Ordering Alternatives under Fuzzy Multiple Criteria Decision Making via a Fuzzy Number Dominance Based Ranking Approach

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Abstract

This work suggests ordering alternatives under fuzzy multiple criteria decision making (MCDM) via a fuzzy number dominance based ranking approach, where the ratings of alternatives versus qualitative criteria and the importance weights of all criteria are assessed in linguistic values represented by fuzzy numbers. The difference of one final fuzzy evaluation value over another is used to represent the dominance degree of one alternative over another. The membership function for the final fuzzy evaluation value of each alternative and the difference between each pair of final fuzzy evaluation values can be developed through $\alpha$-cuts and interval arithmetic of fuzzy numbers. Formulas for the dominance degree can be clearly derived based on the developed membership function via integral development. A simple ranking procedure based on these dominance degrees is then proposed to order the alternatives. Finally a numerical example demonstrates the feasibility of the proposed model.

Keywords: Fuzzy MCDM, fuzzy number dominance, ranking, integral.

1. Introduction

A fuzzy multiple criteria decision making (MCDM)[1] model is to assess alternatives versus selected criteria through a committee of decision makers, where suitability of alternatives versus criteria and the importance weights of criteria can be evaluated in linguistic values represented by fuzzy numbers. Fuzzy set theory, initially proposed by Zadeh [2], has been extensively applied to objectively resolve the uncertainties in human judgment and effectively reflect the ambiguities in the available information in an ill-defined multiple criteria decision making environment. A comparison and review of fuzzy MCDM methods can be found in Carlsson and Fuller [3], Ribeiro [4] and Triantaphyllou and Lin [5]. Some recent applications can be found in [6-20]. The final evaluation values of alternatives in most of the above fuzzy MCDM problems are usually still fuzzy numbers and these fuzzy numbers need a proper ranking approach to defuzzify them into crisp values for decision making. A comparison and review of many of the fuzzy number ranking methods can be seen in [21-31]. However, in spite of the merits, some of the applied ranking methods are computational complex and others are difficult to implement the connection between the ranking procedure and the final fuzzy evaluation values, limiting the applicability of the fuzzy MCDM model.

To resolve the above limitations, this work suggests a fuzzy MCDM model with a fuzzy number dominance based ranking procedure, where ratings of alternatives versus qualitative criteria and the importance weights of all criteria are assessed in linguistic values [32] represented by triangular fuzzy numbers. Criteria are classified to qualitative and quantitative ones. The dominance degree of one alternative over another is a concept from Chen [9], which is developed based on measuring the difference of one final fuzzy evaluation value of the corresponding alternative over another. Through $\alpha$-cuts and interval arithmetic of fuzzy numbers, membership functions for the final fuzzy evaluation value of each alternative and the difference between each pair of final fuzzy evaluation values can be developed. Formulas for the dominance degree of one alternative over another can be clearly derived via integral development based on the developed membership functions. A simple ranking procedure is then proposed to order the alternatives based on these dominance degrees. Finally a numerical example demonstrates the feasibility of the proposed model.

The rest of this work is organized as follows. Section 2 briefly introduces fuzzy set theory. Section 3 introduces the suggested model. Meanwhile, an example is presented in Section 4 to demonstrate the computational process of the proposed model and conclusions are finally made in Section 5.
2. Fuzzy Set Theory

A. Fuzzy sets

\[ A = \{(x, f_A(x)) \mid x \in U\} \], where \( U \) is the universe of discourse, \( x \) is an element in \( U \), \( A \) is a fuzzy set in \( U \), \( f_A(x) \) is the membership function of \( A \) at \( x \) [33]. The larger \( f_A(x) \), the stronger the grade of membership for \( x \) in \( A \).

B. Fuzzy numbers

A real fuzzy number \( A \) is described as any fuzzy subset of the real line \( R \) with membership function \( f_A \) which possesses the following properties [34]:

(a) \( f_A \) is a continuous mapping from \( R \) to \([0,1]\);
(b) \( f_A(x) = 0, \forall x \in (-\infty, a] \);
(c) \( f_A \) is strictly increasing on \([a, b]\);
(d) \( f_A(x) = 1, x \in [b, c] \);
(e) \( f_A \) is strictly decreasing on \([c, d]\);
(f) \( f_A(x) = 0, \forall x \in [d, \infty) \);

where \( a \leq b \leq c \leq d \), \( A \) can be denoted as \([a, b, c, d]\).

The membership function \( f_A \) of the fuzzy number \( A \) can also be expressed as:

\[
 f_A(x) = \begin{cases} 
 f^L_A(x), & a \leq x \leq b \\
 1, & b \leq x \leq c \\
 f^R_A(x), & c \leq x \leq d \\
 0, & \text{otherwise} 
\end{cases} 
\]

where \( f^L_A(x) \) and \( f^R_A(x) \) are the left and right membership functions of \( A \), respectively [33]. A fuzzy triangular number can be denoted as \((a, b, c)\) [35].

C. \( \alpha \)-cuts

The \( \alpha \)-cuts of fuzzy number \( A \) can be defined as \( A^\alpha = \{x \mid f_A(x) \geq \alpha\}, \alpha \in [0,1] \), where \( A^\alpha \) is a non-empty bounded closed interval contained in \( R \) and can be denoted by \( A^\alpha = [A^\alpha_L, A^\alpha_U] \), where \( A^\alpha_L \) and \( A^\alpha_U \) are its lower and upper bounds, respectively [33].

D. Arithmetic operations on fuzzy numbers

Given fuzzy numbers \( A \) and \( B \), \( A, B \in R^+ \), the \( \alpha \)-cuts of \( A \) and \( B \) are \( A^\alpha = [A^\alpha_L, A^\alpha_U] \) and \( B^\alpha = [B^\alpha_L, B^\alpha_U] \), respectively. By the interval arithmetic, some main operations of \( A \) and \( B \) can be expressed as follows [33]:

\[
(A \oplus r)^\alpha = [A^\alpha + r^\alpha, A^\alpha + r^\alpha] \quad \text{and} \quad (A \otimes r)^\alpha = [A^\alpha \cdot r^\alpha, A^\alpha \cdot r^\alpha] \quad r \in R^+ \tag{5}
\]

E. Linguistic values

A linguistic variable is a variable whose values are expressed in linguistic terms. Linguistic variable is a very helpful concept for dealing with situations which are too complex or not well-defined to be reasonably described by traditional quantitative expressions [32]. For example, the linguistic rating set \{\( B, B.B&G, G, VG, E \)\}, where \( B=\text{Bad}, B.B&G=\text{Between Bad and Good}, G=\text{Good}, VG=\text{Very Good}, E=\text{Excellent}\), can be used to evaluate the suitability of alternatives versus qualitative criteria. These linguistic values can be further represented by triangular fuzzy numbers such as \( B=(0,0.1,0.3), B.B&G=(0.1,0.3,0.5), G=(0.3,0.5,0.7), VG=(0.5,0.7,0.9), E=(0.7,0.9,1.0) \) as shown in Fig. 1.

![Figure 1. Linguistic values and their fuzzy numbers for alternatives versus qualitative criteria.](image)

3. Model Development

Assume that a committee of \( k \) decision-makers (i.e. \( D_t \), \( t=1 \sim k \)) is responsible for the selection of \( m \) alternative (i.e. \( A_i \), \( i=1 \sim m \)) under \( n \) criteria \( (C_j, j=1 \sim n) \). Criteria are classified to qualitative, \( C_{ij} \), \( j=1 \sim g \), and quantitative; quantitative criteria are further classified to benefit, \( C_{ij} \), \( j=g+1 \sim h \) and cost, \( C_{ij} \), \( j=h+1 \sim n \) ones. Moreover, we assume that the importance weights of the criteria and the performance ratings under each of the qualitative criteria are assessed in linguistic values represented by positive triangular fuzzy numbers.

A. Average importance weights

Let \( w_{ij} = (o_{ij}, p_{ij}, q_{ij}) \), \( w_{ij} \in R^+ \), \( j=1 \sim n \), \( t=1 \sim k \), be the importance weight assigned by decision maker \( D_t \), to criterion \( C_{ij} \), \( w_{ij} = (o_{ij}, p_{ij}, q_{ij}) \) is the averaged importance weight of criterion \( C_{ij} \) assessed by the committee of \( k \) decision makers.

\[
w_{ij} = \frac{1}{k} \oplus (w_{j1} \oplus w_{j2} \oplus \ldots \oplus w_{jk}) \tag{6}
\]

where

\( \oplus \)
\[ a_j = \frac{1}{k} \sum_{i=1}^{k} a_{ij}, \quad p_j = \frac{1}{k} \sum_{i=1}^{k} p_{ij}, \quad q_j = \frac{1}{k} \sum_{i=1}^{k} q_{ij}. \]

### B. Average ratings of alternatives versus qualitative criteria

Let \( x_{ij} = (a_{ij}, b_{ij}, c_{ij}), i=1, \ldots, m, j=1, \ldots, g, t=1, \ldots, k, \) be the rating assigned to alternative \( A_i \) by decision maker \( D_t \) under criterion \( C_j \). \( x_{ij} = (a_{ij}, b_{ij}, c_{ij}) \) is the averaged rating of alternative \( A_i \) versus criterion \( C_j \) assessed by the committee of decision makers.

\[
x_{ij} = \frac{1}{k} \otimes \left( x_{ijk} \otimes \cdots x_{i1k} \otimes \cdots x_{ij1} \right) \quad (7)
\]

where

\[
a_j = \frac{1}{k} \sum_{i=1}^{k} a_{ij}, \quad b_j = \frac{1}{k} \sum_{i=1}^{k} b_{ij}, \quad c_j = \frac{1}{k} \sum_{i=1}^{k} c_{ij}, \quad j=1 \sim g.
\]

### C. Normalization of ratings of alternatives versus quantitative criteria

Quantitative criteria can be classified as benefit and cost ones. The benefit criterion has the characteristic: the larger the better. Cost criterion has the characteristic: the smaller the better. Values under both benefit and cost quantitative criteria can be either crisp or fuzzy. Values under quantitative criteria may have different units and then must be normalized into a comparable scale for calculation rationale. The following approach [36] is used to complete the normalization. This approach preserves by property where the ranges of normalized triangular fuzzy numbers belong to \([0, 1]\).

Suppose \( r_{ij} = (e_{ij}, f_{ij}, g_{ij}) \) is the performance value of alternative \( A_i \) versus criterion \( C_j \), \( j=g+1 \sim n \). The normalization of the \( r_{ij} \) is as follows:

\[
x_{ij} = \begin{cases} 
\frac{e_{ij} - e_{ij}^-}{d^+}, & \text{if } j \in B, \\
\frac{g_{ij}^- - f_{ij}^-}{d^+}, & \text{if } j \in C,
\end{cases}
\]

where

\[
g_j^+ = \max_i g_{ij}, \quad e_j^- = \min_i e_{ij}, \quad d^+ = g_j^+ - e_j^- - g_j^+ \quad j = g+1 \sim n, \quad B = g+1 \sim h, \quad C = h+1 \sim n.
\]

### D. Final fuzzy evaluation value of each alternative

The final fuzzy evaluation value of each alternative \( A_i \) can be obtained by using the Simple Addictive Weighting [1] concept as follows:

\[
S_i = \frac{1}{n} \sum_{j=1}^{n} x_{ij} \otimes w_j, \quad i=1, 2, \ldots, m.
\]

Here, \( S_i \) is the final fuzzy evaluation values of each alternative \( A_i \). The membership functions of the \( S_i \) can be developed as follows:

Through Eqs. (2)-(4), we obtain:

\[
x_{ij}^a \otimes w_j^a = \left[ (b_{ij} - a_{ij}) \alpha (p_j - o_j) \alpha^2 + (a_{ij} - o_j) \alpha + a_{ij} \alpha \right] + \left[ (b_{ij} - c_{ij}) \alpha (p_j - q_j) \alpha^2 + (c_{ij} - o_j) \alpha + c_{ij} \alpha \right]
\]

By applying Eq. (10) to Eq. (9), we obtain the \( \alpha \)-cut of \( S_i \) as follows:

\[
S_i^\alpha = \sum_{j=1}^{n} x_{ij}^a \otimes w_j^a
\]

Assume:

\[
E_1 = \sum_{j=1}^{n} (b_{ij} - a_{ij}) (p_j - o_j), \\
F_1 = \sum_{j=1}^{n} (a_{ij} (p_j - o_j) + o_j (b_{ij} - a_{ij})), \\
E_2 = \sum_{j=1}^{n} (b_{ij} - c_{ij}) (p_j - q_j), \\
F_2 = \sum_{j=1}^{n} (c_{ij} (p_j - q_j) + q_j (b_{ij} - c_{ij})), \\
V_i = \sum_{j=1}^{n} a_{ij} o_j, \quad Y_i = \sum_{j=1}^{n} b_{ij} p_j, \quad Z_i = \sum_{j=1}^{n} c_{ij} q_j.
\]

Applying the above assumption to Eq. (11), we have the following two equations to solve:

\[
E_1 \alpha^2 + F_1 \alpha + V_i - x = 0 \quad (12)
\]

\[
E_2 \alpha^2 + F_2 \alpha + Z_i - x = 0 \quad (13)
\]

Only root in \([0, 1]\) are retained in Eq. (12) and (13). The left membership function, i.e. \( f_{S_i}^L(x) \), and the right membership function \( f_{S_i}^R(x) \), of the final fuzzy evaluation value \( S_i \) can be produced as follows:

\[
f_{S_i}^L(x) = \frac{F_1^2 + 4E_1(x - Y_i)}{2E_1}, \quad V_i \leq x \leq Y_i,
\]

\[
f_{S_i}^R(x) = \frac{F_2^2 + 4E_2(x - Z_i)}{2E_2}, \quad Y_i \leq x \leq Z_i.
\]

For convenience, \( S_i \) can be displayed as:

\[
S_i = \{ V_i, Y_i, Z_i ; E_1, F_1, E_2, F_2 \}, \quad i=1 \sim m.
\]
E. Difference of two fuzzy numbers

The difference of two fuzzy numbers, that is, the difference of one final fuzzy evaluation value over another from Chen [9], is applied to represent the dominance degree of one alternative over another. The membership function for the difference between each pair of final fuzzy evaluation values can be developed. Suppose $S_h$ and $S_k$, \( h, k = 1 \sim m \), are two fuzzy numbers from the developed model. The difference of these two fuzzy numbers can be denoted as $T_{hk} = S_h \ominus S_k$. The membership function is developed through $\alpha$-cuts and arithmetic operations as follows:

$$T_{hk}^\alpha = [T_{hkl}^\alpha, T_{hku}^\alpha]$$ (17)

where

$$S_h^\alpha = [S_{hl}^\alpha, S_{ku}^\alpha], S_k^\alpha = [S_{hq}^\alpha, S_{ku}^\alpha]$$

Assume:

$$E_{hk1} = \sum_{j=1}^{n} (a_{hj} - a_{kj})(p_j - o_j)^2 - \sum_{j=1}^{n} (b_{hj} - c_{kj})(p_j - q_j)^2$$

$$E_{hk2} = \sum_{j=1}^{n} (b_{hj} - c_{kj})(p_j - q_j)^2 - \sum_{j=1}^{n} (b_{hj} - a_{kj})(p_j - o_j)^2$$

$$F_{hk1} = \sum_{j=1}^{n} (a_{hj} - a_{kj})(p_j - o_j)^2 - \sum_{j=1}^{n} (b_{hj} - c_{kj})(p_j - q_j)^2$$

$$F_{hk2} = \sum_{j=1}^{n} (b_{hj} - c_{kj})(p_j - q_j)^2 - \sum_{j=1}^{n} (b_{hj} - a_{kj})(p_j - o_j)^2$$

$$V_h = \sum_{j=1}^{n} a_{hj}p_j, \quad Y_h = \sum_{j=1}^{n} b_{hj}p_j, \quad Z_h = \sum_{j=1}^{n} c_{hj}q_j$$

$$V_k = \sum_{j=1}^{n} a_{kj}p_j, \quad Y_k = \sum_{j=1}^{n} b_{kj}p_j, \quad Z_k = \sum_{j=1}^{n} c_{kj}q_j$$

By applying the above assumption and Eqs. (11)-(16) to Eq. (17), we have the following two simplified equations to solve:

$$E_{hk1} \alpha^2 + F_{hk1} \alpha + V_h - Z_k - x = 0$$

(18)

$$E_{hk2} \alpha^2 + F_{hk2} \alpha + Z_h - V_k - x = 0$$

(19)

The left and right membership functions of the difference $T_{hk}$ can be produced as follows:

$$f^L_{T_{hk}}(x) = \frac{-F_{hk1} + \sqrt{F_{hk1}^2 + 4E_{hk1}(x-V_h+Z_k)^2}}{2E_{hk1}}$$

$$f^R_{T_{hk}}(x) = \frac{-F_{hk2} + \sqrt{F_{hk2}^2 + 4E_{hk2}(x-Z_h+V_k)^2}}{2E_{hk2}}$$

(20)

(21)

Only two roots in $[0, 1]$ are retained in Eqs. (20) and (21). For convenience, $\alpha$-cuts between $T_{hk}$ can be denoted by:

$$T_{hk} = [V_h-Z_k, Z_h-V_k]$$

Proposition 1: \( f^L_{T_{hk}}(x) = 0 \) when \( x = V_h - Z_k \).

Proof:

$$f^L_{T_{hk}}(x) = \frac{-F_{hk1} + \sqrt{F_{hk1}^2 + 4E_{hk1}(x-V_h+Z_k)^2}}{2E_{hk1}}$$

(22)

Proposition 2: \( F_{hk1} + E_{hk1} = Y_h - Y_k - V_h + Z_k \).

Proof:

$$E_{hk1} + F_{hk1} = Y_h - Y_k - V_h + Z_k$$

(23)

Proposition 3: \( f^L_{T_{hk}}(x) = 1 \) when \( x = Y_h - Y_k \).

Proof:

$$f^L_{T_{hk}}(x) = \frac{-F_{hk1} + \sqrt{F_{hk1}^2 + 4E_{hk1}(x-V_h+Z_k)^2}}{2E_{hk1}}$$

(24)

(25)
Proposition 4: \( f^R_{ihk}(x) = 0 \) when \( x = Z_h - V_k \).

**Proof:** Similar to proposition 1. \( \square \)

Proposition 5: \( E_{hhk} + F_{hhk} = Y_h - Y_k - Z_h + V_k \).

**Proof:**

\[
E_{hhk} + F_{hhk} = \sum_{j=1}^{n} (b_{hj} - c_{hj})(p_j - q_j) - \sum_{j=1}^{n} (b_{hj} - a_{hj})(p_j - o_j) + \sum_{j=1}^{n} (c_{hj}(p_j - q_j) + q_j(b_{hj} - c_{hj})) - \sum_{j=1}^{n} a_{hj}(p_j - o_j) + o_j(b_{hj} - a_{hj})
\]

\[
= \sum_{j=1}^{n} b_{hj}p_j - \sum_{j=1}^{n} c_{hj}p_j - \sum_{j=1}^{n} b_{hj}q_j + \sum_{j=1}^{n} c_{hj}q_j
\]

\[
= \sum_{j=1}^{n} b_{hj}p_j + \sum_{j=1}^{n} a_{hj}o_j - \sum_{j=1}^{n} b_{hj}o_j - \sum_{j=1}^{n} a_{hj}o_j
\]

\[
= \sum_{j=1}^{n} b_{hj}p_j - \sum_{j=1}^{n} b_{hj}q_j - \sum_{j=1}^{n} c_{hj}q_j
\]

\[
= \sum_{j=1}^{n} b_{hj}p_j - \sum_{j=1}^{n} b_{hj}p_j - \sum_{j=1}^{n} c_{hj}q_j
\]

\[
= Y_h - Y_k - Z_h + V_k. \quad \square
\]

Proposition 6: \( f^R_{ihk}(x) = 1 \) when \( x = Y_h - Y_k \).

**Proof:**

\[
f^R_{ihk}(x) = \frac{-F^2_{hhk} - 4E_{hhk}^2(x-Z_h+V_k)}{2E_{hhk}}
\]

\[
= \frac{-F^2_{hhk} - 4E_{hhk}^2(Y_h-Y_k-Z_h+V_k)}{2E_{hhk}}
\]

\[
= \frac{-F^2_{hhk} - 4E_{hhk}^2(E_{hhk} + F_{hhk})}{2E_{hhk}}
\]

\[
= \frac{-F^2_{hhk} - 4E_{hhk}^2(F_{hhk} + F_{hhk})}{2E_{hhk}}
\]

\[
= \frac{-F^2_{hhk} - 4E_{hhk}^2(F_{hhk} + F_{hhk})}{2E_{hhk}} \quad \text{(by proposition 5)}
\]

\[
= \frac{-F^2_{hhk} - 4E_{hhk}^2(F_{hhk} + F_{hhk})}{2E_{hhk}} = 1. \quad \square
\]

**F. Develop dominance degree**

The dominance degree of one alternative over another is developed based on measuring the difference of one final fuzzy evaluation value of the corresponding alternative over another. Formulas for the dominance degree of one alternative over another can be clearly derived via integral procedure based on the developed membership functions in order to enhance applicability of the suggested model.

Herein the concept for measuring the difference of two triangular fuzzy numbers from the Chen work [9] is used. If \( T_{hhk} > 0 \) for \( \alpha \in [0,1] \), alternative \( A_h \) absolutely dominates \( A_k \). If \( T_{hhk} < 0 \) for \( \alpha \in [0,1] \), alternative \( A_h \) is absolutely dominated by \( A_k \). If \( T_{hhk} < 0 \) and \( T_{hhu} > 0 \) for some \( \alpha \) values, formulae for a dominance degree of \( A_h \) over \( A_k \), denoted by \( d_{hhk} \), can be produced through integral procedure [36, 37] as follows:

\[
d_{hhk} = \frac{P_1}{P} \quad (23)
\]

where

\[
P_1 = \int_{x < 0} f^R_{ihk}(x)dx, \quad P_2 = \int_{x > 0} f^R_{ihk}(x)dx, \quad P = P_1 + P_2, \quad P > 0.
\]

Obviously \( P_1 \) indicates the area that alternative \( A_h \) dominates alternative \( A_k \) and \( P_2 \) indicates the area that alternative \( A_h \) is dominated by alternative \( A_k \), and clearly \( d_{hhk} = 1 \). To develop the dominance degree, the following situations must be discussed.

**Situation 1:** \( Y_h - Y_k < 0 \), \( d_{hhk} = \frac{P_1}{P_1 + P_2} \).

\[
\text{Proof:}
\]

\[
f^R_{ihk}(x) = \frac{-F^2_{hhk} - 4E_{hhk}^2(x-Z_h+V_k)}{2E_{hhk}}
\]

\[
= \frac{-F^2_{hhk} - 4E_{hhk}^2(Y_h-Y_k-Z_h+V_k)}{2E_{hhk}}
\]

\[
= \frac{-F^2_{hhk} - 4E_{hhk}^2(E_{hhk} + F_{hhk})}{2E_{hhk}}
\]

\[
= \frac{-F^2_{hhk} - 4E_{hhk}^2(F_{hhk} + F_{hhk})}{2E_{hhk}} \quad \text{(by proposition 5)}
\]

\[
= \frac{-F^2_{hhk} - 4E_{hhk}^2(F_{hhk} + F_{hhk})}{2E_{hhk}}
\]

\[
= \frac{-F^2_{hhk} - 4E_{hhk}^2(F_{hhk} + F_{hhk})}{2E_{hhk}} = 1. \quad \square
\]

**Figure 2:** \( Y_h - Y_k < 0 \).

Situation 1 can be shown as in Figure 2. Formulas for \( P_1 \) and \( P_2 \) in situation 1 can be developed as follows:

\[
P_1 = \int_{x < 0} f^R_{ihk}(x)dx, \quad P_2 = \int_{x > 0} f^R_{ihk}(x)dx, \quad P = P_1 + P_2, \quad P > 0.
\]
\[
R_1 = \int_{0}^{Z_k-V_k} \frac{Z_k-V_k}{F_{hk1}^{-1}(x)} \, dx
\]

\[
= \frac{Z_k-V_k}{f_{hk1}^{-1}(x)} \left[ \frac{F_{hk1}^{2} + 4E_{hk1}(x-V_k + Z_k)}{2E_{hk1}} \right]^{1/2} \, dx
\]

\[
= \int_{0}^{Z_k-V_k} \frac{Z_k-V_k}{2E_{hk1}} \, dx - \int_{0}^{Z_k-V_k} \frac{Z_k-V_k}{2E_{hk1}} \, dx
\]

\[
= \frac{F_{hk1}^{3} - F_{hk1}^{3}(-z_h + V_k)}{2E_{hk1}^{3/2}} + \frac{F_{hk1}^{2} + 4E_{hk1}(Y_h - Y_k - V_h + Z_k)}{2E_{hk1}} \, dy
\]

Thus \( R_1 = \frac{-F_{hk1}(Z_h - V_k)}{2E_{hk1}} \)

\[
F_{hk1}^{3} - F_{hk1}^{3}(-z_h + V_k)
\]

\[
= 12E_{hk1}^{3/2}
\]

Similarly, \( P_2 \) can be developed as follows:

\[
P_2 = \int_{V_h-Z_k}^{Y_h-Y_k} \frac{V_h-Z_k}{f_{hk1}^{-1}(x)} - \int_{V_h-Z_k}^{Y_h-Y_k} \frac{V_h-Z_k}{f_{hk1}^{-1}(x)} \, dx
\]

\[
= \frac{Y_h-Y_k}{V_h-Z_k} - \frac{Y_h-Y_k}{V_h-Z_k} + \left[ \frac{F_{hk1}^{2} + 4E_{hk1}(x-V_h + Z_h)}{2E_{hk1}} \right]^{1/2} \, dy
\]

\[
= \frac{-F_{hk1}(Y_h - Y_k - V_h + Z_k)}{2E_{hk1}} \, dy
\]

\[
P_2 = \frac{-F_{hk1}(Y_h - Y_k - V_h + Z_k)}{2E_{hk1}} + \frac{F_{hk1}^{2} + 4E_{hk1}(Y_h - Y_k - V_h + Z_k)}{2E_{hk1}} \, dy
\]

Therefore, the dominance degree, \( d_{hk} = \frac{R_1}{P} = \frac{R_1}{R_1 + P_2} \), for situation 1, can be easily obtained by using Eq. (24) and Eq. (25).

Situation 2: \( Y_h - Y_k > 0 \), \( d_{hk} = \frac{R_1}{R_1 + P_2} \)

where

\[
R_1 = \frac{-F_{hk1}(Y_h - Y_k)}{2E_{hk1}} \]

\[
+ \left[ \frac{F_{hk1}^{2} + 4E_{hk1}(Y_h - Y_k - V_h + Z_k)}{2E_{hk1}} \right]^{3/2} - \frac{F_{hk1}^{2}}{2E_{hk1}} \]

\[
P_2 = \frac{-F_{hk1}(V_h - Z_h + Y_k)}{2E_{hk1}} \]

\[
+ \left[ \frac{F_{hk1}^{2} + 4E_{hk1}(Y_h - Y_k - V_h + Z_k)}{2E_{hk1}} \right]^{3/2} - \frac{F_{hk1}^{2}}{2E_{hk1}} \]

\[
= \frac{-F_{hk1}(V_h - Z_h + Y_k)}{2E_{hk1}} \]

\[
+ \left[ \frac{F_{hk1}^{2} + 4E_{hk1}(Y_h - Y_k - V_h + Z_k)}{2E_{hk1}} \right]^{3/2} - \frac{F_{hk1}^{2}}{2E_{hk1}} \]

\[
= \frac{-F_{hk1}(V_h - Z_h + Y_k)}{2E_{hk1}} \]

\[
+ \left[ \frac{F_{hk1}^{2} + 4E_{hk1}(Y_h - Y_k - V_h + Z_k)}{2E_{hk1}} \right]^{3/2} - \frac{F_{hk1}^{2}}{2E_{hk1}} \]

\[
= \frac{-F_{hk1}(V_h - Z_h + Y_k)}{2E_{hk1}} \]

\[
+ \left[ \frac{F_{hk1}^{2} + 4E_{hk1}(Y_h - Y_k - V_h + Z_k)}{2E_{hk1}} \right]^{3/2} - \frac{F_{hk1}^{2}}{2E_{hk1}} \]
Proof:

Situation 2 can be shown as in Figure 3. Formulas for $P_1$ and $P_2$ in situation 2 can be developed as follows:

\[
P_1 = \int_0^{Y_h - Y_k} f_{h_{kk}}^L (x) dx + \int_{Y_h - Y_k}^{\infty} f_{h_{kk}}^R (x) dx
\]

\[
P_2 = \int_{0}^{Y_h - Z_k} f_{h_{kk}}^L (x) dx
\]

\[
P_2 = \int_{Y_h - Z_k}^{\infty} f_{h_{kk}}^R (x) dx
\]

\[
\frac{1}{2}E_{h_{kk}}^2 - \left[ \frac{F_{h_{kk}}^2 + 4E_{h_{kk}} (Y_h - Y_k - Z_k + V_k)}{12E_{h_{kk}}^2} \right]^{3/2}
\]

\[
\frac{1}{2}E_{h_{kk}}^2 - \left[ \frac{F_{h_{kk}}^2 + 4E_{h_{kk}} (Y_h - Y_k - Z_k + V_k)}{12E_{h_{kk}}^2} \right]^{3/2}
\]

Therefore, the dominance degree, $d_{hk} = \frac{P_1}{P} = \frac{R_1}{R_1 + P_2}$, for situation 2, can be easily obtained by using Eq. (26) and Eq. (27).

Situation 3: $Y_h - Y_k = 0$, $d_{hk} = \frac{R_1}{R_1 + P_2}$.

Proof:

Situation 3 can be shown as in Figure 4. Formulas for $P_1$ and $P_2$ in situation 3 can be developed as follows:

\[
P_1 = \int_0^{Z_h - V_k} f_{h_{kk}}^L (x) dx
\]

\[
P_2 = \int_{0}^{Z_h - V_k} f_{h_{kk}}^R (x) dx
\]

\[
\frac{1}{2}E_{h_{kk}}^2 - \left[ \frac{F_{h_{kk}}^2 + 4E_{h_{kk}} (Y_h - Y_k - Z_k + V_k)}{12E_{h_{kk}}^2} \right]^{3/2}
\]
committee of three decision-makers,

After an initial screening, four alternative locations are considered. The company needs to select a location to construct a plant. A hypothetical facility location selection problem is formed to determine the most suitable alternative location. Further assume that five criteria are selected and categorized by the decision makers as shown in Table 1. Furthermore, assume that the decision-makers employ the linguistic values and their corresponding fuzzy numbers shown in Fig. 5 to evaluate the suitability of alternative locations versus different qualitative criteria as shown in Tables 3-5. By Eq. (7), the average ratings of criteria can be produced and are also displayed in Tables 3-5. The evaluation data of alternative locations versus quantitative criteria are shown in Table 6. And by Eq. (8), the normalized data is also shown in Table 6.

By Eq. (6), the average weights of criteria can be obtained and also shown in Table 2. Meanwhile, the decision-makers employ the linguistic values and corresponding fuzzy numbers shown in Fig. 1 in Section 2.5 to evaluate the suitability of alternative locations versus different qualitative criteria as shown in Tables 3-5. By Eq. (7), the average ratings of criteria can be produced and are also displayed in Tables 3-5. The evaluation data of alternative locations versus quantitative criteria are shown in Table 6. And by Eq. (8), the normalized data is also shown in Table 6.

4. Numerical Example

A hypothetical facility location selection problem is applied to demonstrate the computational process of the proposed model. Assume that a high-tech manufacturing company needs to select a location to construct a plant. After an initial screening, four alternative locations A₁, A₂, A₃ and A₄ are chosen for further evaluation. A committee of three decision-makers, D₁, D₂ and D₃ is formed to determine the most suitable alternative location. Further assume that five criteria are selected and categorized by the decision makers as shown in Table 1. Furthermore, assume that the decision-makers employ the linguistic values and their corresponding fuzzy numbers shown in Fig. 5 to evaluate the importance weights of the criteria as shown in Table 2.

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By Eqs. (9)-(16), $P_i$, $i=1\sim4$, can be obtained and displayed as in Table 7. By Eqs. (17)-(22), the difference of two final fuzzy evaluation values $T_{hk}$, $h,k=1\sim4$, can be obtained and displayed as in Table 8. By Eqs. (23)-(29), the dominance degree, $d_{hk}$, $h,k=1\sim4$, can be obtained and displayed as in Table 9.

Table 5. Ratings of locations under $C_3$ and average ratings.

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>Decision makers</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>Average ratings ($x_{i}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>VG</td>
<td>$E$</td>
<td>$VG$</td>
<td></td>
<td>(0.5667, 0.7667, 0.9333)</td>
</tr>
<tr>
<td>$A_2$</td>
<td>VG</td>
<td>$E$</td>
<td>$G$</td>
<td></td>
<td>(0.5000, 0.7000, 0.8667)</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$G$</td>
<td>$VG$</td>
<td>$B$</td>
<td>$B&amp;G$</td>
<td>(0.3000, 0.5000, 0.7000)</td>
</tr>
<tr>
<td>$A_4$</td>
<td>$G$</td>
<td>$E$</td>
<td>$G$</td>
<td></td>
<td>(0.4333, 0.6333, 0.8000)</td>
</tr>
</tbody>
</table>

By Eqs. (9)-(16), $P_i$, $i=1\sim4$, can be obtained and displayed as in Table 7. By Eqs. (17)-(22), the difference of two final fuzzy evaluation values $T_{hk}$, $h,k=1\sim4$, can be obtained and displayed as in Table 8. By Eqs. (23)-(29), the dominance degree, $d_{hk}$, $h,k=1\sim4$, can be obtained and displayed as in Table 9.

Table 6. Quantitative values of locations and normalized values.

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>Community attitude (0-100 points), $C_i$</th>
<th>Investment cost (million $), C_i$</th>
<th>Normalized points, $x_{i}$</th>
<th>Normalized costs, $y_{i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>93</td>
<td>(48,53,58)</td>
<td>0.168</td>
<td>0.880</td>
</tr>
<tr>
<td>$A_2$</td>
<td>88</td>
<td>(46,50,54)</td>
<td>0.5</td>
<td>0.381</td>
</tr>
<tr>
<td>$A_3$</td>
<td>83</td>
<td>(52,56,62)</td>
<td>0</td>
<td>0.000</td>
</tr>
<tr>
<td>$A_4$</td>
<td>89</td>
<td>(41,45,48)</td>
<td>0.6</td>
<td>0.667</td>
</tr>
</tbody>
</table>

Table 7. Final fuzzy evaluation values of alternatives.

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>$S_1$</th>
<th>$V_1$</th>
<th>$Y_1$</th>
<th>$Z_1$</th>
<th>$A_1$</th>
<th>$E_1$</th>
<th>$F_1$</th>
<th>$E_2$</th>
<th>$F_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>1.470</td>
<td>2.518</td>
<td>3.583</td>
<td>0.168</td>
<td>0.880</td>
<td>0.132</td>
<td>-1.197</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_2$</td>
<td>1.158</td>
<td>2.110</td>
<td>3.088</td>
<td>0.158</td>
<td>0.794</td>
<td>0.120</td>
<td>-1.099</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.392</td>
<td>1.089</td>
<td>1.996</td>
<td>0.177</td>
<td>0.520</td>
<td>0.131</td>
<td>-1.038</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_4$</td>
<td>1.210</td>
<td>2.192</td>
<td>3.220</td>
<td>0.149</td>
<td>0.833</td>
<td>0.116</td>
<td>-1.144</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 8. The difference of two final fuzzy evaluation values.

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>$Y_k - Y_{k-1}$</th>
<th>$Z_k - Z_{k-1}$</th>
<th>$E_{ik1}$</th>
<th>$E_{ik2}$</th>
<th>$F_{ik1}$</th>
<th>$F_{ik2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{12}$</td>
<td>-1.618</td>
<td>0.408</td>
<td>2.426</td>
<td>0.047</td>
<td>1.979</td>
<td>-0.026</td>
</tr>
<tr>
<td>$T_{13}$</td>
<td>-0.526</td>
<td>1.429</td>
<td>3.191</td>
<td>0.036</td>
<td>1.918</td>
<td>-0.045</td>
</tr>
<tr>
<td>$T_{14}$</td>
<td>-1.750</td>
<td>0.326</td>
<td>2.373</td>
<td>0.052</td>
<td>2.024</td>
<td>-0.017</td>
</tr>
<tr>
<td>$T_{23}$</td>
<td>-0.838</td>
<td>1.020</td>
<td>2.696</td>
<td>0.027</td>
<td>1.832</td>
<td>-0.057</td>
</tr>
<tr>
<td>$T_{24}$</td>
<td>-2.062</td>
<td>-0.082</td>
<td>1.878</td>
<td>0.042</td>
<td>1.938</td>
<td>-0.028</td>
</tr>
<tr>
<td>$T_{34}$</td>
<td>-2.828</td>
<td>-1.103</td>
<td>0.786</td>
<td>0.061</td>
<td>1.664</td>
<td>-0.017</td>
</tr>
</tbody>
</table>

Table 9. Dominance degree of one alternative over another.

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>$Y_k - Y_{k-1}$</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P$</th>
<th>$n_{i1}/P_i$</th>
<th>$n_{i2}/P_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{i2}$</td>
<td>&gt;0</td>
<td>1.381</td>
<td>0.653</td>
<td>2.034</td>
<td>0.679</td>
<td>0.321</td>
</tr>
<tr>
<td>$d_{i3}$</td>
<td>&gt;0</td>
<td>1.800</td>
<td>0.072</td>
<td>1.872</td>
<td>0.962</td>
<td>0.038</td>
</tr>
<tr>
<td>$d_{i4}$</td>
<td>&gt;0</td>
<td>1.590</td>
<td>0.191</td>
<td>1.781</td>
<td>0.893</td>
<td>0.107</td>
</tr>
<tr>
<td>$d_{i4}$</td>
<td>&lt;0</td>
<td>1.378</td>
<td>0.695</td>
<td>2.073</td>
<td>0.665</td>
<td>0.335</td>
</tr>
<tr>
<td>$d_{i4}$</td>
<td>&lt;0</td>
<td>0.904</td>
<td>1.077</td>
<td>1.982</td>
<td>0.456</td>
<td>0.544</td>
</tr>
<tr>
<td>$d_{i4}$</td>
<td>&lt;0</td>
<td>0.165</td>
<td>1.655</td>
<td>1.820</td>
<td>0.090</td>
<td>0.910</td>
</tr>
</tbody>
</table>

By the ranking approach in Section 3.7, the ordering of the four alternatives can be determined as follows:

$$d(A_1) = \text{Max} \{d_{12}, d_{13}, d_{14}\} = \text{Max} \{0.679, 0.962, 0.665\} = 0.962;$$
$$d(A_2) = \text{Max} \{d_{21}, d_{23}, d_{24}\} = \text{Max} \{0.321, 0.893, 0.456\} = 0.893;$$
$$d(A_3) = \text{Max} \{d_{31}, d_{32}, d_{34}\} = \text{Max} \{0.038, 0.107, 0.090\} = 0.107;$$
$$d(A_4) = \text{Max} \{d_{41}, d_{42}, d_{43}\} = \text{Max} \{0.335, 0.544, 0.910\} = 0.910.$$

Obviously, $d(A_1) > d(A_2) > d(A_3)$ because 0.962 > 0.910 > 0.893 > 0.107. Therefore the ranking ordering of the four alternatives is $A_1 > A_4 > A_2 > A_3$.

5. Conclusions

A fuzzy MCDM model with a fuzzy number dominance based ranking procedure has been proposed to order alternatives. Formulas for the dominance degree of one alternative over another can be clearly derived via integral process based on the developed membership functions for the difference of each pair of final fuzzy evaluation values of the corresponding alternatives. The ranking ordering of the alternatives can then be determined by a proposed simple ranking procedure based on those dominance degrees. The proposed model connects the final fuzzy evaluation values from a fuzzy MCDM and the ranking procedure through clearly developed formulae, efficiently ordering the alternatives. Finally a numerical example of facility location selection problem has demonstrated the feasibility of the proposed model. The proposed model can also be applied to other management problems under fuzzy MCDM environment.

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