On Interval-Valued Fuzzy Metric Spaces

Yonghong Shen, Haifeng Li, and Faxing Wang

Abstract

In this paper, following the ideas of (continuous) \(t\)-norm and interval numbers, a concept of (continuous) interval-valued \(t\)-norm is proposed. Based on the interval-valued fuzzy set and the continuous interval-valued \(t\)-norm, we propose a notion of interval-valued fuzzy metric space, which is a generalization of fuzzy metric space in the sense of George and Veeramani [Fuzzy Sets and Systems 64 (1994): 395-399]. Meanwhile, we show that each metric induces an interval-valued fuzzy metric in certain conditions. Finally, we define a Hausdorff topology on an interval-valued fuzzy metric space and generalize some well-known conclusions of general metric spaces.

Keywords: Interval numbers, Interval-valued fuzzy sets, Interval-valued \(t\)-norm, Interval-valued fuzzy metric spaces, Topology, Complete.

1. Introduction

Fuzzy metric is regarded as a generalization of usual metric and it plays an important role in the study of fuzzy topology. In 1975, Kramosil and Michalek [1] first introduced the notion of fuzzy metric inspired by statistical metric spaces. Afterwards, Erceg [2] presented two kinds of notions of pseudo-metric and metric on the fuzzy set as another generalization of usual metric in 1979. In 1982, Deng [3] extended pseudo metric to fuzzy pseudo metric and introduced the concept of pseudo metric space. Soon after, the concept of fuzzy metric space was formally proposed by Kaleva and Seikkala [4], and by Abu Osman [5], respectively, by using two different ways. In [4], Kaleva and Seikkala made use of a non-negative, upper semi-continuous, normal and convex fuzzy number to measure the distance between two points in a fuzzy metric space. In [5], Abu Osman defined the fuzzy metric between two fuzzy sets by means of their membership functions. Obviously, the above notions are different from that proposed by Kramosil and Michalek. Later, some problems related to the fuzzy metric space has been widely and deeply investigated by many authors [6-9] from different points of view. With the help of continuous \(t\)-norm, Grabiec [9] redefined the fuzzy metric introduced by Kramosil and Michalek and extended the well-known fixed point theorems of Banach and Edelstein to fuzzy metric spaces. In [7, 8], in order to introduce a Hausdor topology on the fuzzy metric space, George and Veeramani modified the definition of the fuzzy metric space given by Grabiec and proved some known results of metric spaces including Baire's theorem for fuzzy metric spaces. Furthermore, a necessary and sufficient condition for the fuzzy metric space to be complete is presented, and the uniform limit theorem is generalized to fuzzy metric spaces. In [6], Chakrabarty defined some concepts associated with fuzzy topology and presented some properties for fuzzy metric spaces in the sense of Abu Osman.

As a natural generalization of the fuzzy metric space, in recent years, the concept of intuitionistic fuzzy metric space was proposed and studied by many authors [10-18]. We noticed that this space is constructed on the basis of continuous \(t\)-norm. Meanwhile, some main results of fuzzy metric spaces are extended in previous literature. In particular, Park [10] first defined the notion of intuitionistic fuzzy metric space and generalized some well-known results of fuzzy metric spaces, such as Baire's theorem, uniform limit theorem, etc.

The concept of interval-valued fuzzy set was introduced by Zadeh in 1975 [19]. An interval-valued fuzzy set is characterized by an interval-valued membership function, and it is taken as another generalization of the fuzzy set. In 2009, Li [20] introduced three kinds of distances between two interval-valued fuzzy sets (or numbers) defined on real line \(\mathbb{R}\). Moreover, he noted that each kind of distance is a metric on the corresponding sets and each interval-valued fuzzy numbers metric space is complete. Obviously, the idea of above defined metric space is different from that of George and Veeramani [7]. It should be noted that George and Veeramani applied fuzzy set to express the uncertainty of the dis-
tance between two points in a fuzzy metric space, and then proposed the concept of the continuous $t$-norm, which generalizes the triangle inequality of general metric spaces. Therefore, motivated by the idea of fuzzy metric space, we put forward the concept of continuous interval-valued $t$-norm and define an interval-valued fuzzy metric space. Furthermore, we discuss some topological properties of this kind of metric space.

2. Interval numbers and interval-valued $t$-norm

For completeness and clarity, in this section, some related concepts and conclusions from [21, 22] are summarized below. Moreover, the concept of an interval-valued $t$-norm is presented as a generalization of the ordinary $t$-norm proposed by Kelment and Schweizer, etc. [23, 24].

Let $I$ be a closed unit interval, i.e., $I=[0,1]$, let $(\mathbf{I},\leq,\land,\lor,\land,\lor)$, $(\mathbf{I})\leq(\mathbf{I})\land(\mathbf{I})\lor(\mathbf{I})$.

\begin{itemize}
  \item [(O1)] $(\mathbf{I})\land(\mathbf{I})\land(\mathbf{I})$ for all $x \in X$, then the ordinary fuzzy sets $A^*:X \rightarrow I$ and $A^*:X \rightarrow I$ are called the lower fuzzy set and upper fuzzy set of $A$, respectively. Especially, $A$ is called a degenerate fuzzy set if $A^*(x) = A^*(x)$ for all $x \in X$.
  
  Based on the relevant operations of interval numbers, we introduce the following operations including equality, intersection, union and complement of interval-valued fuzzy sets.

  For any $A,B \in IVF(X)$, define
  \begin{itemize}
    \item [(O4)] $A = B$ iff $A^*(x) = B^*(x)$ and $A^*(x) = B^*(x)$ for all $x \in X$;
    \item [(O5)] $(A \land B)(x) = A(x) \land B(x)$
      
      $\exists[A^*(x) \land B^*(x), A^*(x) \land B^*(x)]$;
    \item [(O6)] $(A \lor B)(x) = A(x) \lor B(x)$
      
      $\exists[A^*(x) \lor B^*(x), A^*(x) \lor B^*(x)]$;
    \item [(O7)] $A^*(x) = (A(x))^* = \overline{1 - A(x)}$
      
      $\exists[1 - A^*(x), 1 - A^*(x)]$.
  \end{itemize}

  Now, we extend $t$-norm to interval-valued $t$-norm for the definition of interval-valued metric space.

  Definition 2: An interval-valued $t$-norm is a binary operation on $[I]$, i.e., an operation $*: [I] \times [I] \rightarrow [I]$ such that for all $\overline{a}, \overline{b}, \overline{c} \in [I]$ the following four conditions are satisfied:

  \begin{itemize}
    \item [(T1)] Commutativity: $\overline{a} * \overline{b} = \overline{b} * \overline{a}$;
    \item [(T2)] Associativity: $\overline{a} * (\overline{b} * \overline{c}) = (\overline{a} * \overline{b}) * \overline{c}$;
    \item [(T3)] Monotonicity: $\overline{a} * \overline{b} \leq \overline{a} * \overline{c}$ whenever $\overline{b} \leq \overline{c}$;
    \item [(T4)] Boundary condition;
      
      $\overline{a} * \overline{1} = \overline{a}, \overline{a} * \overline{0} = [a^-, a^+]^* \in [0,1]$.
  \end{itemize}

Example 1: (i) $\overline{a} * \overline{b} = [a^- \land b^-, a^+ \land b^+]$;

(ii) $\overline{a} * \overline{b} = [a^- \lor b^-, a^+ \lor b^+]$.

It is easy to see that several additional useful properties follow immediately for an interval-valued $t$-norm $*$ on $[I]$.

Remark 1: (i) Directly from Definition 2 we can deduce that, for all $\overline{a} \in [I]$, each interval-valued $t$-norm $*$ satisfies the following additional boundary conditions:
\[ \overline{0} \ast_i a = \overline{a} \ast_i \overline{0} = \overline{0}, \]
\[ \overline{1} \ast_i a = [0, 1] \ast_i [a^-, a^+] = \overline{1}, \]
\[ \overline{1} \ast_i a = \overline{1}. \]

(ii) In essence, according to commutativity (T1), it can easily be verified that the monotonicity of an interval-valued t-norm \( \ast_i \) in its second component described by (T3) is equivalent to the (joint) monotonicity in both components, namely, \( \overline{a} \ast_i \overline{b} \leq \overline{c} \ast_i \overline{d} \) whenever \( \overline{a} \leq \overline{c} \) and \( \overline{b} \leq \overline{d} \).

Remark 2: For all \( \overline{a} \in [I] \), it is obvious that \( \overline{0} \leq \overline{a} \leq \overline{1} \).

As a consequence of (ii) we obtain for every \( \overline{a}, \overline{b} \in [I] \),
\[ \overline{0} = \overline{a} \ast_i \overline{0} \leq \overline{a} \ast_i \overline{b} \leq \overline{a} \ast_i \overline{b} \overline{b} = \overline{b} \overline{b} \]   
Hence \( \overline{0} = \overline{a} \ast_i \overline{0} \leq \overline{a} \ast_i \overline{b} \leq \overline{a} \wedge \overline{b} \).

Definition 3: Let \( \{\overline{a}_n\} = \{[a^+_n, a^-_n]\} (n \in \mathbb{N}^+) \) be a sequence of interval numbers in \([I] \), \( \overline{a} = [a^-, a^+] \in [I] \), if \( \lim_{n \to \infty} a^-_n = a^- \) and \( \lim_{n \to \infty} a^+_n = a^+ \), then we call the sequence \( \{\overline{a}_n\} \) is convergent to \( \overline{a} \), and which is denoted by \( \lim_{n \to \infty} \overline{a}_n = \overline{a} \).

In general, most of interval-valued t-norms we will consider are continuous, so we will define the continuous interval-valued t-norm using the idea of continuous t-norm [23, 25].

Definition 4: An interval-valued t-norm \( \ast_i \) is continuous if and only if it is continuous in its first component, i.e., for each \( \overline{b} \in [I] \), if \( \lim_{n \to \infty} a^+_n = \overline{a}^+ \), then
\[ \lim_{n \to \infty} (\overline{a}^+_n \ast_i \overline{b}) = (\lim_{n \to \infty} \overline{a}^+_n \ast_i \overline{b}) = \overline{a}^+_i \ast_i \overline{b}, \]
where \( \{\overline{a}_n\} \subseteq [I], \overline{a} \in [I] \).

Remark 3: Similar to the properties of continuous t-norm, if \( \ast_i \) is a continuous interval-valued t-norm, the following conclusions can be obtained:
(i) For any \( \overline{r}_1, \overline{r}_2 \in [I] \) with \( \overline{r}_1 > \overline{r}_2 \), there exists a \( \overline{r}_3 \in [I] \) such that \( \overline{r}_1 \ast_i \overline{r}_3 \geq \overline{r}_2 \),
(ii) For any \( \overline{r} \in [I] \), there exists a \( \overline{r}_5 \in [I] \) such that \( \overline{r} \ast_i \overline{r}_5 \geq \overline{r}_3 \).

3. Interval-valued fuzzy metric spaces

As a natural generalization of the fuzzy metric space in the sense of George and Veeramani [7, 8], in this section, we put forward the concept of interval-valued fuzzy metric space using the previous definitions.

Definition 5: The triple \((X, M, \ast_i)\) is said to be an interval-valued fuzzy metric space if \( X \) is an arbitrary set, \( \ast_i \) is a continuous interval-valued t-norm on \([I]\) and \( M \) is an interval-valued fuzzy set on \( X^2 \times (0, \infty) \) satisfying the following conditions:
(C1) \( M(x, y, t) > \overline{0} \);
(C2) \( M(x, y, t) = \overline{1} \) if and only if \( x = y \);
(C3) \( M(x, y, t) = M(y, x, t) \);
(C4) \( M(x, y, t) \ast_i M(y, z, s) \leq M(x, z, t + s) \);
(C5) \( M(x, y, t) \ast_i : (0, \infty) \to [I] \) is continuous;
(C6) \( \lim_{t, s \to \infty} M(x, y, t) = \overline{1} \), where \( x, y, z \in X \) and \( t, s > 0 \).

In the above definition, \( M = [M^-, M^+] \) is called an interval-valued fuzzy metric on \( X \). The functions \( M^-(x, y, t) \) and \( M^+(x, y, t) \) denote the lower nearness degree and the upper nearness degree between \( x \) and \( y \) with respect to \( t \), respectively. Besides, it should be pointed out that the condition (C5) implies the functions \( M^+(x, y, \cdot) \) and \( M^-(x, y, \cdot) \) are continuous from \((0, \infty)\) to \([0, 1]\) satisfying \( M^-(x, y, t) \leq M^+(x, y, t) \) for all \( t > 0 \). The combination of conditions (C1) and (C6) shows that the lower nearness degree and upper nearness degree are non-decreasing and tend to gradually 1, respectively.

Note that an interval-valued fuzzy metric space will degenerate into an ordinary fuzzy metric space if \( M^-(x, y, t) = M^+(x, y, t) \) for all \( t > 0 \).

Example 2: Let \( X = \mathbb{R} \). Define \( \overline{a} \ast_i \overline{b} = [a^+ \ast b^+, a^- \ast b^-] \) and the interval-valued fuzzy metric function is taken as
\[ M(x, y, t) = [M^-(x, y, t), M^+(x, y, t)] = [e^{-\frac{k(t-x)}{1-t}}, e^{-\frac{k(t+y)}{1-t}}] \]
(k \( \geq 1 \)). For any \( x, y \in X \) and \( t \in (0, \infty) \). Then \((X, M, \ast_i)\) is an interval-valued fuzzy metric space.

Proof:
(a) Obviously, \( M(x, y, t) > \overline{0} \) for all \( x, y \in X \), \( t > 0 \).
(b) \( M(x, y, t) = \overline{1} \) if and only if \( x = y \).
(c) \( M(x, y, t) = M(y, x, t) \).
(d) Now we start to prove
\[ M(x, y, t) \ast_i M(y, z, s) \leq M(x, z, t+s) \].
For all \( t, s > 0 \), since
\[ |x - z| \leq \frac{(l + s)}{t} |x - y| + \frac{(l + s)}{t} |y - z|, \] we have
\[ |x - z| \leq \frac{|x - y| + |y - z|}{t + s} \]
\[ k \cdot |x - z| \leq \frac{k \cdot |x - y| + k \cdot |y - z|}{t + s} \quad (k \geq 1). \]

Therefore, we can obtain
\[ e^{t+e} \leq e^{-t} \cdot e^{-t}, \quad e^{t+e} \leq e^{-t} \cdot e^{-t}. \]

Thus $M^-(x, y, t) \cdot M^-(x, y, s) \leq M^-(x, y, t + s)$ and $M^+(x, y, t) \cdot M^+(x, y, s) \leq M^+(x, y, t + s)$, i.e., $M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)$.

(c) $M(x, y, \cdot) : (0, \infty) \to [I]$ is continuous.

(f) $\lim_{t \to \infty} e^{-\frac{|x - y|}{t}} = \lim_{t \to \infty} \frac{-k \cdot |x - y|}{t} = 1$ for all $x, y \in X$.

Hence $(X, M^-, \ast)$ is an interval-valued fuzzy metric space.

Note: It is clear that this example is a direct extension of example 2.7 in [7]. In this example, $k \geq 1$ is a fixed constant, if $k > 1$, then $(X, M^-, \ast)$ is a classical interval-valued fuzzy metric space. In particular, if $k = 1$, then $(X, M^-, \ast)$ becomes a fuzzy metric space given by George and Veeramani [7].

Remark 4: In the example 2, we take
\[ M^-(x, y, t) = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}, \quad M^+(x, y, t) = e^{-\frac{|x - y|}{t}}. \]

By Definition 5, it is easy to verify that the conditions (C1-C5) are satisfied. However, the condition (C6) cannot be satisfied. We can say that $(X, M^-, \ast)$ is not an interval-valued fuzzy metric space under this situation. In fact, for every $x, y \in X$ and $x \neq y$, we can obtain
\[ \lim_{t \to \infty} M(x, y, t) = \lim_{t \to \infty} M^-(x, y, t) \lim_{t \to \infty} M^+(x, y, t) = [0, 1]. \]

The result shows that the lower nearness degree between $x$ and $y$ is always 0 for sufficiently large positive number $t$. $M^-(x, y, t)$ and $M^+(x, y, t)$ can also be regarded as the lower limit and upper limit of the probability that the distance between $x$ and $y$ is smaller than $t$, respectively. Evidently, we know that $M^-(x, y, t) \to 1$ and $M^+(x, y, t) \to 1$ as $t \to \infty$.

Therefore, the example indicates that the condition (C6) is nonsuperfluous and the affixation of the condition is also necessary and reasonable on the basis of the original definition of the fuzzy metric space introduced by Kramosil, Grabiec and George, etc. [1, 7, 9].

Essentially, in the above example, $\mathbb{R}$ and $|x - y|$ can be replaced by any metric space $X$ and an ordinary metric or distance $d(x, y)$, respectively. Furthermore, the conclusion still holds if we replace the interval-valued $t$-norm with $\bar{a} \ast \bar{b} = [a \cdot b^-, a \cdot b^+]$.

Next, we show that every metric can induce an interval-valued fuzzy metric.

Example 3: (Induced interval-valued fuzzy metric) Let $(X, d)$ be a general metric space. Define the interval-valued $t$-norm $\bar{a} \ast \bar{b} = [a \cdot b^-, a \cdot b^+]$ and interval-valued fuzzy metric
\[ M(x, y, t) = [M^-(x, y, t), M^+(x, y, t)] \]
\[ = [\frac{ht}{ht^* + ld(x, y)}, \frac{ht}{ht^* + md(x, y)}]. \]

For all $h, l, m, n \in \mathbb{R}^+$ and $l \geq m$. Then $(X, M^-, \ast)$ is an interval-valued fuzzy metric space.

Remark 5: Note that the above example holds even if the interval-valued $t$-norm $\bar{a} \ast \bar{b} = [a \cdot b^-, a \cdot b^+]$. Especially, if $h = n = 1$ and $l \geq m$, we then obtain
\[ M(x, y, t) = [M^-(x, y, t), M^+(x, y, t)] \]
\[ = [\frac{t}{t + ld(x, y)}, \frac{t}{t + md(x, y)}]. \]

We call this interval-valued fuzzy metric induced by a metric $d$ the standard interval-valued fuzzy metric. Moreover, if $l = m = 1$, it will reduce to the standard fuzzy metric.

Theorem 1: Let $(X, M^-, \ast)$ be an interval-valued fuzzy metric space, for all $x, y \in X$, if $s > t > 0$, then $M(x, y, t) \leq M(x, y, s)$.

Proof: According to Definition 3.1, if $s > t > 0$, then we know that
\[ M(x, y, t) \ast M(y, y, s - t) \leq M(x, y, s). \]

Since $M(y, y, s - t) = \bar{1}$, we have
\[ M(x, y, t) \ast M(y, y, s - t) = M(x, y, t) \ast \bar{1} = M(x, y, t). \]

Therefore, $M(x, y, t) \leq M(x, y, s)$.

Remark 6: In an interval-valued fuzzy metric space $(X, M^-, \ast)$, whenever $M(x, y, t) > \bar{1} - \bar{1}$ for all $x, y \in X$, $t > 0$, $\bar{1} \in (I)$, there exists a $t_0$ with $0 < t_0 < t$ such that $M(x, y, t_0) > \bar{1} - \bar{1}$.

4. Topology induced by an interval-valued fuzzy metric space

In this section, we define the topology induced by an
interval-valued fuzzy metric space, and investigate some relevant topological properties.

**Definition 6:** Let \((X, M^*, *)\) be an interval-valued fuzzy metric space. The open ball \(B(x, \bar{r}, t)\) with center \(x \in X\) and the interval number \(\bar{r}\), \(0 < \bar{r} < \bar{r}_1\), \(t > 0\), is defined as follows:

\[
B(x, \bar{r}, t) = \{ y \in X | M(x, y, t) > \bar{r} - \bar{r}_1 \}.
\]

**Theorem 2:** Every open ball \(B(x, \bar{r}, t)\) is an open set.

**Proof:** Let \(B(x, \bar{r}, t)\) be an open ball with center \(x\) and interval number radius \(\bar{r}\) with respect to \(t\). For any \(y \in B(x, \bar{r}, t)\), we know that \(M(x, y, t) > \bar{r} - \bar{r}_1\). Therefore, there exists a \(t_0 \in (0, t)\) such that \(M(x, y, t_0) > \bar{r} - \bar{r}_1\). Set \(\bar{r}_0 = M(x, y, t_0)\). Since \(\bar{r}_0 > \bar{r} - \bar{r}_1\), there exists a \(\bar{r} \in (I, \bar{r})\) such that \(\bar{r}_0 > \bar{r} - \bar{r}_1\). Now for given \(\bar{r}_0\) and \(\bar{r}\) with \(\bar{r}_0 > \bar{r} - \bar{r}_1\), there exists a \(\bar{r}_1 \in (I, \bar{r})\) such that \(\bar{r}_0 * \bar{r}_1 > \bar{r} - \bar{r}_1\). Consider the open ball \(B(y, \bar{r} - \bar{r}_1, t - t_0)\), we will obtain that \(B(y, \bar{r} - \bar{r}_1, t - t_0) \subset B(x, \bar{r}, t)\). In fact, for every \(z \in B(y, \bar{r} - \bar{r}_1, t - t_0)\), we have \(M(y, z, t - t_0) > \bar{r}_1\). Therefore,

\[
M(y, z, t) \geq M(y, z, t - t_0) * M(y, z, t_0) \geq \bar{r}_0 * \bar{r}_1 > \bar{r} - \bar{r}_1 > \bar{r} - \bar{r}_1.
\]

Thus \(z \in B(x, \bar{r}, t)\) and hence \(B(y, \bar{r} - \bar{r}_1, t - t_0) \subset B(x, \bar{r}, t)\).

**Theorem 3:** Let \((X, M^*, *)\) be an interval-valued fuzzy metric space. Define

\[
T_M = \{ A \subset X \mid \forall x \in A, \exists t > 0, \bar{r} \in (I) \text{ such that } B(x, \bar{r}, t) \subset A \}.
\]

Then \(T_M\) is a topology on \(X\).

**Proof:** (i) \(\emptyset, X \in T_M\);

(ii) For all \(A_1, A_2 \in T_M\), firstly, if \(A_1 \cap A_2 = \emptyset\), then \(A_1 \cap A_2 = T_M\). On the other hand, for every \(x \in A_1 \cap A_2\), there exist \(t_1 > 0\) and \(\bar{r}_1 \in (I)\) such that \(B(x, \bar{r}_1, t_1) \subset A_1\). Meanwhile, there also exist \(t_2 > 0\) and \(\bar{r}_2 \in (I)\) such that \(B(x, \bar{r}_2, t_2) \subset A_2\). Set \(\bar{r} = \bar{r}_1 \wedge \bar{r}_2\) and \(t = \min\{t_1, t_2\}\). Clearly, we have \(\bar{r} - \bar{r} \geq \bar{r}_1 - \bar{r}_1\) and \(\bar{r} - \bar{r} \geq \bar{r}_2 - \bar{r}_2\). According to Theorem 1, we can obtain

\[
B(x, \bar{r}, t) = \{ y \in X | M(x, y, t) > \bar{r} - \bar{r}_1 \} \subset \{ y \in X | M(x, y, t) > \bar{r}_1 - \bar{r}_1 \} \subset \{ y \in X | M(x, y, t) > \bar{r}_2 - \bar{r}_2 \} = B(x, \bar{r}_1, t_1) \subset A_1.
\]

Analogously, \(B(x, \bar{r}, t) \subset B(x, \bar{r}_2, t_2) \subset A_2\). Thus \(B(x, \bar{r}, t) \subset A_1 \cap A_2\) and hence \(A_1 \cap A_2 \in T_M\).

(iii) For all \(A_j \in T_M(g \in \Gamma)\), \(\Gamma\) is an index set. For every \(x \in \bigcup_{\gamma \in \Gamma} A_j\), therefore, there exist \(t > 0\) and \(\bar{r} \in (I)\) such that \(B(x, \bar{r}, t) \subset A_{\gamma}\). Thus \(B(x, \bar{r}, t) \subset \bigcup_{\gamma \in \Gamma} A_{\gamma}\). It means that \(\bigcup_{\gamma \in \Gamma} A_j \in T_M\). The proof of the theorem is now completed.

**Remark 7:** From Theorem 2 and Theorem 3, each interval-valued fuzzy metric \(M\) in an interval-valued fuzzy metric space \((X, M^*, *)\) can generate a topology \(T_M\) on \(X\), and which has a base such as the family of open sets of the form \(\{B(x, \bar{r}, t) : x \in X, \bar{r} \in (I), t > 0\}\).

**Theorem 4:** Let \((X, M^*, *)\) be an interval-valued fuzzy metric space, the topology \(T_M\) is first countable.

**Proof:** For all \(x \in X\), define \(B_x = \{ B(x, \bar{r}_n, 1/n) \mid n = 1, 2, 3, \ldots \}\), where \(\bar{r}_n = \{\frac{1}{n}, \frac{1}{n}\}\).

By Theorem 2, \(B_x\) is a family of open sets with center \(x\). Obviously, \(B_x\) constitutes a local base at \(x\). Therefore, the topology \(T_M\) is first countable.

**Remark 8:** Notice that \(B_x\) is a nested local base at \(x\).

Set \(B_n = B(x, \bar{r}_n, \frac{1}{n}) \mid n = 1, 2, 3, \ldots\). For all \(n \in \mathbb{N}^+\), by Theorem 1, since \(\bar{r}_n - \bar{r}_n \leq \bar{r} - \bar{r}_{n+1}\), we have

\[
B(x, \bar{r}_n, \frac{1}{n}) \supset B(x, \bar{r}_{n+1}, \frac{1}{n+1}) \supset B(x, \bar{r}_{n+1}, \frac{1}{n+1}).
\]

Hence, it is obvious that \(B_n \supset B_{n+1}\).

**Theorem 5:** Every interval-valued fuzzy metric space is a Hausdorff space \((T_2\text{-space})\).

**Proof:** Let \((X, M^*, *)\) be an interval-valued fuzzy metric space. Suppose that \(x\) and \(y\) two distinct points in \(X\). Clearly, we know that \(\bar{0} < M(x, y, t) < \bar{r}\) for all \(t > 0\). Put \(M(x, y, t) = \bar{r}\), for some \(\bar{r}\),
\[ 0 < \bar{r} < \bar{t}. \] For each \( \bar{r}_0, \bar{r} < \bar{r}_0 < \bar{t}, \) there exist a \( \bar{r}_1 \in (\bar{I}) \) such that \( \bar{r}_1 \geq \bar{r}_0. \) Now consider open balls \( B(x, \bar{t} - \bar{r}_1, 1) \) and \( B(y, \bar{t} - \bar{r}_1, 1) \). Actually, \( B(x, \bar{t} - \bar{r}_1, 1) \cap B(x, \bar{t} - \bar{r}_1, 1) = \emptyset. \) Conversely, if there exists a \( z \in B(x, \bar{t} - \bar{r}_1, 1) \cap B(x, \bar{t} - \bar{r}_1, 1), \) then
\[
\bar{r} = M(x, y, t) \geq M(x, z, \bar{t}_1) \bar{r}_1, M(z, y, \bar{t}_1) \geq \bar{r}_1 \bar{r}_0 > \bar{r}.
\]
This leads to a contradiction. Hence \((X, M, \cdot)\) is a Hausdorff space \((T_d - \text{space})\).

**Corollary 1:** Every interval-valued fuzzy metric space is a \( T_d \)-space.

**Remark 9:** Let \((X, d)\) be a metric space. Let
\[
M(x, y, t) = \left[ M^{-}(x, y, t), M^{+}(x, y, t) \right] = \left[ \frac{t}{t + \delta d(x, y)}, \frac{t}{t + \delta d(x, y)} \right] (l \geq m)
\]
be an interval-valued fuzzy metric defined on \( X \). Then the topology \( T_d \) induced by the metric \( d \) and the topology \( T_m \) induced by the interval-valued fuzzy metric \( M \) are the same.

**Definition 7:** Let \((X, M, \cdot)\) be an interval-valued fuzzy metric space. A subset \( A \) of \( X \) is said to be IVF-bounded if and only if there exist \( t > 0 \) and \( \bar{r} \in (\bar{I}) \) such that \( M(x, y, t) > \bar{t} - \bar{r} \) for all \( x, y \in A \).

**Remark 10:** Let \((X, M, \cdot)\) be an interval-valued fuzzy metric space induced by a metric \( d \) on \( X \). Then \( A \subseteq X \) is IVF-bounded if and only if it is bounded.

**Theorem 6:** Every compact subset \( A \) of an interval-valued fuzzy metric space \((X, M, \cdot)\) is IVF-bounded.

**Proof:** Given \( A \) is a compact subset of \( X \). Set \( t > 0 \) and \( 0 < \bar{r} < \bar{t} \). Obviously, the family of subsets \( T_x = \{B(x, \bar{r}, t) | x \in A\} \) constitutes an open covering of \( A \). Since \( A \) is a compact set, there exist \( x_1, x_2, ..., x_n \in A \) such that \( A \subseteq \bigcup_{i=1}^{n} B(x, \bar{r}, t) \). For every \( x, y \in A \). Then \( x \in B(x, \bar{r}, t) \) and \( y \in B(x, \bar{r}, t) \) for some \( i, j \). Therefore, we can obtain \( M(x, x, t) > \bar{t} - \bar{r} \) and \( M(y, x, t) > \bar{t} - \bar{r} \). Now let \( \bar{a} = \left[ \min \left\{ M^{-}(x, x, t) \right\}, \min \left\{ M^{+}(x, x, t) \right\} \right] \). Then \( \bar{a} > 0 \). Thus, we have
\[
M(x, y, 3t) \geq M(x, x, t) \cdot M(y, y, t) \geq (\bar{t} - \bar{r}) \cdot (\bar{t} - \bar{r}) \bar{a} > \bar{t} - \bar{s}
\]
for some \( \bar{s} \in (\bar{I}) \).

Taking \( t' = 3t \), we get \( M(x, y, t') > \bar{t} - \bar{s} \) for all \( x, y \in A \). Hence \( A \) is IVF-bounded.

**Corollary 2:** Every compact set \( A \) of an interval-valued fuzzy metric space \((X, M, \cdot)\) is closed and bounded.

**Definition 8:** An interval-valued fuzzy metric space \((X, M, \cdot)\) is compact if \( X \) is a compact set.

**Theorem 7:** Every closed subset \( A \) of a compact interval-valued fuzzy metric space \((X, M, \cdot)\) is compact.

**Proof:** Suppose that the family of open sets \( T = \{G_i\} \) is an arbitrary open covering of \( A \), i.e., \( A \subseteq \bigcup_{i=1}^{n} G_i \). Then \( X = (\bigcup_{i=1}^{n} G_i) \cup A^c (A^c \text{ denotes the complementary set of } A) \). Since \( A \) is a closed set, \( T^c = \{G_i\} \cup \{A^c\} \) constitutes an open covering of \( X \). In addition, since \((X, M, \cdot)\) is a compact space, there exist \( G_i, G_{i+1}, ..., G_m \in T \) such that \( X = G_i \cup G_{i+1} \cup ... \cup G_m \cup A^c \). Therefore, we have \( A \subseteq G_i \cup G_{i+1} \cup ... \cup G_m, \) namely, for arbitrary open covering of \( A \), there is a finite sub-covering of it. Hence \( A \) is a compact set.

**Corollary 3:** Let \((X, M, \cdot)\) be a compact interval-valued fuzzy metric space. A subset \( A \) of \( X \) is a closed set if and only if \( A \) is a compact subset.

**Proof:** It follows from Theorems 5, 7 and Corollary 2.
there exists a $n_0 \in \mathbb{N}$ such that $x_n \in B(x, \overline{r}, t)$ for all $n \geq n_0$. Clearly, $M(x_n, x, t) > \overline{1} - \overline{r}$, i.e., $\overline{1} - M(x_n, x, t) < \overline{r}$. Therefore, $M(x, x_n, t) \rightarrow \overline{1}$ as $n \rightarrow \infty$. On the other hand, if for each $t > 0$, $M(x, x_n, t) \rightarrow \overline{1}$ as $n \rightarrow \infty$, then for $\overline{r} \in (I)$, there exists a $n_0 \in \mathbb{N}$ such that $\overline{1} - M(x_n, x, t) < \overline{r}$ for all $n \geq n_0$. It follows that $M(x_n, x, t) > \overline{1} - \overline{r}$ for all $n \geq n_0$. Thus, $x_n \in B(x, \overline{r}, t)$ for all $n \geq n_0$ and hence $x_n \rightarrow x$.

**Definition 9:** Let $(X, M, *_{I})$ be an interval-valued fuzzy metric space. A sequence $\{x_n\}$ in $X$ is called a Cauchy sequence if and only if $\{x_n\}$ converges as $n \rightarrow \infty$.

**Definition 10:** An interval-valued fuzzy metric space $(X, M, *_{I})$ in which every Cauchy sequence is convergent is called a complete interval-valued fuzzy metric space.

**Remark 11:** Similar to the definition of Cauchy sequence in the fuzzy metric space [9], we can also introduce another definition of Cauchy sequence in the interval-valued fuzzy metric space, i.e., a sequence $\{x_n\}$ is a Cauchy sequence if and only if

$$\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, t) = \overline{1}$$

for each $p > 0$ and $t > 0$.

However, we note that with this definition, even $\mathbb{R}$ fails to be complete. For example, let $x_n = \frac{1}{n} + \frac{1}{3} + \cdots + \frac{1}{n}$ in $(\mathbb{R}, M, *_{I})$, where

$$M(x, y, t) = \frac{t}{t + \overline{1} - d(x, y)}$$

for each $t > 0$ and $t > 0$, we obtain

$$M(x_{n+p}, x_n, t) = \frac{t}{t + \overline{1} - \overline{r}} \rightarrow \overline{1}$$

This sequence $\{x_n\}$ in the interval-valued fuzzy metric space $(\mathbb{R}, M, *_{I})$. Additionally, if $\mathbb{R}$ is interval-valued fuzzy complete, then there exists $x \in \mathbb{R}$ such that $M(x_{n+p}, x_n, t) \rightarrow \overline{1}$ as $n \rightarrow \infty$. From this if follow that

$$M(x_n, x, t) = \frac{t}{t + \overline{1} - \overline{r} \cdot t + m \overline{1} - \overline{r}} \rightarrow \overline{1}$$

Furthermore, we have $|x_n - x| \rightarrow 0$ as $n \rightarrow \infty$. So $x_n \rightarrow x$ in $\mathbb{R}$ which is not true. The example shows that $\{x_n\}$ is a Cauchy sequence, and $\mathbb{R}$ is not complete under the previous definition. Whereas, one can see that Definition 9 guarantees $\mathbb{R}$ is a complete interval-valued fuzzy metric space.

**Theorem 9:** Let $(X, M, *_{I})$ be a compact interval-valued fuzzy metric space. If every Cauchy sequence $\{x_n\}$ in $X$ has a convergent subsequence, then $(X, M, *_{I})$ is complete.

**Proof:** Let $\{x_n\}$ be a subsequence of the Cauchy sequence $\{x_n\}$ that converges to $x$. Now we prove that $x_n \rightarrow x$. For any $t > 0$ and $\overline{r} \in (I)$, choose $\overline{r} \in (I)$ such that $(\overline{1} - \overline{r}) \overline{r} \geq 1 - \overline{r}$. Since $\{x_n\}$ is a Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that $M(x_m, x_n, \frac{1}{2}) > (\overline{1} - \overline{r}) \overline{r}$ for all $m, n \geq n_0$. Since $x_n \rightarrow x$, there is a positive integer $i_p$ such that $M(x_p, x_n, \frac{1}{2}) > (\overline{1} - \overline{r})$ for all $n \geq n_0$ and $i_p \geq n_0$. We have

$$M(x_n, x, t) \geq M(x_n, x_p, \frac{t}{2} \overline{r}) \geq (\overline{1} - \overline{r}) \overline{r} \geq 1 - \overline{r}$$

Therefore $x_n \rightarrow x$ and hence $(X, M, *_{I})$ is complete.

**Theorem 10:** Let $(X, M, *_{I})$ be a compact interval-valued fuzzy metric space and let $A$ be a subset of $X$. $M_\delta = M|_{\mathcal{F}(\delta(0,\infty))}$ denotes the restriction of the interval-valued fuzzy metric $M$ on $A$. Then $(A, M_\delta, *_{I})$ is complete if and only if $A$ is a closed subset of $X$.

**Proof:** Suppose that $A$ is a closed subset of $X$ and let $\{x_n\}$ be a Cauchy sequence in $A$. Clearly, $A \subset X$, $\{x_n\}$ is also a Cauchy sequence in $X$. Since
If every Cauchy sequence of \( \{x_n\} \) converges in \( A \), then \((X,M,\ast)\) is complete. Conversely, suppose that \((A,M_A,\ast)\) is complete and \( A \) is not closed. Let \( x \in \overline{A} \) and \( x \not\in A \). Then there is a sequence \( \{x_n\} \) of points in \( A \) that converges to \( x \). It is obvious that \( \{x_n\} \) is a Cauchy sequence. Thus, for each \( \epsilon > 0 \) and each \( t > 0 \), there exists \( N \in \mathbb{N} \) such that \( M(x_n,x_m,t) > \epsilon \) for all \( m,n \geq N \). Since \( \{x_n\} \subset A \), we know that \( M(x_n,x_n,t) = M_A(x_n,x_n,t) \). Therefore \( M_A(x_n,x_n,t) > \epsilon \) for all \( m,n \geq n_0 \). Thus \( \{x_n\} \) is a Cauchy sequence in \( A \). Since \((A,M_A,\ast)\) is complete, there exists a \( y \in A \) such that \( x_n \to y \). That is, for each \( \epsilon > 0 \) and each \( t > 0 \), there exists an \( n_0 \in \mathbb{N} \) such that \( M_A(x_n,y,t) > \epsilon \) for all \( n \geq n_0 \). Since \( \{x_n\} \subset A \) and \( y \in A \), \( M(y,x_n,t) = M_A(y,x_n,t) \). Hence \( \{x_n\} \) converges to both \( x \) and \( y \) in \( X \). But we know that \( x \not\in A \) and \( y \in A \), i.e., \( x \neq y \). This is a contradiction.

Remark 12: Theorem 10 shows that every closed subspace of a complete interval-valued fuzzy metric space is complete.

Theorem 11: Let \((X,M,\ast)\) be an interval-valued fuzzy metric space and let \( A \) be a subset of \( X \). If every Cauchy sequence of \( A \) converges in \( X \), then every Cauchy sequence of \( A \) converges in \( X \).

Proof: Let \( \{x_n\} \) be a Cauchy sequence of \( A \). For any \( \epsilon > 0 \) and \( t > 0 \), choose \( r,s \in \mathbb{R}^+ \) such that \( (\overline{1-r}) \ast (\overline{1-s}) \geq \overline{1-\epsilon} \). For each \( x_n \in A \), \( \overline{0} < \overline{r} < \overline{1} \) and each \( t > 0 \), there exists a \( y_n \in A \) such that \( M(x_n,y_n,t) > \overline{1-r} \). Since \( \{x_n\} \) is a Cauchy sequence, for any \( \epsilon > 0 \) and \( t > 0 \), there exist a \( n_0 \in \mathbb{N} \) such that \( M(x_n,x_m,t) > \overline{1-\epsilon} \) for all \( m,n \geq n_0 \). Then

\[
M(y_n,y_m,3t) \geq M(y_n,x_n,t) \ast M(x_n,x_m,t) \ast M(x_m,y_m,t) > (\overline{1-r}) \ast (\overline{1-s}) \geq 1-\epsilon
\]

Hence \( \{x_n\} \) is convergent in \( X \).

Corollary 4: Let \((X,M,\ast)\) be an interval-valued fuzzy metric space and let \( A \) be a dense subset of \( X \). If every Cauchy sequence of \( A \) is convergent in \( X \), then \((X,M,\ast)\) is complete.

Theorem 12: Let \((X,M,\ast)\) be an interval-valued fuzzy metric space. For any \( t > 0 \) and \( \overline{r},\overline{s} \in \mathbb{R}^+ \), if \( (\overline{1-r}) \ast (\overline{1-s}) \geq (\overline{1-\epsilon}) \), then \( B(x,\overline{r},\frac{t}{2}) \subset B(x,\overline{s},t) \).

Proof: Let \( B(y,\overline{r},\frac{t}{2}) \) be an open ball with center \( y \) and interval number radius \( \overline{r} \). For all \( y \in B(x,\overline{r},\frac{t}{2}) \), we know that \( B(y,\overline{r},\frac{t}{2}) \cap B(x,\overline{s},\frac{t}{2}) \neq \emptyset \). Assume that \( z \in B(x,\overline{r},\frac{t}{2}) \cap B(x,\overline{s},\frac{t}{2}) \), then we have

\[
M(x,y,t) \geq M(x,z,\frac{t}{2}) \ast M(x,z,\frac{t}{2})M(y,z,\overline{r}) > (\overline{1-r}) \ast (\overline{1-s}) \geq \overline{1-\epsilon}
\]

Hence \( y \in B(x,\overline{s},t) \) and thus \( B(x,\overline{r},\frac{t}{2}) \subset B(x,\overline{s},t) \).

Theorem 13: A subset of an interval-valued fuzzy metric space \((X,M,\ast)\) is nowhere dense if and only if every nonempty open set in \( X \) contains an open ball whose closure is disjoint from \( A \).

Proof: Let \( U \) be a nonempty open subset of \( X \). Since \( A \) is a nowhere dense subset, there exists a nonempty open set \( V \subset U \) such that \( V \cap A = \emptyset \). Let \( x \in V \). Then there exist \( \overline{s} \in (I) \) and \( t > 0 \) such that \( B(x,\overline{s},t) \subset V \). Choose \( \overline{r} \in (I) \) such that \( (\overline{1-r}) \ast (\overline{1-s}) \geq (\overline{1-\epsilon}) \). By Theorem 12, \( B(x,\overline{r},\frac{t}{2}) \subset B(x,\overline{s},t) \). Thus \( B(x,\overline{r},\frac{t}{2}) \subset V \) and \( B(x,\overline{r},\frac{t}{2}) \cap A = \emptyset \). Conversely, if \( A \) is not nowhere dense, i.e., \( Int(A) \neq \emptyset \), then there exists a nonempty open set \( U \) such that \( U \subset A \subset \overline{A} \). Let \( B(x,\overline{r},t) \) be an open ball such that \( B(x,\overline{r},t) \subset U \). Then \( B(x,\overline{r},\frac{t}{2}) \cap A = \emptyset \). This leads to a contradiction.

Theorem 14: (Baire’s Theorem) Let \( U_n \) (\( n = 1,2,3,\ldots \)) be a countable number of dense open sets in the com-
complete interval-valued fuzzy metric space \((X, M_\ast, \ast)\).

Then \(\bigcap_{n=1}^{\infty} U_n\) is dense in \(X\).

**Proof:** Let \(V_0\) be a nonempty open set of \(X\). Since \(U_1\) is dense in \(X\), we know that \(V_0 \cap U_1 = \emptyset\). Let \(x \in V_0 \cap U_1\). Since \(V_0 \cap U_1\) is open, there exist \(\overline{0} < \overline{r} < \overline{1}\) and \(t_1 > 0\) such that \(B(x, \overline{r}, t_1) \subset V_0 \cap U_1\).

Choose \(\overline{r}' < \overline{r}\) and \(t'_1 = \min\{t_1, 1\}\) such that \(B(x, \overline{r}', t'_1) \subset V_0 \cap U_1\). Let \(V_1 = B(x, \overline{r}', t'_1)\). Since \(U_2\) is dense in \(X\), we have \(V_1 \cap U_2 \neq \emptyset\). Let \(x \in V_1 \cap U_2\). Since \(V_1 \cap U_2\) is open, there exist \(\overline{0} < \overline{r}_2 < \overline{1}\) and \(t_2 > 0\) such that \(B(x, \overline{r}_2, t_2) \subset V_1 \cap U_2\). Choose \(\overline{r}'_2 < \overline{r}_2\) and \(t'_2 = \min\{t_2, \frac{1}{2}\}\) such that \(B(x, \overline{r}'_2, t'_2) \subset V_1 \cap U_2\). Let \(V_2 = B(x, \overline{r}'_2, t'_2)\).

Similarly continuing in this manner we can obtain a sequence \(\{x_n\}\) and a sequence \(\{t'_n\}\) such that \(0 < t'_n < \frac{1}{n}\) and \(V_n = B(x, \overline{r}'_n, t'_n) \subset V_{n-1} \cap U_n\). Now we claim that \(\{x_n\}\) is a Cauchy sequence.

For a given \(t > 0\) and \(\overline{0} < \overline{r} < \overline{1}\), choose \(n_0\) such that \(\frac{1}{n_0} < t\) and \((\overline{1}) = \frac{1}{n_0}\). Then for all \(n \geq n_0\) and \(m \geq n\), we have

\[
M(x_n, x_m, t) \geq M(x_n, x_m, \frac{1}{n}) \geq \overline{1} - \left(\frac{1}{n}\right) > \overline{1} - \overline{r}.
\]

Therefore \(\{x_n\}\) is a Cauchy sequence. Since \((X, M_\ast, \ast)\) is complete, there exists a \(x \in X\) such that \(x_n \to x\). Since \(x_k \in B(x_n, \overline{r}'_n, t'_n)\) for all \(k \geq n\), we know that \(x \in B(x_n, \overline{r}'_n, t'_n) = V_n\). Hence \(x \in V_n \subset V_{n-1} \cap U_n\) for all \(n \in \mathbb{N}^+\). Therefore \(V_0 \cap (\bigcap_{n=1}^{\infty} U_n) \neq \emptyset\). Thus \(\bigcap_{n=1}^{\infty} U_n\) is dense in \(X\).

**Remark 13:** Since each metric induces an interval-valued metric and interval-valued fuzzy metric is a generalization of fuzzy metric, Baire’s theorem for complete metric space [26] and Baire’s theorem for complete fuzzy metric space [7] are two particular cases of the above theorem.

5. Conclusions

As a natural generalization of the fuzzy metric space, the main contributions of this paper are: (1) A new concept of the interval-valued \(t\)-norm was introduced, and the sequence of interval numbers was given. Meantime, the continuity of an interval-valued \(t\)-norm was defined by means of the convergence of sequence of interval numbers. (2) The notion of interval-valued fuzzy metric space was presented on the basis of the continuous interval-valued \(t\)-norm. Specially, it should be pointed out that the condition (C6) must be appended in this definition, and the main reason was explained in Remark 4. That is to say, there are some differences between the notion of interval-valued fuzzy metric space and the one of fuzzy metric space. (3) Some related properties of the topology by an interval-valued fuzzy metric space were examined. Moreover, it is worth mentioning that Baire’s Theorem has also been extended to interval-valued fuzzy metric spaces.

The interval-valued fuzzy metric space will be proved to be an important basis for the construction of interval-valued fuzzy topology. At the same time, it will enrich the contents of fuzzy mathematics. In our future research, we intend to establish some fixed point theorems for contraction type mappings in interval-valued fuzzy metric spaces.

**Acknowledgments**

The authors would like to express their sincere thanks to Editor-in-Chief, Associate Editor and two anonymous referees for their valuable comments and suggestions which have helped immensely in improving the quality of the paper.

This work was supported by “QingLan” Talent Engineering Funds by Tianshui Normal University, the Funding Project for Academic Human Resources Development in Institutions of Higher Learning under the Jurisdiction of Beijing Municipality (No.PHR201007117) and the Beijing Municipal Education Commission Foundation of China (No. KM201210038001).

**References**


