The Entropy and Similarity Measure of Interval Valued Intuitionistic Fuzzy Sets and Their Relationship

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Abstract

In this paper, we research the relationship between entropy and similarity of interval valued intuitionistic fuzzy sets in detail and prove eight theorems that entropy and similarity of interval valued intuitionistic fuzzy sets can be transformed to each other based on their axiomatic definitions. Finally, we propose some formulae to calculate entropy and similarity of interval valued intuitionistic fuzzy sets.

Keywords: Interval valued intuitionistic fuzzy set, entropy, similarity measure, uncertainty measure.

1. Introduction

Since the fuzzy set theory was introduced by Zadeh [37], many new approaches and theories treating imprecision and uncertainty have been proposed, such as interval valued fuzzy set introduced by Zadeh [38], intuitionistic fuzzy set theory introduced by Atanassov [1] and so on. Among these theories, a well-known extension of the classic fuzzy set theory is interval valued intuitionistic fuzzy set theory, which was first introduced by Atanassov [2]. After that, many researchers have investigated this topic and have obtained some meaningful conclusions. For example, H. Bustince et al at [39] correlation of interval-valued intuitionistic fuzzy sets, Zhang Zhenhua et al at [45] Generalized Interval Valued Intuitionistic Fuzzy Sets theory, Jun Ye at [4], Shyi-Ming Chen et al at [29] and Zeshui Xu at [32, 34, 35] multicriteria fuzzy decision-making of interval-valued intuitionistic fuzzy sets and so on [3, 6, 9-12, 16, 20, 24].

Entropy and similarity measure of fuzzy set are two important topics in fuzzy set theory and have many successful applications. Entropy of fuzzy set describes the fuzziness degree of a fuzzy set and was first introduced by Zadeh [37]. Since then, many scholars have studied it from different points of view. For example, in 1972, De Luca and Termini [8] introduced some axioms to describe the fuzziness degree of a fuzzy set. Kaufmann [17] put forward a method to measure the fuzziness degree of a fuzzy set by a metric distance between its membership function and the membership function of its nearest crisp set. Yager [36] proposed a method to measure fuzziness degree of fuzzy set by a lack of distinction between the fuzzy set and its complement. Parkash et al. [25] investigated the new measures of weighted fuzzy entropy and their applications for the study of maximum weighted fuzzy entropy principle. Xiaoxia Huang [48] investigated the entropy method for diversified fuzzy portfolio selection. Mendel [33] investigated uncertainty measures for interval type-2 fuzzy sets. Yonghong Shen et al. [47] investigated the interval-valued fuzzy metric space. On the other hand, similarity measure of fuzzy set pioneered by Wang [30] indicates the similarity degree of fuzzy sets and has extensively been applied in many fields such as fuzzy clustering, image processing, fuzzy reasoning and fuzzy neural network and so on [7, 30, 40].

Recently, the uncertainty measures for interval type-2 fuzzy sets are studied by Wu and Mendel [22]. The relationship between similarity measure and entropy of fuzzy sets is studied by some researchers such as Liu [21], Fan [13] and Zeng et al [41-44].

Similarly, the entropy and similarity of intuitionistic fuzzy sets are very important in theory and applications in which the intuitionistic fuzzy sets are used to describe the imprecision and uncertainty. So many researchers studied these two topics and obtained some meaningful results. For instance, Szmidt and Kacprzyk [27] applied similarity measure of intuitionistic fuzzy sets in group decision making. Jun

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Ye [46] studied the fuzzy cross entropy of the interval-valued intuitionistic fuzzy sets and its optimal decision-making method based on the weights of alternatives. Burillo and Bustince [5] and Szmidt and Kacprzyk [26] studied the entropy of intuitionistic fuzzy sets from different points of views, respectively.

However, the researches of similarity measure and entropy for interval valued intuitionistic fuzzy sets are also empty, it is necessary to investigate the concepts and relationship between similarity measure and entropy for interval valued intuitionistic fuzzy sets. How to systematically study the transformation method from entropy to similarity measure of interval valued intuitionistic fuzzy sets and vice versa based on their axiomatic definitions becomes a challenge. In this paper, we study the axiomatic definitions of similarity measure and entropy for interval valued intuitionistic fuzzy sets and investigate the relationships between entropy and similarity measure of interval valued intuitionistic fuzzy sets and vice versa based on their axiomatic definitions.

The rest of this paper is organized as follows. In section 2 and 3, we recall some basic notions. In section 4, we investigate the relationship between entropy and similarity measure of intuitionistic fuzzy sets and prove six theorems that entropy and similarity can be transformed to each other based on their axiomatic definitions, and propose some new formulae to calculate entropy and similarity measures of interval valued intuitionistic fuzzy sets. The final is the conclusion.

2. Preliminaries

Throughout this paper, we use $X$ to denote the universal set, and IVIFSs($X$) to stand for the set of all interval valued intuitionistic fuzzy subsets over $X$. We write $\mathcal{F}(X)$ and $\mathcal{P}(X)$ for the set of all ordinary fuzzy sets and crisp sets in $X$, respectively. $\emptyset$ stands for the empty set, and the operation “c” denotes the complement operation.

Let $L=[0,1]$ and $[L]$ be the set of all closed subintervals of the interval $[0,1]$. Especially for an arbitrary element $a, b \in [0,1]$, we assume that $a$ is the same as $[a, a]$, namely, $a=[a, a]$. Then, according to Zadeh’s extension principle, for any $a=[a', a'], b=[b', b']$, and $\bar{a} \bar{b} \in [L]$ we can popularize some operators such as $\lor, \land$, and $c$ to $[L]$ and have

$\bar{a} \lor \bar{b} = [a' \lor b', a' \lor b']$,  
$\bar{a} \land \bar{b} = [a' \land b', a' \land b']$,  
$\lor_{c \in \mathbb{R}} \bar{a} = [\lor_{c \in \mathbb{R}} a', \lor_{c \in \mathbb{R}} a']$,  
$\land_{c \in \mathbb{R}} \bar{a} = [\land_{c \in \mathbb{R}} a', \land_{c \in \mathbb{R}} a']$,  
$\bar{a} = [1-a', 1-a']$.

where $W$ denotes an arbitrary index set. Furthermore, we have

$\bar{a} = \bar{b} \iff a' = b'$,  
$\bar{a} \leq \bar{b} \iff a' \leq b'$,  
$\bar{a} \geq \bar{b} \iff a' \geq b'$,  
$\bar{a} > \bar{b} \iff a' > b'$,  
$\bar{a} > \bar{b} \iff a' > b'$.

Obviously, there exist a minimal element $\bar{0} = [0,0]$ and a maximal element $\bar{1} = [1,1]$ in $[L]$. For $K \subseteq [0,1]$,  
$\inf K = \inf \{x \mid x \in K\}$,  
and $\sup K = \sup \{x \mid x \in K\}$.

Definition 1 [1]: An interval valued intuitionistic fuzzy set (IVIFS) $A$ over $X$ is defined as an object of the form

$A = \{x, \mu_A(x), \nu_A(x) \mid x \in X\}$,  
where

$\mu_A(x) = [\mu_A^0(x), \mu_A^1(x)] \subseteq [0,1]$,  
and

$\nu_A(x) = [\nu_A^0(x), \nu_A^1(x)] \subseteq [0,1]$,  
are closed intervals, and for all $x \in X$,  
$\mu_A^0(x) + \nu_A^0(x) \leq 1$.

Let $A \in$ IVIFSs($X$). If for every $x \in X$,  
$\mu_A^0(x) = \mu_A^1(x)$,  
$\nu_A^0(x) = \nu_A^1(x)$.

Then $A$ is an intuitionistic fuzzy set.

If $A, B \in$ IVIFSs($X$),  
$M_A : X \to [L]$ is the membership function and $N_A : X \to [L]$ is the non-membership function and the following operations can be found in [2]:

$A \subseteq_{a} B$ iff $(\forall x \in X)(\mu_A(x) \leq \mu_B(x))$,  
$A \subseteq_{\bar{a}} B$ iff $(\forall x \in X)(\mu_A^0(x) \leq \mu_B^0(x))$,  
$A \subseteq_{\bar{0}} B$ iff $(\forall x \in X)(\nu_A^0(x) \geq \nu_B^0(x))$,  
$A \subseteq_{\bar{0}} B$ iff $(\forall x \in X)(\nu_A^0(x) \geq \nu_B^0(x))$,  
$A \subseteq B$ iff $A \subseteq_{\bar{0}} B$ and $A \subseteq_{\bar{0} +} B$,  
$A \subseteq B$ iff $A \subseteq_{\bar{0}} B$ and $A \subseteq_{\bar{0} -} B$,  
$A = B$ iff $A \subseteq B, B \subseteq A$,  
$A' = \{x, N_A(x), M_A(x) \mid x \in X\}$.  


3. Entropy and Similarity of IVIFSSs($X$)

In this section, we introduce the definitions of entropy and similarity measure of IVIFSSs($X$), which are the extensions of entropy and similarity measure of classical fuzzy sets.

De Luca and Termini [8] first axiomatized fuzzy entropy. They were formulated in the following way.

Let $E$ be a mapping $E:F(X) \rightarrow [0,1]$. Hence $E$ is a fuzzy set defined on fuzzy sets. $E$ is an entropy measure if it satisfies the four axioms:

(E1) $E(A)=0$ if $A$ is a crisp set,

(E2) $E(A)=1$ if $\mu_A(x) = 0.5$ for all $x \in X$,

(E3) $E(A) \leq E(B)$ if $A$ is less fuzzy than $B$, i.e., $\mu_A(x) \leq \mu_B(x)$ when $\mu_B(x) \leq 0.5$. Or $\mu_A(x) \geq \mu_B(x)$ when $\mu_B(x) \geq 0.5$,

(E4) $E(A)=E(A')$.

In De Luca and Termini’s axioms (E1)-(E4), $A$ is less fuzzy than $B$ is an important conception, and a geometric interpretation of “$A$ is less fuzzy than $B$” is presented in Fig. 1.

![Figure 1. Geometric interpretation of $A$ is less fuzzy than $B$.](image1.png)

**Remark:** Straightforward, let $A$ be an interval valued intuitionistic fuzzy set, then $<\mu^A_i,v^A_i>$ and $<\mu^A_i,v^A_i>$ are two intuitionistic fuzzy sets.

**Definition 2:** Let $A,B$ be two interval valued intuitionistic fuzzy sets. If $<\mu^B_i,v^B_i>$ is less fuzzy than $<\mu^A_i,v^A_i>$ is less fuzzy than $<\mu^B_i,v^B_i>$ then we say that $B$ is a sharpened version of $A$.

Let $A,B$ be two interval valued intuitionistic fuzzy sets. A geometric interpretation of “$B$ is a sharpened version of $A$” is presented in Fig. 2 and Fig. 3.

![Figure 2. Geometric interpretation of “$B$ is a sharp version of $A$”.](image2.png)

![Figure 3. Geometric interpretation of “$B$ is a sharp version of $A$”.](image3.png)

**Definition 3:** A real function $E:IVIFSSs(X) \rightarrow [0,1]$ is called an entropy on IVIFSSs($X$), if $E$ satisfies the following properties:

(E1): $E(A)=0$ if $A$ is a crisp set;

(E2): $E(A)=1$ if $\mu_A(x) = v_A(x)$ for all $x \in X$;

(E3): $E(A) \leq E(B)$ if $B$ is a sharpened version of $A$;

(E4): $E(A)=E(A')$.

**Remark:** It is obvious that the Definition 3 is the extension of the De Luca and Termini’s definition. Furthermore, form the Definition 2, we can get the entropy on interval valued fuzzy sets (shortly, IVFSs), as the following definition.

**Definition 4:** Let $A=[A^-_i,A^+_i]$ and $B=[B^-_i,B^+_i]$ be two interval valued fuzzy sets. If $A^-$ is less fuzzy than $B^-$, then we say that $B$ is a sharpened version of $A$.

**Definition 5:** A real function $E:IVFSs(X) \rightarrow [0,1]$ is called an entropy on IVFSs($X$), if $E$ satisfies the following properties:

(E1): $E(A)=0$ if $A$ is a crisp set;

(E2): $E(A)=1$ if $\mu_A(x) = v_A(x)$ for all $x \in X$;

(E3): $E(A) \leq E(B)$ if $B$ is a sharpened version of $A$;

(E4): $E(A)=E(A')$.

For example, let $X = \{x_1,x_2,\cdots,x_n\}$ and $A = \{<x,\mu_A(x),v_A(x)>|x \in X\} \in IVIFSs(X)$, then

$$E(A) = \frac{\sum_{i=1}^{n}((\mu^A_i(x_i) + \mu^A_i(x_i)) \wedge (v^A_i(x_i) + v^A_i(x_i)))}{\sum_{i=1}^{n}((\mu^A_i(x_i) + \mu^A_i(x_i)) \vee (v^A_i(x_i) + v^A_i(x_i)))},$$

is an entropy on IVIFSs($X$).

Moreover, another important information measure of interval valued intuitionistic fuzzy sets is similarity measure which plays an important role in many fields, such as pattern recognition, approximate reasoning and so on.

**Definition 6:** A real function $S:IVIFSs(X) \times IVIFSs(X) \rightarrow [0,1]$ is called a similarity measure on IVIFSs($X$), if $S$
satisfies the following properties:

(S1) \( S(A, A') = 0 \) if \( A \) is a crisp set;

(S2) \( S(A, B) = 1 \Leftrightarrow A = B \);

(S3) \( S(A, B) = S(B, A) \);

(S4) For all \( A, B, C \in \text{IVIFSs}(X) \), if \( A \subseteq B \subseteq C \), then \( S(A, C) \leq S(A, B) \) and \( S(A, C) \leq S(B, C) \).

For example, let \( X = \{x_1, x_2, \ldots, x_n\} \),
\[ A = \{< \mu_a(x), \nu_a(x) > | x \in X \} \in \text{IVIFSs}(X) \]
and \( B = \{< \mu_b(x), \nu_b(x) > | x \in X \} \in \text{IVIFSs}(X) \), then
\[
S(A, B) = 1 - \frac{1}{4n} \sum_{i=1}^{n} (|\mu_a(x_i) - \mu_b(x_i)| + |\mu_a(x_i) - \mu_b(x_i)| + |\nu_a(x_i) - \nu_b(x_i)|)
\]
(11)
is a similarity measure on \( \text{IVIFSs}(X) \).

4. Relationship between Similarity Measure and Entropy of Interval Valued Intuitionistic Fuzzy Sets

4.1 Transformation methods from entropy to similarity measure

In this section, we discuss on the relationship between similarity measure and entropy of interval valued intuitionistic fuzzy sets based on their axiomatic definitions.

For \( A, B \in \text{IVIFSs}(X) \), we define \( f(A, B) \in \text{IVIFSs}(X) \) as follows:

\[
\forall x \in X, \\
\mu_{f(A,B)}(x) = [\mu_{f(A,B)}(x), \mu_{f(A,B)}^+(x)], \\
\nu_{f(A,B)}(x) = [\nu_{f(A,B)}(x), \nu_{f(A,B)}^+(x)],
\]
where
\[
\mu_{f(A,B)}(x) = 1 - \frac{1}{2} \sum_{i=1}^{n} (|\mu_a(x_i) - \mu_b(x_i)| + |\nu_a(x_i) - \nu_b(x_i)|)
\]
and
\[
\nu_{f(A,B)}(x) = 1 - \frac{1}{2} \sum_{i=1}^{n} (|\mu_a(x_i) - \mu_b(x_i)| + |\nu_a(x_i) - \nu_b(x_i)|)
\]
(12)

Then, we have the following theorem.

**Theorem 1:** Let \( E \) be an entropy on \( \text{IVIFSs}(X) \). Then \( E_{f} : \text{IVIFSs}(X) \times \text{IVIFSs}(X) \rightarrow [0,1] \)
\[
( A, B ) \quad \Rightarrow \quad E(f(A, B))
\]
is a similarity measure on \( \text{IVIFSs}(X) \).

**Proof:**

(S1): If \( A \) is crisp set, then
\[
\mu_{f(A,B)}(x) = \mu_{f(A,B)}^+(x) = 1,
\]
that is \( f(A, A') = X \). Then \( E(f(A, A')) = 0 \).

(S2): \( E(f(A, B)) = 1 \Leftrightarrow \mu_{f(A,B)} = \nu_{f(A,B)}; \)

(S3): According to (12), \( f(A, B) = f(B, A) \) is obvious, therefore, \( E(f(A, B)) = E(f(B, A)) \).

(S4): If \( A \subseteq B \subseteq C \), then for all \( x \in X \), we have
\[
\mu_{f(A,B)}(x) \leq \mu_{f(A,B)}^+(x), \quad \mu_{f(A,B)}^-(x) \leq \mu_{f(A,B)}^+(x),
\]
\[
\nu_{f(A,B)}(x) \geq \nu_{f(A,B)}^-(x), \quad v_{f(A,B)}^-(x) \geq v_{f(A,B)}^+(x).
\]
So we can get
\[
\mu_{f(A,B)}(x) \geq \mu_{f(A,B)}^-(x) \geq v_{f(A,B)}^-(x), \quad \mu_{f(A,B)}^+(x) \geq v_{f(A,B)}^+(x).
\]
It follows that \( f(A, B) \) is a sharpened version of \( f(A, C) \). Hence \( E(f(A, C)) \leq E(f(A, B)) \).

With the same reason, we can prove
\[
E(f(A, C)) \leq E(f(B, C)).
\]
So, we complete the proof of Theorem 1.

**Corollary 1:** Let \( E \) be an entropy on \( \text{IVIFSs}(X) \) and \( f(A, B) \) is defined as above. Then
\[
E_{f} : \text{IVIFSs}(X) \times \text{IVIFSs}(X) \rightarrow [0,1]
\]
\[
( A, B ) \quad \Rightarrow \quad E(f(A, B))
\]
is a similarity measure on \( \text{IVIFSs}(X) \).

For \( A, B \in \text{IVIFSs}(X) \), we define \( g(A, B) \) as follows:

\[
\forall x \in X, \\
\mu_{g(A,B)}(x) = [\mu_{g(A,B)}(x), \mu_{g(A,B)}^+(x)], \\
\nu_{g(A,B)}(x) = [\nu_{g(A,B)}(x), \nu_{g(A,B)}^+(x)],
\]
where
\[
\mu_{g(A,B)}(x) = 8 + [\mu_a(x) - \mu_b(x)] + |v_a(x) - v_b(x)|
\]
(13)
and
\[
\nu_{g(A,B)}(x) = 8 + 2[\mu_a(x) - \mu_b(x)] + |v_a(x) - v_b(x)|
\]
Then, we have the following theorem.

**Theorem 2:** Let \( E \) be an entropy on \( \text{IVIFSs}(X) \), and assume that \( g(A, B) \) is defined as above. Then \( E_{g} : \text{IVIFSs}(X) \times \text{IVIFSs}(X) \rightarrow [0,1] \)
\[
( A, B ) \quad \Rightarrow \quad E(g(A, B))
\]
is a similarity measure on \( \text{IVIFSs}(X) \).

**Proof:** This proof is similar to that of Theorem 1.
Corollary 2: Let \( E \) be an entropy on IVIFSs(\( X \)), and assume that \( g(A,B) \) is defined as above. Then 
\[
E'_g: \text{IVIFSs}(X) \times \text{IVIFSs}(X) \rightarrow [0,1] 
\]
\[
(A,B) \mapsto E(g(A,B)) 
\]
is a similarity measure on IVIFSs(\( X \)).

Example 1: Let \( X = \{x_1, x_2, \cdots, x_n\} \) and assume that \( E \) is defined as in (10). For \( A, B \in \text{IVIFSs}(X) \),
\[
E_f(A,B) = E(f(A,B)) \quad \text{(14)} 
\]
\[
= \frac{\sum_{x \in X} (2 - [\mu_f(x) - \mu_g(x)]^2 + [\nu_f(x) - \nu_g(x)]^2)}{\sum_{x \in X} (2 + [\mu_f(x) - \mu_g(x)]^2 + [\nu_f(x) - \nu_g(x)]^2)} 
\]
are similarity measures on IVIFSs(\( X \)). It is obvious that 
\[
E_f(A,B) = E'_f(A,B) 
\]
and 
\[
E_g(A,B) = E'_g(A,B) 
\]
For \( A, B \in \text{IVIFSs}(X) \), \( p > 0 \) and for every \( x \in X \), we define \( f^p(A,B), g^p(A,B) \in \text{IVIFSs}(X) \) as follows:
\[
\mu_{f^p}(x) = \left[ \mu_{f^p}(x), \mu_{f^p}(x) \right], \quad \nu_{f^p}(x) = \left[ \nu_{f^p}(x), \nu_{f^p}(x) \right], 
\]
where
\[
\mu_{f^p}(x) = \left[ \frac{1 + [\mu_f(x) - \mu_g(x)]^2 + [\nu_f(x) - \nu_g(x)]^2}{2} \right]^{\frac{1}{p}} 
\]
\[
\nu_{f^p}(x) = \left[ \frac{1 + [\mu_f(x) - \mu_g(x)]^2 + [\nu_f(x) - \nu_g(x)]^2}{2} \right]^{\frac{1}{p}} 
\]
\[
\mu_{g^p}(x) = \left[ \frac{1 - [\mu_f(x) - \mu_g(x)]^2 + [\nu_f(x) - \nu_g(x)]^2}{2} \right]^{\frac{1}{p}} 
\]
\[
\nu_{g^p}(x) = \left[ \frac{1 - [\mu_f(x) - \mu_g(x)]^2 + [\nu_f(x) - \nu_g(x)]^2}{2} \right]^{\frac{1}{p}} 
\]
are similarity measures on IVIFSs(\( X \)). It is obvious that 
\[
E_{f^p}(A,B) = E'_{f^p}(A,B), \quad E_{g^p}(A,B) = E'_{g^p}(A,B) 
\]
4.2 Transformation methods from similarity measure to entropy of interval valued intuitionistic fuzzy sets
In this section, we propose other transform methods of
setting up entropy of interval valued intuitionistic fuzzy sets based on similarity measure of interval valued intuitionistic fuzzy sets.

For $A \in \text{IVIFSs}(X)$ and $x \in X$, we define $R(A), Z(A)$ as follows:

$\mu_{R(A)}(x) = [\mu_{R(A)}^-(x), \mu_{R(A)}^+(x)],$
$\nu_{R(A)}(x) = [\nu_{R(A)}^-(x), \nu_{R(A)}^+(x)],$
$\mu_{Z(A)}(x) = [\mu_{Z(A)}^-(x), \mu_{Z(A)}^+(x)],$
$\nu_{Z(A)}(x) = [\nu_{Z(A)}^-(x), \nu_{Z(A)}^+(x)],$

where

$\mu_{R(A)}^-(x) = \frac{1 + [\mu_{R(A)}^-(x) - \nu_{R(A)}^+(x)]}{2},$
$\mu_{R(A)}^+(x) = \frac{1 + [\mu_{R(A)}^+(x) - \nu_{R(A)}^-(x)]}{2},$
$\nu_{R(A)}^-(x) = \frac{1 - [\mu_{R(A)}-(x) - \nu_{R(A)}^-(x)]}{2},$
$\nu_{R(A)}^+(x) = \frac{1 - [\mu_{R(A)}^-(x) - \nu_{R(A)}^+(x)]}{2},$
$\mu_{Z(A)}^-(x) = \frac{1 + [\mu_{Z(A)}^-(x) - \nu_{Z(A)}^+(x)]}{2},$
$\mu_{Z(A)}^+(x) = \frac{1 + [\mu_{Z(A)}^+(x) - \nu_{Z(A)}^-(x)]}{2},$
$\nu_{Z(A)}^-(x) = \frac{1 - [\mu_{Z(A)}^-(x) - \nu_{Z(A)}^-(x)]}{2},$
$\nu_{Z(A)}^+(x) = \frac{1 - [\mu_{Z(A)}^-(x) - \nu_{Z(A)}^+(x)]}{2}.$

$\mu_{R(A)}(x) = [\mu_{R(A)}^+(x), \mu_{R(A)}^-(x)],$
$\nu_{R(A)}(x) = [\nu_{R(A)}^+(x), \nu_{R(A)}^-(x)],$
$\mu_{Z(A)}(x) = [\mu_{Z(A)}^+(x), \mu_{Z(A)}^-(x)],$
$\nu_{Z(A)}(x) = [\nu_{Z(A)}^+(x), \nu_{Z(A)}^-(x)],$

Theorem 4: Let $S$ be similarity measure on IVIFSs$(X)$. Then

$S_{\text{rz}} : \text{IVIFSs}(X) \rightarrow [0, 1]$ $A \mapsto S(R(A), Z(A))$

is an entropy on IVIFSs$(X)$.

Proof:

(P1): If $A$ is a crisp set, then for each $x \in X$, $\mu_{R(A)}(x) = [1, 1], \nu_{R(A)}(x) = [0, 0]$,
$\mu_{Z(A)}(x) = [0, 0], \nu_{Z(A)}(x) = [1, 1]$,
i.e. $R(A) = X, Z(A) = \emptyset$.

By the definition of similarity measure, we can get $S(R(A), Z(A)) = 0$.

(P2): $S(R(A), Z(A)) = 1 \iff R(A) = Z(A)$,
for all $x \in X$, i.e. $S(R(A), Z(A)) = 1 \iff \mu_x = \nu_x$.

(P3): If $B$ is a sharpened version of $A$, it follows that

$\mu_B^-(x) \leq \mu_A^-(x) \leq \mu_B^+(x)$,
$\mu_B^+(x) \leq \mu_A^+(x) \leq \mu_B^+(x),$

or

$\nu_B^-(x) \leq \nu_A^-(x) \leq \mu_B^+(x)$,
$\mu_B^+(x) \leq \nu_A^+(x) \leq \nu_B^+(x),$

So we have

$\mu_B^+(x) - \nu_B^-(x) \leq \mu_A^+(x) - \nu_A^-(x)$,

$\mu_B^+(x) - \nu_B^+(x) \geq \mu_A^+(x) - \nu_A^+(x)$.

It follows that

$\mu_B^+(x) \geq \mu_A^+(x) \geq \mu_B^+(x)$,

$\mu_B^+(x) \leq \mu_A^+(x) \leq \mu_B^+(x),$

$\nu_B^-(x) \leq \nu_A^-(x) \leq \nu_B^+(x)$,

$\nu_B^+(x) \leq \nu_A^+(x) \leq \nu_B^+(x)$.

Therefore,

$R(A) \supseteq R(B) \supseteq Z(B) \supseteq Z(A)$.

So

$S(R(A), Z(A)) \leq S(R(B), Z(B))$.

(P4): Known by the definitions of $R(A), Z(A)$, we have

$R(A) = R(A')$,
$Z(A) = Z(A').$

Therefore, $S(R(A), Z(A)) = S(R(A'), Z(A'))$.

Corollary 3: Let $S$ be similarity measure on IVIFSs$(X)$. Then

$S_{\text{rz}} : \text{IVIFSs}(X) \rightarrow [0, 1]$ $A \mapsto S(R(A), Z(A))$

is an entropy on IVIFSs$(X)$.

Example 3: Let $X = \{x_1, x_2, \cdots, x_n\}$ and assume that $S$ is defined as in (11). For $A \in \text{IVIFSs}(X)$,

$S_{\text{rz}} (A, B) = S(R(A), Z(A))$

$= -1 \sum_{i=1}^{n} \frac{1}{4n} \left[ (\mu_{i}(x) - \mu_{i}(x)) \vee (\mu_{i}(x) - \mu_{i}(x)) \right]$

$= \frac{1}{2} \left[ (\mu_{i}(x) - \mu_{i}(x)) \vee (\mu_{i}(x) - \mu_{i}(x)) \right]$

$= \frac{1}{2} \left[ (\mu_{i}(x) - \mu_{i}(x)) \vee (\mu_{i}(x) - \mu_{i}(x)) \right]$

$= \frac{1}{2} \left[ (\mu_{i}(x) - \mu_{i}(x)) \vee (\mu_{i}(x) - \mu_{i}(x)) \right]$

is an entropy on IVIFSs$(X)$. We also have

$S_{\text{rz}} (A) = S_{\text{rz}} (A').$
where

\[
\begin{align*}
\mu_{p(x)}(x) &= \frac{1 + \left[ \frac{\mu_{\ell}(x) - \mu_{\ell}(x)}{\mu_{\ell}(x) - \mu_{\ell}(x)} \right]^p}{2}, \\
\mu_{p(x)}(x) &= \frac{1 + \left[ \frac{\mu_{\ell}(x) - \mu_{\ell}(x)}{\mu_{\ell}(x) - \mu_{\ell}(x)} \right]^p}{2}, \\
\nu_{p(x)}(x) &= \frac{1 - \left[ \frac{\mu_{\ell}(x) - \mu_{\ell}(x)}{\mu_{\ell}(x) - \mu_{\ell}(x)} \right]^p}{2}, \\
\nu_{p(x)}(x) &= \frac{1 - \left[ \frac{\mu_{\ell}(x) - \mu_{\ell}(x)}{\mu_{\ell}(x) - \mu_{\ell}(x)} \right]^p}{2}, \\
\nu_{p(x)}(x) &= \frac{1 + \left[ \frac{\mu_{\ell}(x) - \mu_{\ell}(x)}{\mu_{\ell}(x) - \mu_{\ell}(x)} \right]^p}{2}, \\
\nu_{p(x)}(x) &= \frac{1 + \left[ \frac{\mu_{\ell}(x) - \mu_{\ell}(x)}{\mu_{\ell}(x) - \mu_{\ell}(x)} \right]^p}{2}, \\
\nu_{p(x)}(x) &= \frac{1 + \left[ \frac{\mu_{\ell}(x) - \mu_{\ell}(x)}{\mu_{\ell}(x) - \mu_{\ell}(x)} \right]^p}{2}, \\
\nu_{p(x)}(x) &= \frac{1 + \left[ \frac{\mu_{\ell}(x) - \mu_{\ell}(x)}{\mu_{\ell}(x) - \mu_{\ell}(x)} \right]^p}{2}.
\end{align*}
\]

**Theorem 5:** Let \( S \) be a similarity measure on IVIFSs(X). Then

\[
S_{p(x)} : \text{IVIFSs}(X) \rightarrow [0,1]
\]

is an entropy on IVIFSs(X).

**Proof:** This proof is similar with Theorem 4.

**Corollary 4:** Let \( S \) be a similarity measure on IVIFSs(X). Then

\[
S_{p(x)} : \text{IVIFSs}(X) \rightarrow [0,1]
\]

is an entropy on IVIFSs(X).

**Example 4:** Let \( X = \{x_1, x_2, \cdots, x_n\} \) and assume that \( S \) is defined as in (11). For \( A \in \text{IVIFSs}(X) \),

\[
S_{p(x)}(A) = S(R(A), Z(A))
\]

is an entropy on IVIFSs(X). We also have \( S_{p(x)}(A) = S_{p(x)}(A) \).

**Theorem 6:** Let \( S \) be a similarity measure on IVIFSs(X). Then

\[
S_{p(x)}(A) = S_{p(x)}(A)
\]

is an entropy on IVIFSs(X).

**Proof:**

(P1): It is obvious that \( S_{p(x)}(A) = S_{p(x)}(A') = 0 \) when \( A \) is a crisp set.

(P2): Known by the definition of \( S \), for all \( x \in X \)

\[
S(A, A') = 1 \iff A = A' \iff \mu_1(x) = \nu_1(x).
\]

(P3): If \( B \) is a sharpened version of \( A \), we have

\[
\mu(x) \leq \mu_B(x) \leq \nu_B(x) \leq \nu(x),
\]

\[
\mu(x) \leq \mu_B(x) \leq \nu_B(x) \leq \nu(x),
\]

and

\[
\nu(x) \leq \nu_B(x) \leq \mu_B(x) \leq \mu(x),
\]

or

\[
\nu(x) \leq \nu_B(x) \leq \mu_B(x) \leq \mu(x),
\]

According to (1), (2), (3) and (4), we have

\[
A \subseteq B \subseteq A', B \subseteq A \subseteq A',
\]

or

\[
A' \subseteq B \subseteq A', B \subseteq A \subseteq A',
\]

It follows that

\[
A \subseteq B \subseteq B' \subseteq A',
\]

or

\[
A' \subseteq B' \subseteq B \subseteq A.
\]

Hence, \( S(A, A') \leq S(B, B') \).

(P4): \( S(A, A') = S(A', A) \) is obvious.

### 4.3 Further Discussion

How to systematically investigate the transformation methods from entropy to similarity measure and vice versa for interval valued intuitionistic fuzzy sets is an interesting topic. In this section, we investigate some sufficient conditions for transformations from an entropy measure to a similarity measure and vice versa for interval valued intuitionistic fuzzy sets.

Let \( \mathcal{E} = \{ E | E \text{ is an entropy on IVIFSs}(X) \} \) and \( S = \{ S | S \text{ is a similarity measure on IVIFSs}(X) \} \).

The mappings

\[
\theta : \text{IVIFSs}(X) \rightarrow \mathcal{E}(X)
\]

\[
\phi : \text{IVIFSs}(X) \rightarrow \mathcal{E}(X)
\]

satisfy the following properties:

(1) For \( A \in \mathcal{P}(X) \), \( \theta(A) \) and \( \phi(A) \) are crisp sets and \( \theta(A) \cup \phi(A) = X \);

(2) \( \theta(A) = \phi(A) \iff M_A = N_A \);

(3) If \( | \mu_1(x) - \nu_1(x) | \geq | \mu_2(x) - \nu_2(x) | \)

and \( | \mu_1(x) - \nu_1(x) | \geq | \mu_2(x) - \nu_2(x) | \) for all \( x \in X \),
one of the following four formulae holds:
\( \varphi(A) \subseteq \varphi(B) \subseteq \theta(B) \subseteq \theta(A) \),
\( \varphi(A) \subseteq \theta(B) \subseteq \varphi(B) \subseteq \theta(A) \),
\( \varphi(A) \supseteq \varphi(B) \supseteq \theta(B) \supseteq \theta(A) \),
\( \varphi(A) \supseteq \theta(B) \supseteq \varphi(B) \supseteq M(A) \).

(4) \( \varphi(A) = \varphi(A') \) and \( \theta(A) = \theta(A') \).

Let \( \mathcal{Y} = \{ (\theta, \varphi) \mid \theta, \varphi \) are mappings from IVIFSs(\( X \)) to IVIFSs(\( X \)) and satisfying the above four properties\}. It is easy to verify that the mappings \( \theta \) and \( \varphi \) defined as ones of Theorem 4 and Theorem 5 satisfying the properties above. So \( \mathcal{Y} \neq \emptyset \).

**Theorem 7:** For any \( (\theta, \varphi) \in \mathcal{Y} \), there exists a mapping \( F_{\theta \varphi} \) from \( S \) to \( \mathcal{E} \).

**Proof:** Let construct \( F_{\theta \varphi} \) as follows:

\[
F_{\theta \varphi} : S \rightarrow \mathcal{E} \quad S \mapsto F_{\theta \varphi}(S)
\]

where

\[
F_{\theta \varphi}(S)(A) \triangleq S(\theta(A), \varphi(A)) \text{ for } A \in \text{IVIFSs}(X).
\]

We only show \( F_{\theta \varphi}(S) \) is an entropy on IVIFSs(\( X \)).

(P1) Since \( (\theta, \varphi) \in \mathcal{Y} \), we have \( \theta(A) \cup \varphi(A) = X \) for any \( A \in \mathcal{P}(X) \).

It follows that \( F_{\theta \varphi}(S)(A) = S(\theta(A), \varphi(A)) = 0 \).

(P2) \( F_{\theta \varphi}(S)(A) = 1 \Leftrightarrow \mu_\theta = \nu_\varphi \) according to \( (\theta, \varphi) \in \mathcal{Y} \).

(P3) If \( A \) is less fuzzy than \( B \), so we have

\[
| \mu_\theta(x) - \nu_\varphi(x) | \geq | \mu_\theta(x) - \nu_\varphi(x) |,
\]

and

\[
| \mu_\theta(x) - \nu_\varphi(x) | \geq | \mu_\theta(x) - \nu_\varphi(x) |,
\]

for all \( x \in X \). Due to the properties (3) of \( (\theta, \varphi) \), we have

\[
F_{\theta \varphi}(S)(A) = S(\theta(A), \varphi(A)) \subseteq S(\theta(B), \varphi(B)) = F_{\theta \varphi}(B).
\]

(P4) It is obvious that \( F_{\theta \varphi}(S)(A) = F_{\theta \varphi}(S)(A') \).

So \( F_{\theta \varphi}(S) \in \mathcal{E} \), namely, \( F_{\theta \varphi} \) is well-defined.

Hence, we complete the proof of the theorem.

Next, we give a sufficient condition for the transformation from the entropy to similarity measure of interval valued intuitionistic fuzzy sets.

Let \( f : \text{IVIFSs}(X) \times \text{IVIFSs}(X) \rightarrow \text{IVIFSs}(X) \)

\[
(A, B) \mapsto f(A, B) \triangleq \langle < x, \mu_{f(A,B)}(x), \nu_{f(A,B)}(x) > \mid x \in X \rangle
\]

satisfy the following properties:

(1) For \( A \in \mathcal{P}(X) \), \( f(A, A') \in \mathcal{P}(X) \);

(2) \( \mu_{f(A,B)} = \nu_{f(A,B)} \Leftrightarrow A = B \);

(3) \( f(A, B) = f(B, A) \);

(4) If \( A \subseteq B \subseteq C \), then \( f(A, C) \) is less fuzzy than \( f(A, B) \) and \( f(B, C) \) is a sharpened version of \( f(A, C) \).

Let \( \mathcal{L} = \{ f \mid f \) is a mapping from IVIFSs(\( X \)) to IVIFSs(\( X \)) \} and satisfies the above properties. It is easy to verify that the mapping \( f \) defined as in one of Theorem 1 satisfies the properties above. Therefore, \( \mathcal{L} \neq \emptyset \).

**Theorem 8:** For any \( f \in \mathcal{L} \), there exists a mapping \( G_f \) from \( S \) to \( \mathcal{S} \).

**Proof:** Let construct \( G_f \) for any \( f \in \mathcal{L} \) as follows:

\[
G_f : \mathcal{E} \rightarrow \mathcal{F} \quad E \mapsto G_f(E),
\]

where

\[
G_f(E)(A, B) \triangleq E(f(A, B)),
\]

For \( A, B \in \text{IVIFSs}(X) \). We will show \( E(f(A, B)) \) is a similarity measure on IVIFSs(\( X \)).

(S1) Due to \( f \in \mathcal{L} \), we have \( f(A, A') \in \mathcal{P}(X) \). It follows that \( G_f(E)(A, A') = E(f(A, A')) = 0 \) for any \( E \in \mathcal{E} \).

(S2) When \( A = B \), we have \( M_{f(A,B)} = N_{f(A,B)} \) for \( f \in \mathcal{L} \).

It follows that \( E(f(A, B)) = 1 \), i.e., \( G_f(E)(A, B) = 1 \).

(S3) \( G_f(E)(A, B) = E(f(A, B)) = E(f(B, A)) = G_f(E)(B, A) \).

(S4) For \( f \in \mathcal{L} \), we have \( f(A, B) \) is a sharpened version of \( f(A, C) \) and \( f(B, C) \) is a sharpened version of \( f(A, C) \) when \( A \subseteq B \subseteq C \).

So \( G_f(E)(A, C) = E(f(A, C)) \leq E(f(A, B)) = G_f(E)(A, B) \), and

\( G_f(E)(A, C) = E(f(A, C)) \leq E(f(B, C)) = G_f(E)(B, C) \),

for \( A \subseteq B \subseteq C \).

So \( G_f(E) \in \mathcal{S} \), namely, \( G_f \) is well-defined.

Hence, we complete the proof of the theorem.

### 5. Conclusions

Both entropy and similarity are important conceptions in handling uncertainty. In this paper, we explore one aspect of their interactions and investigate the relationship between entropy and similarity of interval valued intuitionistic fuzzy sets and some methods are given to describe entropy of interval valued intuitionistic fuzzy sets based on its similarity measure. Firstly, obtain eight theorems that entropy and similarity of interval valued intuitionistic fuzzy sets can be transformed by each other based on their axiomatic definitions, and then propose some formulas to calculate entropy and similarity measure of interval valued intuitionistic fuzzy sets. Finally, we investigate some sufficient conditions for transformations from an entropy measure to a similarity measure and vice versa for interval valued...
intuitionistic fuzzy sets. We believe many these results can be applied in many fields such as image processing, pattern recognition, and fuzzy reasoning and some applications will be given in the next study.

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