20$. The simulation results are shown in Fig. 1 and Fig. 2. From the observation of the simulation results, the adaptive fuzzy controller guarantees the stabilization performance.

6. Conclusion

In this paper, the stabilization problem has been studied for a class of perturbed completely pure-feedback uncertain nonlinear systems. We use Takagi-Sugeno type fuzzy logic systems to approximate
uncertain functions. A stable adaptive fuzzy control algorithm with $H_{\infty}$ performance has been proposed by use of ISS theory and by combining backstepping technique with generalized small-gain approach. Finally, a simulation example has been presented to illustrate the stabilization performance of the closed-loop systems by use of the proposed algorithm.

![Figure 1. The response of $x_1$.](image1)

![Figure 2. Control signal $u$.](image2)

**Acknowledgment**

This work was partially supported by the Natural Science Foundation of China (No. 61273106, No. 11202180) and the Natural Science Foundation of Jiangsu Province of China (No. BK2010293).

**References**


Jianjiang Yu received his Ph.D degree in Control Theory and Control Engineering from Southeast University, China, in 2010. He is currently an associate professor in School of Information Science and Technology, Yancheng Teachers University, China. His research interests include fuzzy control, time-delay systems and networked control.