Fuzzy Splines and Their Applications to Interpolate Fuzzy Data

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Abstract

One of the important methods in many practical areas is interpolation method. Clearly, in many real world problems, the input and output data are involved with uncertainty. So, we must develop interpolation methods for fuzzy data. In this paper, first, we define a new set of spline functions to interpolate given fuzzy data. Then we discuss some important theorems on these splines together with their existence and uniqueness properties. Also, we present numerical examples to illustrate the differences between using our spline and other interpolations that have been studied before.

Keywords: Fuzzy interpolation, Fuzzy Spline, Extension principle.

1. Introduction

The following problem was first posed by L. A. Zadeh, see for example [11]. Suppose that we have \( n+1 \) distinct real numbers \( x_0, x_1, \ldots, x_n \), and for each of these numbers a fuzzy value in \( \mathbb{R} \), rather than a crisp value, is given. Zadeh asked the question whether it is possible to construct some kind of smooth function on \( \mathbb{R} \) to fit the collection of fuzzy data at these \( n+1 \) points.

Lagrange interpolation for fuzzy data was first investigated by Lowen [11]. Later, Kaleva [8] avoided the well-known computational troubles associated with crisp Lagrange interpolation by using linear and not-a-knot cubic spline approximations. If the fuzzy data are not convex, then a technical difficulty arises and in this case, the Bernstein approximation can be constructed (see, for example, Diamond and Ramer [5]). The interpolation of fuzzy data by using spline functions of odd degree was considered in [1] with complete splines, in [2] with natural splines, and in [4] with fuzzy splines, and finally in [3] with \( E(3) \) cubic splines. Constructing consistent fuzzy surfaces from fuzzy data in the sense of Lagrange polynomials, linear splines and not-a-knot cubic splines were described in [10]. Recently, the authors of [14] applied piecewise quartic polynomials induced from a \( E(3) \) cubic spline to approximate fuzzy data. For other works, see [15-18].

In this paper, in Section 2, we will introduce a new set of fuzzy splines to interpolate the fuzzy data. Then some important theorems on these splines together with their existence and uniqueness properties will be discussed. Finally, in Section 4, to illustrate the differences between of using our splines and other interpolations that have been studied before, some numerical examples will be presented.

2. Preliminaries

In this section, we recall some Fundamental results of fuzzy numbers and fuzzy interpolations.

Definition 1: A fuzzy number is a mapping \( u: \mathbb{R} \rightarrow I = [0,1] \) with the following properties, see [9, 13]:

1. \( u \) is an upper semi-continuous function on \( \mathbb{R} \),
2. \( u(x) = 0 \) outside of some interval \([c,d] \subset \mathbb{R}\), and
3. there exist real numbers \( a, b \), such that \( c \leq a \leq b \leq d \), and
   1. \( u(x) \) is a monotonic increasing function on \([c,a]\)
   2. \( u(x) \) is a monotonic decreasing function on \([b,d]\)
   3. \( u(x) = 1 \), for all \( x \) in \([a,b]\).

The set of all fuzzy numbers is denoted by \( \mathcal{F} \). The popular type of fuzzy number is the set of triangular fuzzy number \( u = (c, \alpha, \beta) \) defined by

\[
\begin{align*}
\alpha & \leq x \leq c, \\
c - \alpha & \leq x < c, \\
\frac{c - x}{\beta} & \leq x < c + \beta, \\
0 & \quad \text{otherwise},
\end{align*}
\]

where \( \alpha > 0 \) and \( \beta > 0 \). Note that the triangular fuzzy numbers are special cases of \( L-L \) fuzzy numbers, see[6].

Definition 2: If \( u \in \mathcal{F} \) then the \( \alpha \)-level set of \( u \) is denoted by \( [u]^\alpha \) and defined by \( [u]^\alpha = \{ x \in \mathbb{R} | u(x) \geq \alpha \} \), where \( 0 < \alpha \leq 1 \). Also, \( [u]^0 \) is called the support of \( u \) and it is
given by \( \bigcup_{\alpha \in [0,1]} [u]^\alpha \). It follows that the level sets of \( u \) are closed and bounded intervals in \( R \).

It is well-known that the addition and multiplication operations of real numbers can be extended to \( \mathcal{F} \). In other words, for any \( 0 < \alpha \leq 1 \), \( \lambda \in R \) and \( u, v \in \mathcal{F} \), we have:
\[
[u + v]^\alpha = [u]^\alpha + [v]^\alpha \quad \text{and} \quad [\lambda u]^\alpha = \lambda [u]^\alpha.
\]
Consider \( n+1 \) distinct real numbers \( x_0 \leq x_1 \leq \ldots \leq x_n \). For each \( x_i \) we associate a fuzzy number \( u_i \in \mathcal{F} \). To solve Zadeh’s problem, we must find a continuous interpolating fuzzy number \( [u] \). Let \( P_{y_0, y_1, \ldots, y_n}(x) \) be the Lagrange interpolation polynomial of degree \( n \) which interpolates the data \((x_i, y_i)\); \( i = 0, 1, \ldots, n \). According to the extension principle [6], we can write the membership function \( F(x) \) for each \( x \in R \) as follows:
\[
\mu_{F(x)}(t) = \begin{cases} 
\sup_{t \in P_{y_0, y_1, \ldots, y_n}(x)} \min_{0 \leq \alpha \leq 1} \mu_\alpha(y_i); & \text{if } P_{y_0, y_1, \ldots, y_n}(t) \neq 0, \\
0, & \text{otherwise}
\end{cases}
\]
where \( \mu_\alpha \) is the membership function of \( u_\alpha \). For each \( \alpha \in (0,1] \) and \( i = 0, 1, \ldots, n \), let \( J_\alpha^i = [u_\alpha]^\alpha = \mu_\alpha(\alpha, 1], \) and \( F^\alpha(x) \) be the \( \alpha \)-level sets of \( u_\alpha \) and \( F(x) \), respectively. Hence,
\[
F^\alpha(x) = \{ t \in R \mid \mu_{F(x)}(t) \geq \alpha \} = \{ t \in R \mid \exists y \in \prod_{i=0}^{n} J_\alpha^i : P_{y_0, y_1, \ldots, y_n}(x) = t \}
\]
where \( P_{y_0, y_1, \ldots, y_n}(x) = t \) is an \( n \)-fold zero of \( F(x) \).

3. Fuzzy Splines of Odd Degree

Definition 3: A function \( s : [x_0, x_n] \rightarrow R \) is called a spline of odd degree \( l = 2m - 1 \) with \( m = 2k \), \( k \geq 1 \), provided that it possesses the following properties:

a) \( s \in C^{l-1}([x_0, x_n]) \)

b) \( s(x) \) is a polynomial of degree \( L \) for \( x \in [x_i, x_{i+1}] \), \( i = 0, 1, \ldots, n - 1 \)

c) \( s^{(i)}(x_i) = 0, \nu = 1, \ldots, m - 1 \)

d) \( s^{(i)}(x_n) = 0, \nu = \frac{m - 1}{2}, \ldots, \frac{3m - 2}{2} - 2 \)

We denote the family of these splines by \( S_i \in [x_0, x_n] \). If \( s_i \in S_i \) interpolates the data \((x_j, f_j), j = 0, \ldots, n \), where \( f_j = \delta_j \), Kronecker delta, then
\[
S_{j_0, j_1, \ldots, j_n}(x) = \sum_{i=0}^{n} s_i(x) y_i,
\]
where \( S_{j_0, j_1, \ldots, j_n}(x) \in S_i(x_0, x_n) \), interpolates \((x_j, f_j), i = 0, \ldots, n \). Hence from Section 2 we have
\[
F^\alpha(x) = \{ t \in R \mid \exists y \in \prod_{i=0}^{n} J_\alpha^i : S_{j_0, j_1, \ldots, j_n}(x) = t \} = \sum_{i=0}^{n} s_i(x) J_\alpha^i,
\]
where \( F(x) = \sum_{i=0}^{n} s_i(x) u_i \) if all \( u_i \) are \( L-L \) fuzzy numbers, then \( F(x) \) is an \( L-L \) fuzzy number for all \( x \in [x_0, x_n] \). Here we take \( u_i = (m, l, r_i) \) as a triangular fuzzy number.

Definition 4: A point \( \alpha \in [x_i, x_{i+1}] \subset [x_0, x_n] \), \( 0 \leq \nu \leq n - 1 \) is called an essential zero of a spline \( s \in S_i(x_0, x_n) \), provided that \( s(\alpha) = 0 \), but \( s(x) \neq 0 \) on \([x_0, x_{i+1}]\), see [7].

Definition 5: The number of all essential zeros of \( s \) in \([x_0, x_n]\) is denoted by \( Z(s) \), where each zero is counted according to its multiplicity, see [7].

Theorem 1 [7]: If \( s \in S_i(x_0, x_n) \), \( Z(s) \), the number of essential zeros of \( s \) in \([x_0, x_n]\) is at most \( n + l - 1 \), where each zero is counted according to its multiplicity. If \([x_0, x_{i+1}]\) is a maximal subinterval where \( s \) vanishes, then \( x_{i+1} \) is an \( l \)-fold essential zero of \( s \).

Theorem 2: If \([x_0, x_n]\) is a maximal interval where \( s_i \in S_i(x_0, x_n) \) vanishes identically and nowhere inside subintervals \([x_k, x_{k+1}]\) for \( k > \sigma \), then \( s_i^{(2m-2)} \) has at least \( n - \sigma + m - 1 \) essential zeros on \([x_0, x_n]\).
Proof: Since \( s_i(x_i) = 1 \) we have \( \sigma < i \) and hence \( s_j(x) = 0 \) for \( x = x_j, j = 0, \ldots, i-1, i+1, \ldots, n \).

On the other hand, Theorem 1 expresses \( x_\alpha \) as an \( l \)-fold zero of \( s_i \) and using Rolle’s Theorem \( s_i \) has at least \( n-\sigma \) essential zeros on \([x_\sigma, x_n] \). By repeating this argument it can be seen \( s_i^{(k)} \) has at least \( n-\sigma \) zeros in the interval \([x_\sigma, x_n] \) for \( k = 2, \ldots, \frac{m}{2} - 1 \). On one side, using Rolle’s Theorem and knowing \( x_\sigma \) is an \( l \)-fold zero of \( s_i \), \( s_i^{(k)} \) has at least \( n-\sigma + k + 1 \) zero in the interval \([x_\sigma, x_n] \) for \( k = 0, 1, \ldots, m-2 \), because \( x_\sigma \) justifies in the part of (d) of the Definition 3. By virtue of Rolle’s Theorem, the functions \( s_i^{(2m-2)}, \ldots, s_i^{(2m-2)-1} \) have at least \( n-\sigma + m - 1 \) essential zeros on \([x_\sigma, x_n] \) because \( x_\sigma \) is an \( l \)-fold zero of \( s_i \).

Theorem 3: For all \( s_i \in S_i(x_\sigma, x_n), i=0, \ldots, n \):

(a) \( s_i \) is not identically zero on any subinterval \([x_j, x_{j+1}] \);
(b) the sign of \( s_i \) does not change on \([x_j, x_{j+1}] \);
(c) the sign of \( s_i \) changes at \( x_j \) for all \( j \neq i \).

Proof: Suppose that \( s_i(x_i) = 0 \) for each \( x \in [x_j, x_{j+1}] \). We assume \( j+1 < i \) and prove the theorem. We have a similar proof for \( j > i \). Let

\[
s(x) = \begin{cases} 
0, & x_0 \leq x \leq x_{j+1}; \\
s_i(x), & x_{j+1} < x \leq x_n.
\end{cases}
\]

Clearly \( s \in S_i(x_\sigma, x_n) \). By the uniqueness of the spline, \( s_i(x) = 0 \) for all \( x \in [x_\sigma, x_{j+1}] \). Let \( x_\sigma, x_n \) be the maximal interval where \( s_i(x) \) vanishes. Since \( s_i(x) = 1 \), we have \( \sigma < i \).

Similarly suppose \( [x_\tau, x_\alpha], \tau > i \), be the maximal interval such that \( s_i(x) = 0 \), for all \( x \in [x_\tau, x_\alpha] \). Theorem 1 is applied to \( s_i \), restricted to \([x_\sigma, x_\tau] \). We consider two following cases:

If \( \tau < n \), then \( x_\sigma, x_\tau \) are \( l \)-fold zeros of \( s_i \) and since \( x_i \) is not a zero, by Theorem 1,

\[
2l + (\tau - \sigma - 1) - 1 \leq Z(s_i(\tau)) \leq (\tau - \sigma) + l - 1
\]

which gives \( l \leq 1 \) contradicting \( l \geq 3 \).

If \( \tau = n \), by Theorem 2, \( s_i^{(2m-2)} = s_i(x_\sigma, x_n) \) and by Theorem 1,

\[
n - \sigma + m - 1 \leq Z(s_i^{(2m-2)}) \leq n - \sigma,
\]

which gives \( m \leq 1 \), a contradiction. Hence (a) is proved.

Suppose \( r \) be the number of zeros of \( s_i \) on \([x_\sigma, x_n] \), \( r \geq n \). By Rolle’s Theorem and knowing \( x_0 \) is an \((m-1)\)-fold zero of \( s_i \), the function \( s_i^{(k)} \) has at least \( r \) essential zeros on \([x_\sigma, x_n] \) for \( k = 1, \ldots, m \). Now by using Rolle’s Theorem and parts (c), (d) of Definition 3, \( s_i^{(k+1)} \) has at least \( r + k + 1 \) essential zeros on \([x_\sigma, x_n] \) for \( k = 0, \ldots, m - 1 \). The functions \( s_i^{(m)}, \ldots, s_i^{(2m-2)} \) have at least \( r + \frac{m}{2} \) zeros on \([x_\sigma, x_n] \) because of Rolle’s Theorem and part (d) of Definition 3. From \( s_i^{(2m-2)} \) to \( s_i^{(2m-2)-1} \), because of Rolle’s Theorem in each time one zero is omitted. So \( s_i^{(2m-2)} \) has at least \( r \) essential zero on \([x_\sigma, x_n] \). On other hand, \( s_i \in S_i(x_\sigma, x_n) \) and by Theorem 1

\[
n \leq r \leq Z(s_i^{(2m-2)}) \leq n.
\]

We conclude that the essential zeros of \( s_i \) are \( x_j \) for \( i \neq j \) with multiplicity one. This gives (b) and (c). \( \Box \)

Theorem 4: If \( F(x) = \sum_{j=0}^{n} s_i(x)u_i \) is an interpolating fuzzy spline, then for \( x \in (x_i, x_{i+1}) \) and for all \( \alpha \in [0,1] \),

\[
\text{len}^\alpha F(x) \geq \min \{ \text{len}^\alpha s_i(x), \text{len}^\alpha F(x_{i+1}) \},
\]

where ‘len’ denotes the length of an interval.

Proof: Since the addition does not decrease the length of an interval we have

\[
\text{len}^\alpha F(x) \geq \min \left\{ \text{len}^\alpha \sum_{j=0}^{n} s_i(x)J_j^\alpha \right\}
\]

\[
\geq \min \left\{ \text{len}^\alpha s_i(x), \text{len}^\alpha F(x_{i+1}) \right\} \sum_{j=0}^{n} \left[ s_j(x) \right] J^\alpha
\]

If we show \( s(x) = s_i(x) + s_{i+1} \geq 1 \), for all \( x \in (x_i, x_{i+1}) \), the proof will be complete.

The polynomial \( s(x) \in S_i(x_\sigma, x_n) \) interpolates the data \((x_j, f_j)\), where \( f_j = 1 \) for \( j = i, i+1 \) and zero otherwise. Suppose that \( 0 < i < n - 1 \) and \( s(x) < 1 \) for some \( x \in (x_i, x_{i+1}) \), then \( s'(x) \) has at least three zeros in \((x_i, x_{i+1}) \) and using Rolle’s Theorem and Definition 3, the functions \( s^{(k)}(s) \) has at least \( n+1 \) zeros on \([x_\sigma, x_n] \) for \( k = 1, \ldots, \frac{m}{2} - 1 \). By considering Rolle’s Theorem and Definition 3, \( s^{(k+1)}(s) \) has at least \( n+k+2 \) zeros on
\[ [x_0, x_n] \text{ for } k = 0, 1, \ldots, \frac{m-1}{2}. \] The functions 
\[ S^{(n)}(x), \ldots, S^{(\frac{m-2}{2})}(x) \] have at least \( n + \frac{m}{2} + 1 \) zeros on 
\[ [x_0, x_n], \] because of Rolle’s Theorem and parts (c), (d) of Definition 3. From \( S^{(\frac{m-2}{2})}(x) \) to \( S^{(\frac{m-2}{2})}(x) \), because of Rolle’s Theorem, in each time one zero is omitted. So 
\( S^{(\frac{m-2}{2})}(x) \) has at least \( n + 1 \) essential zero on 
\[ [x_0, x_n], \] which is a contradiction, since \( (2,2,10)(x,y) \) and \( l = 3 \).

**Theorem 5:** For any given function \( y = f(x) \) defined at 
\[ \{x_i\}_{i=0}^{n}, \] there exist a unique spline function 
\[ S(x) \in S_i(x_0, x_n), \] which interpolates the function values 
\( y_i = f(x_i). \)

**Proof:** The proof is similar to the proof of Theorem 5 in [4].

### 4. Numerical Examples

Let \( J^a = [a_i^a, b_i^a] \), then the upper end point of 
\( F^a(x) \) is the solution of following problem:

Maximize \( S_{x_0,\ldots,x_n}(x) \)

subject to \( a_i^a \leq y_j \leq b_i^a; \) \( i = 0, \ldots, n. \)

where the optimal solution is

\[
y_i = \begin{cases} 
  b_i^a & \text{if } s(x) \geq 0 \\
  a_i^a & \text{if } s(x) < 0 .
\end{cases}
\]

Similarly the lower end point of \( F^a(x) \) is obtained. Hence if \( u_i = (m_i, l_i, r_i) \) and \( F(x) = (m(x), l(x), r(x)) \), we have

\[
m(x) = \sum_{i=0}^n s_i(x) m_i, \\
l(x) = \sum_{i=0}^n s_i(x) l_i - \sum_{i=0}^n s_i(x) r_i, \\
r(x) = \sum_{i=0}^n s_i(x) r_i - \sum_{i=0}^n s_i(x) l_i,
\]

as in [4].

**Example 1:** Suppose we have the data \((x_i, u_i)\) in Table 1 and \( l = 3. \) For example,
\[
F(2.2) = (2.62744, 3.89523, 4.89523), \\
F(3.1) = (-2.30624, -0.369895, 0.630105).
\]
Fig. 1 shows the zero, 0.5 and one level sets.

**Example 2:** Suppose we have the data \((x_i, u_i)\) in Table 2 and \( l = 3, \) i.e. using cubic spline, as Example 2.1 of Kaleva [8]. For more details, see Fig. 2.

**Example 3:** Suppose we have the data \((x_i, u_i)\) and \( l = 3, \)

The solid lines, the dashed lines and the thick line represent the support, 0.5-level set and 1-level set of \( F(x). \)

![Figure 1](image1.png)

Figure 1. The solid lines, the dashed lines and the thick line represent the support, 0.5-level set and 1-level set of \( F(x). \)

**Example 2:** Suppose we have the data \((x_i, u_i)\) in Table 2 and \( l = 3, \) i.e. using cubic spline, as Example 2.1 of Kaleva [8]. For more details, see Fig. 2.

**Table 2. The data of Example 2.**

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>1</th>
<th>1.1</th>
<th>1.2</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_i )</td>
<td>0</td>
<td>5</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( l_i )</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>( r_i )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

![Figure 2](image2.png)

Figure 2. The solid lines, the dashed lines and the thick line represent the support, 0.5-level set and 1-level set of \( F(x). \)

**Example 3:** Suppose we have the data \((x_i, u_i)\) and \( l = 3, \)

To compare the results of \( F(x) \) with fuzzy cubic spline with the not-a-knot condition [8] and \( E(3) \) fuzzy spline [3], see Figs. 3-5.
Figure 3. The solid lines represent the support and the dashed lines represent 0.5-level set and the thick line represents 1-level set of $F(x)$.

Figure 4. The solid lines represent the support and the dashed lines represent 0.5-level set and the thick line represents 1-level set of fuzzy cubic Spline with the not-a-knot condition[8].

Figure 5. The solid lines represent the support and the dashed lines represent 0.5-level set and the thick line represents 1-level set of $E(3)$ fuzzy spline [3].

5. Conclusions

In this paper, a new set of spline functions to interpolate given fuzzy data is defined. It is proved that the sign of defined splines, $s_j$, does not change on subintervals. Also, to illustrate the differences between using defined spline in this paper and other interpolations that have been studied before, numerical examples are presented.

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References


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