The Owen Value for Fuzzy Games with a Coalition Structure

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Abstract

In this paper, we make a study of the Owen value for cooperative fuzzy games with a coalition structure, which can be regarded as the generalization of crisp case. An axiomatic system of the Owen value for fuzzy games with a coalition structure is obtained by extending the corresponding characterizations given by Owen. The relationship between the Owen value and the fuzzy core of fuzzy games with a coalition structure is shown. In order to better understand the Owen value for fuzzy games with a coalition structure, we further research the Owen values for four special kinds of fuzzy games with a coalition structure, and give their explicit forms.

Keywords: fuzzy game, coalition structure, Owen value, fuzzy core.

1. Introduction

As we know, in some cooperative games, such as Economic Community and Military Alliance, the players are joined in coalitions that form a partition or coalitional structure of the set of players. For this kind of games, in 1974, Aumann and Drèze in [1] are the first researchers in this respect. In their paper, they established a model of games with a coalition structure, where the coalitions are independent with each other. Different to the model given by Aumann and Drèze [1], in 1977, Owen [16] introduced games with a priori unions which are known as the Owen coalition structure where the probability of cooperation among coalitions is considered, and provided the Owen value, which is an extension of the Shapley value. Hart and Kurz (see [11]) later proposed an alternative axiomatization of the Owen value for infinite universe of players, which are carrier, additivity, anonymity and the coalitional inessential game property. Several axiomatic systems of the Owen value are considered in [2, 12, 13, 19].

Later, Owen [17] gave another payoff index for games with a coalition structure, which is called the Banzhaf-Owen value, and discussed its axiomatic system. Alonso-Meijide et al. in [3] and Amer et al. in [4] proposed a comparative axiomatic characterization of the Banzhaf-Owen coalitional value, respectively. Alonso-Mejide and Fiestras-Janeiro in [5] pointed the Banzhaf-Owen value dissatisfies symmetry in quotient game, and gave another solution concept for games with a coalition structure, which is known as the symmetric Banzhaf value. But all above are only introduced for crisp games.

There are some situations where some players do not fully participate in a coalition, but to a certain degree. In this situation, a coalition is called a fuzzy coalition, which is formed by some players with partial participation. In 1974, Aubin [6] first discussed in this area. Owen [18] defined a kind of fuzzy games, which is called fuzzy games with multilinear extension form. Tsurumi et al. in [22] defined a kind of fuzzy games with Choquet integral form, and the Shapley function defined on this class of games is given. Butnariu [7] defined a class of fuzzy games with proportional value, and gave the expression of the Shapley function on this limited class of games. Recently, Butnariu and Kroupa in [8] expanded the fuzzy games with proportional value to fuzzy games with weighted function, and gave the corresponding Shapley function. The fuzzy games with multilinear extension form and with Choquet integral form are both monotone nondecreasing and continuous with respect to rates of players’ participation. The Shapley function for fuzzy games is studied in [7, 8, 14, 15, 22]. The fuzzy core for fuzzy games is researched in [23, 25]. The lexicographical solution for fuzzy games is discussed in [20].

The purpose of this paper is to study the Owen value for fuzzy games with a coalition structure. A general form of the Owen value for fuzzy games with a coalition structure is given, the relationship between the Owen value and the fuzzy core of fuzzy games with a coalition structure is discussed. Moreover, we discuss the Owen values for four kinds of fuzzy games with a coalition structure; their existence and uniqueness are also proved.

This paper is organized as follows. In the next section, we recall some notations and basic definitions, which will be used in the following. In section 3, an axiomatic definition of the Owen value for fuzzy games with a coalition structure is offered, and its explicit form is given. The relationship between the Owen value and the
fuzzy core of fuzzy games with a coalition structure is shown. In section 4, we pay a special attention to discuss
the Owen values for four special kinds of fuzzy games
with a coalition structure, and give and investigate the
explicit forms of the Owen values for these kinds of fuzzy
games.

2. Preliminaries

Let $N = \{1, 2, \ldots, n\}$ be the player set, and $P(N)$ be
the set of all crisp subsets in $N$. The coalitions in $P(N)$ are
denoted by $S_0, T_0, \ldots$. For any $S_0 \in P(N)$, the cardinality
of $S_0$ is denoted by the corresponding lower case $s$. A function
$v_0 : P(N) \rightarrow \mathbb{R}_+$, satisfying $v_0(\emptyset) = 0$, is called a crisp
game. Let $G_0(N)$ denote the set of all crisp games in $N$.

Let $L(N)$ denote the set of all fuzzy coalitions in $N$. The coalitions in $L(N)$ are denoted by $S, T, \ldots$. For a
fuzzy coalition $S$ and player $i$, $S(i)$ indicates the mem-
bership grade of $i$ in $S$, i.e., the rate of the $i$th player's
participation in $S$. For any $S \in L(N)$, the support is
denoted by $\text{Supp} S = \{i \in N | S(i) > 0\}$, and the cardinality is
denoted by $|\text{Supp} S|$. We use the notation $S \subseteq T$ if and
only if $S(i) = T(i)$ or $S(i) = 0$ for any $i \in N$. A function
$v : L(N) \rightarrow \mathbb{R}_+$, satisfying $v(\emptyset) = 0$, is called a fuzzy
game. Let $G(N)$ denote the set of all fuzzy games in $N$. For any
$S \in L(N)$, we will use $S = (S(i))_{i \in N}$ to denote it, and we
will omit braces for singletons, e.g., by writing $(v(S), S \vee \wedge T, S \vee \wedge U(i)$ instead of $(v(S)), (v(S)), (S) \backslash \{T\},
\{U(i)\}$ for any $v \in G(N)$ and any $\{U(i)\}, S, T \in L(N)$. In
this paper, union and intersection of two fuzzy coalitions are
defined as usual, i.e., $(S \vee T)(i) = S(i) \vee T(i)$ and
$(S \wedge T)(i) = S(i) \wedge T(i)$.

Remark 1: The fuzzy coalition $S \subseteq T$ denotes the set of players $i \in \text{Supp} S \cap \text{Supp} T$ with $(S \wedge T)(i) = S(i)$.

Definition 1: A game $v_0 \in G_0(N)$ is said to be convex if
$v_0(T_0) + v_0(S_0) - v_0(S_0 \cup T_0) + v_0(S_0 \cap T_0) \forall S_0, T_0 \in P(N)$.

Definition 2: A game $v \in G(N)$ is said to be fuzzy convex
if $v(T) + v(S) \leq v(S \vee T) + v(S \wedge T)$ for any $S, T \in L(N)$.

Definition 3: Let $v \in G(N)$ and $U \in L(N), T \subseteq U$ is called a
fuzzy carrier in $U$ for $v$ if $v(S \vee T) = v(S)$ for any $S \subseteq U$.

Definition 4: Let $v \in G(N)$ and $U \in L(N)$. Player $i$ is called a
fuzzy null player in $U$ for $v$ if $v(S \vee U(i)) = v(S)$ for any $S \subseteq U$ with $i \notin \text{Supp} S$.

A crisp coalition structure $\Gamma = \{B_1^0, B_2^0, \ldots, B_n^0\}$ in $N$ is a
partition of $N$, i.e., $\bigcup_{i \neq j \in N} B_i^0 = N$ and $B_i^0 \cap B_j^0 = \emptyset$ for any
$i \neq j$, where $i, j \in M = \{1, 2, \ldots, m\}$. A crisp coalition
structure in $N$ is denoted by $(N, \Gamma)$. For any $S_0 \in P(N, \Gamma), S_0$ is
called a feasible coalition, where $P(N, \Gamma)$ denotes the set
of all feasible coalitions in $(N, \Gamma)$.

Example 1: Let $N = \{1, 2, 3, 4\}$ and $\Gamma = \{B_0^1, B_0^2\}$, where $B_1^0 = \{1, 2\}$ and $B_2^0 = \{3, 4\}$, then $P(N, \Gamma) = \{\emptyset, \{i\} | i \in N, \{1, 2\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, N\}$.

Similar to crisp case, a fuzzy coalition structure $\Gamma_f = \{B_1, B_2, \ldots, B_n\}$ in $U \in L(N)$ is a partition of $U$, i.e.

$\cup_{i \neq j \in U} \text{Supp} B_i = \text{Supp} U \cap \text{Supp} B_j = \emptyset$

for any $i \neq j, i, j \in M = \{1, 2, \ldots, m\}$, where $B_i \subseteq U$. The fuzzy
coalition structure $\Gamma_f$ in $U$ is denoted by $(U, \Gamma_f)$. For any
$S \in L(U, \Gamma_f), S$ is called a feasible fuzzy coalition, where
$L(U, \Gamma_f)$ denotes the set of all feasible fuzzy coalitions
in $(U, \Gamma_f)$.

Example 2: Let $N = \{1, 2, 3, 4\}$, and $U = \{U(i)\}_{i \in M}$ be a fuzzy
coalition in $L(N)$ such that $(i) - 0$ for any $i \in N$. $\Gamma_f = \{B_1, B_2\}$ is a fuzzy coalition structure in $U$, where $B_1 = \{U(1), U(2)\}$ and $B_2 = \{U(3), U(4)\}$, then

$L(U, \Gamma_f) = \{\emptyset, \{U(i)\} | i \in N, \{U(1), U(2)\}, \{U(3), U(4)\}, \{U(1), U(2), U(3)\}, \{U(1), U(2), U(4)\}, \{U(1), U(3), U(4)\}, \{U(2), U(3), U(4)\}, U\}$.

Remark 2: In this paper, if there is no special explanation
$\Gamma_f = \{B_1, B_2, \ldots, B_n\}$ and $M = \{1, 2, \ldots, m\}$ always hold.
Moreover, we use $(N, v_0, \Gamma)$ and $(U, v, \Gamma_f)$ to denote crisp
game $v_0 \in G_0(N)$ in coalition structure $(N, \Gamma)$ and fuzzy
game $v \in G(N)$ in coalition structure $(U, \Gamma_f)$, respectively.

3. The Owen Value for Fuzzy Games with a Coa-
ition Structure

In this section, we deal with some basic concepts for
crisp games with a coalition structure, and the relation-
ship between the Owen value and the core of crisp
games with a coalition structure. Furthermore, we give
the definition of the Owen value for fuzzy games with a
coalition structure, and provide its explicit form, which
is an extension of the crisp case.

A. Some Concepts for Crisp Games with a Coalition
Structure

Similar to traditional case, we give the following defi-
nitions about crisp games with a coalition structure.

Definition 5: For $(N, v_0, \Gamma)$, if the vector $x = \{x_1, x_2, \ldots, x_n\}$
satisfies the following conditions:

$$
x_i \geq v_0(i) \quad i \in N
\sum_{i \in N} x_i = v_0(N)
$$

then $x$ is called an imputation in $(N, v_0, \Gamma)$, where $x_i$
denotes the player $i$’s payoff. The set of all imputations in
$\Gamma$ is denoted by $E(N,v_0,\Gamma)$. 

**Definition 6:** Let $x, y \in E(N,v_0,\Gamma)$, if there exists $\emptyset \neq R_0 \in P(N,\Gamma)$, which satisfies the following conditions:

$$\begin{align*}
    x_i > y_i & \quad \forall i \in R_0 \\
    v_0(R_0) & \geq \sum_{i \in R_0} x_i
\end{align*}$$

then we say $x$ dominates $y$ by $R_0$, and denoted by $x \succ_{R_0} y$. 

**Definition 7:** The set of all $x \in E(N,v_0,\Gamma)$ is said to be the core of $(N,v_0,\Gamma)$, if there doesn’t exist $\emptyset \neq R_0 \in P(N,\Gamma)$ such that $x \succ_{R_0} y$ for all $y \in E(N,v_0,\Gamma)$. 

From Definition 7, we know the core $C(N,v_0,\Gamma)$ of $(N,v_0,\Gamma)$ can be expressed by

$$C(N,v_0,\Gamma) = \{x \in \mathbb{R}^n_+ \mid \sum_{i \in R_0} x_i = v_0(N), \sum_{i \in R_0} x_i \geq v_0(S_0), \forall S_0 \in P(N,\Gamma)\}.$$

**Theorem 1:** Let $v_0 \in G_0(N)$ be convex in $(N,\Gamma)$, then $C(N,v_0,\Gamma) \neq \emptyset$. 

**Proof:** Similar to Shapley [21], we give the following proof. Let $\Gamma = \{B_0^1, B_0^2, \ldots, B_0^m\}$, we rearrange the order of the players in $(N,\Gamma)$, and get $\pi^\Gamma = \{\pi B_0^1, \pi B_0^2, \ldots, \pi B_0^m\}$, satisfying $\pi B_0^1 = \{i_1, i_2, \ldots, i_r\}$, $\pi B_0^2 = \{i_{r+1}, i_{r+2}, \ldots, i_{r+t}\}$, $\pi B_0^m = \{i_{m-1}, i_{m-2}, \ldots, i_1\}$, where $b_k$ denotes the cardinality of $B_0^k$ for any $k \in M$. Let

$$\begin{align*}
    x_i &= v_0(i_i) \\
    & \quad \ldots \\
    v_0(i_{r+t}) &= v_0(N) - v_0(N \setminus \{i_r, \ldots, i_t\})
\end{align*}$$

It is obvious that $\sum x_i = v_0(N)$. 

For any $S_0 \in P(N,\Gamma)$, without loss of generality, suppose $S_0 = \bigcup_{i \in R_0 \cap M} B_0^i$, where $T_0 \subseteq B_0^i$. If $T_0 \subset B_0^i$, then we have

$$\begin{align*}
    N \setminus S_0 &= (B_0^i \setminus T_0) \cup \bigcup_{i \in R_0 \cap M \setminus \{i\}} B_0^i \\
    &= (i_1^t, i_1^{j+1}, \ldots, i_1^t) \cup \bigcup_{i \in R_0 \cap M \setminus \{i\}} B_0^i
\end{align*}$$

where $(i_1^{j+1}, \ldots, i_1^t) = B_0^i \setminus T_0$ with $1 \leq j + t \leq b_i$. 

Let $H_0 = (i_1^t, \ldots, i_1^t) \cup \bigcup_{i \in R_0 \cap M \setminus \{i\}} B_0^i$, where $(i_1^t, \ldots, i_1^t) \subseteq B_0^i$. Then we have $S_0 \cup H_0 = S_0 \cup i_1^t$ and $S_0 \cap H_0 = H_0 \setminus i_1^t$. 

From the convexity of $v_0$, we get

$$\sum_{i \in R_0} x_i - v_0(S_0) \geq \sum_{i \in R_0 \cap M} x_i - v_0(S_0 \cup i_1^t).$$

If $T_0 = B_0^i$, then we have $S_0 = \bigcup_{i \in R_0 \cap M} B_0^i$ with $k \in R$, and

$$N \setminus S_0 = \bigcup_{i \in R_0 \cap M \setminus \{i\}} B_0^i,$$ 

Let $H_0 = (i_1^t, \ldots, i_1^t) \cup \bigcup_{i \in R_0 \cap M \setminus \{i\}} B_0^i$, where $(i_1^t, \ldots, i_1^t) \subseteq B_0^i$. Then we have $S_0 \cup H_0 = S_0 \cup i_1^t$ and $S_0 \cap H_0 = H_0 \setminus i_1^t$. From the convexity of $v_0$, we obtain

$$\sum_{i \in S_0} x_i - v_0(S_0) \geq \sum_{i \in S_0 \cup i_1^t} x_i - v_0(S_0 \cup i_1^t).$$

Repeat this process and promise $H_0 \in P(N,\Gamma)$, we can obtain $\sum x_i \geq v_0(S_0)$. Thus, $x = (x_1, \ldots, x_n) \in C(N,v_0,\Gamma) \neq \emptyset$. 

Owen [16] gave the explicit form of the Owen value for crisp games with a coalition structure as follows:

$$\beta_j(N,v_0,\Gamma) = \sum_{b_k^i \subseteq \Gamma} \frac{r!(m-r-1)! s!(b_k^i - s - 1)!}{b_k^i!} \times (v_0(S_0 \cup Q_0^i) - v_0(S_0)) \quad \forall i \in N,$$

where $m$ and $r$ denote the cardinalities of $M$ and $R$, respectively. 

**Theorem 2:** Let $v_0 \in G_0(N)$ be convex in $(N,\Gamma)$, then $\{\beta_j(N,v_0,\Gamma)\}_{j=1}^n \in C(N,v_0,\Gamma)$. 

**Proof:** From Eq. (1) and Theorem 1, we know the function $\beta$ is a convex combination of $m! \sum_{k \in M} b_k^i$ elements in $C(N,v_0,\Gamma)$. Since $C(N, v_0, \Gamma)$ is a convex set, we get the conclusion. 

**B. A General Form of the Owen Value for Fuzzy Games with a Coalition Structure**

Similar to crisp case, we give the following definitions for fuzzy games with a coalition structure.

**Definition 8:** For $(U,v,\Gamma)$, if the vector $x = \{x_1, x_2, \ldots, x_n\}$ satisfies the following conditions:

$$\begin{align*}
    x_i &\geq v(U(i)) \quad i \in \text{Supp}U \\
    \sum_{i \in \text{Supp}U} x_i &= v(U)
\end{align*}$$

then $x$ is called an imputation in $(U,v,\Gamma)$, where $x_i$ denotes the player $i$’s payoff. The set of all imputations in $(U,v,\Gamma)$ is denoted by $E(U,v,\Gamma)$. 

**Definition 9:** Let $x, y \in E(U,v,\Gamma)$, if there exists $\emptyset \neq R \in L(U,\Gamma)$, which satisfies the following conditions:

$$\begin{align*}
    x_i &\geq y_i \quad \forall i \in \text{Supp}R \\
    v(R) &\geq \sum_{i \in \text{Supp}R} x_i
\end{align*}$$

then we say $x$ dominates $y$ by $R$, and denoted by $x \succ_R y$. 

**Definition 10:** The set of all $x \in E(U,v,\Gamma)$ is said to be the fuzzy core of $(U,v,\Gamma)$, if there doesn’t exist $\emptyset \neq R \in L(U,\Gamma)$ such that $x \succ_R y$ for all $y \in E(U,v,\Gamma)$. 

From Definition 10, we know the fuzzy core $FC(U,v,\Gamma)$ of $(U,v,\Gamma)$ can be denoted by
FC(U,v,ΓΓ) = \{x ∈ \mathbb{R}^{|\text{Supp}U|}_{+} : \sum_{i ∈ \text{Supp} U} x_i = v(U), \sum_{i ∈ \text{Supp} S} x_i ≥ v(S), ∀ S ∈ L(U,ΓΓ)\}.

Theorem 3: Let v ∈ G(N) be fuzzy convex in (U,ΓΓ), then FC(U,v,ΓΓ) ≠ ∅.

Proof: The proof of Theorem 3 is similar to that of Theorem 1.

Similar to the definitions of the quotient game and the Owen value for crisp games with a coalition structure, we give the definitions of the fuzzy quotient game and the Owen value for fuzzy games with a coalition structure as follows:

Definition 11: Let v ∈ G(N), if we have vβ(R) = v(∪p∈RBp) for any R ⊆ M, then vβ is said to be a fuzzy quotient game in (U,ΓΓ), which is denoted by (M,vβ).

Definition 12: Let v ∈ G(N), a function f : (U,v,ΓΓ) → \mathbb{R}_{+}^{L(U)} is said to be the Owen value in (U,ΓΓ) for v ∈ G(N) if it satisfies

Axiom 1 (efficiency): For v ∈ G(N) we have

v(U) = \sum_{i ∈ \text{Supp} U} f_i(U,v,ΓΓ);

Axiom 2 (fuzzy null player): If i ∈ SuppU is a fuzzy null player in U for v ∈ G(N), then f_i(U,v,ΓΓ) = 0;

Axiom 3 (additivity): For fuzzy games v_1,v_2 ∈ G(N), if there exists a fuzzy game v_1 + v_2 ∈ G(N) such that (v_1 + v_2)(S) = v_1(S) + v_2(S) for any S ∈ L(U,ΓΓ), then

f(U,v_1 + v_2,ΓΓ) = f(U,v_1,ΓΓ) + f(U,v_2,ΓΓ);

Axiom 4 (symmetry within fuzzy coalitions): For v ∈ G(N) and any B_i ∈ ΓΓ such that i,j ∈ SuppB_i and U(i) = U(j), if we have v(S ∪ U(i)) = v(S ∪ U(j)) for any S ∈ L(U,ΓΓ) with i,j ∈ SuppS, then f_i(U,v,ΓΓ) = f_j(U,v,ΓΓ);

Axiom 5 (symmetry in fuzzy quotient game): For v ∈ G(N) and any B_i,B_j ∈ ΓΓ and if we have vβ(R ∪ k) = vβ(R ∪ k) for any R ⊆ M \ \{k,p\}, then

\sum_{i ∈ \text{Supp} B_i} f_i(U,v,ΓΓ) = \sum_{j ∈ \text{Supp} B_j} f_j(U,v,ΓΓ).

Theorem 4: Let v ∈ G(N), the function β : (U,v,ΓΓ) → \mathbb{R}_{+}^{L(U)} is defined by

\[ β(U,v,ΓΓ) = \sum_{R ⊆ M \setminus {i} \text{Supp} S} \sum_{i ∈ \text{Supp} S} δ_m^\beta δ_n^\beta (v(S ∪ Q) \setminus U(i)) \quad ∀ i ∈ \text{Supp} U, \]

where \[ Q = v_{∪_{p∈RB_P}} , \quad δ_m = r!(m - r - 1)!/m! \quad \text{and} \quad λ_n^\beta = (|\text{Supp} S| - 1)!(|\text{Supp} B_i| - |\text{Supp} S|)!/|\text{Supp} B_i|! \quad \text{m} \quad \text{and} \quad r \] denote the cardinalities of M and R, respectively. Then β is the Owen value of (U,v,ΓΓ).

Proof: Axiom 1: From Eq. (2), we have

\[ \sum_{i ∈ \text{Supp} U} β(U,v,ΓΓ) = \sum_{i ∈ \text{Supp} U} \sum_{R ⊆ M \setminus {i} \text{Supp} S} δ_m^\beta δ_n^\beta (v(S ∪ Q) \setminus U(i)) - v((S ∪ Q) \setminus U(i)). \]

For any S ⊆ B_i, let vβ(S) = v(S ∪ Q) \setminus v(Q). Thus, we get

\[ \sum_{i ∈ \text{Supp} U} β(U,v,ΓΓ) = \sum_{i ∈ \text{Supp} U} \sum_{R ⊆ M \setminus {i} \text{Supp} S} δ_m^\beta δ_n^\beta (v(S ∪ Q) \setminus U(i)). \]
Axiom 5: From the proof of Axiom 1, we have
\[ \sum_{i \in \text{Supp}(U)} \beta(U, v, \Gamma_f) = \sum_{R \subseteq M : k} \delta_m^S (v^p(R \cup k) - v^p(R)). \]
Since \( v^p(R \cup k) = v^p(R \cup p) \) for any \( R \subseteq M \setminus \{k, p\} \), we get
\[ \sum_{R \subseteq M : k} \delta_m^S (v^p(R \cup k) - v^p(R)) = \sum_{R \subseteq M : p} \delta_m^S (v^p(R \cup p) - v^p(R)) \]
and
\[ \beta(U, v, \Gamma_f) = \sum_{i \in \text{Supp}(U)} \beta(U, v, \Gamma_f). \]

Theorem 5: Let \( v \in G(N) \) be fuzzy convex in \((U, \Gamma_f)\), then
\[ (\beta(U, v, \Gamma_f))_{i \in \text{Supp}(U)} \in FC(U, v, \Gamma_f), \]
where \( \beta(U, v, \Gamma_f) \) as shown in Eq. (2).
Proof: The proof of Theorem 5 is similar to that of Theorem 2.

Corollary 1: Let \( v \in G(N) \) be fuzzy convex in \((U, \Gamma_f)\), then
\[ (\beta(U, v, \Gamma_f))_{i \in \text{Supp}(U)} \in E(U, v, \Gamma_f), \]
where \( \beta(U, v, \Gamma_f) \) as shown in Eq. (2).

Theorem 6: Let \( T \) be a fuzzy carrier in \( U \) for \( v \in G(N) \), then \( \beta(U, v, \Gamma_f) = \beta(T, v, \Gamma_f) \) for any \( i \in \text{Supp}(U) \), where \( \Gamma_f \) a fuzzy coalition structure in \( T \) with respect to \( \Gamma_f \), and \( v_T \) is the restriction in \( T \) for \( v \in G(N) \).

Proof: From Definition 3, we have
\[ \beta(U, v, \Gamma_f) = \sum_{i \in \text{Supp}(U)} \sum_{j \in \text{Supp}(U)} \delta_m^S (v(S \cup Q) - v((S \cup Q) \setminus U(i))). \]

Axiom 6: Let \( v \in G(N) \) be fuzzy convex in \((U, \Gamma_f)\), then
\[ (\beta(U, v, \Gamma_f))_{i \in \text{Supp}(U)} \in FC(U, v, \Gamma_f), \]
where \( \beta(U, v, \Gamma_f) \) as shown in Eq. (2).
Proof: From the convexity of \( v \in G(N) \), we have
\[ v(S \cup U(i)) - v(S) \geq v(U(i)) \] for any \( S \in L(U, \Gamma_f) \) with \( i \notin \text{Supp}(S) \). Equation (2), we get
\[ \beta(U, v, \Gamma_f) \geq \sum_{R \subseteq M : k} \sum_{i \in \text{Supp}(U) \setminus \text{Supp}(i)} \delta_m^S (v(U(i))) \]
for any \( i \in \text{Supp}(U) \setminus \text{Supp}(i) \).

Theorem 7: Let \( v \in G(N) \) be fuzzy convex in \((U, \Gamma_f)\), then \( \beta(U, v, \Gamma_f) \) for any \( i \in \text{Supp}(U) \).
Proof: From the convexity of \( v \in G(N) \), we have
\[ v(S \cup U(i)) - v(S) \geq v(U(i)) \] for any \( S \in L(U, \Gamma_f) \) with \( i \notin \text{Supp}(S) \). Equation (2), we get
\[ \beta(U, v, \Gamma_f) \geq \sum_{R \subseteq M : k} \sum_{i \in \text{Supp}(U) \setminus \text{Supp}(i)} \delta_m^S (v(U(i))) \]
for any \( i \in \text{Supp}(U) \setminus \text{Supp}(i) \).

4. The Owen Values for Four Special kinds of Fuzzy Games with a Coalition Structure

In order to better understand the Owen value for fuzzy games with a coalition structure. In this section, we will discuss the Owen values for four special kinds of fuzzy games with a coalition structure, which are proposed by Owen [18], Tsurumi et al. [22], Butnariu [7] and Butnariu and Kroupa [8].

A. The Owen Value for Fuzzy Games with Multilinear Extension Form and a Coalition Structure

The value of fuzzy coalition for fuzzy games with multilinear extension form is expressed by (see [18])
\[ v_o(U) = \sum_{i \in \text{Supp}(U)} \Pi(i) \Pi(1 - U(i)) v_0(T_i). \]

For any \( U \in L(N) \),

Let \( G_o(U) \) denote the set of fuzzy games in \( U \in L(N) \) where the value of any fuzzy coalition \( S \subseteq U \) is given by Eq. (4).

For \( v_o \in G_o(U) \), the value of the fuzzy coalition \( U \) with respect to \((U, \Gamma_f)\) is written as
\[ v_o(U) = \sum_{i \in \text{Supp}(U \setminus \Gamma_f)} \Pi(i) \Pi(1 - U(i)) v_0(T_i). \]

where \( \text{Supp}(U, \Gamma_f) \) denotes the support of \( L(U, \Gamma_f) \).

Remark 3: The value in Eq. (5) is not equal to that given in Eq. (3) since not all fuzzy coalitions in \( U \) are feasible.

When the fuzzy game \( v \in G(N) \) is restricted in the setting of \( v_o \in G_o(U) \), from Definition 12, we get the definition of the Owen value for \( v_o \in G_o(U) \). Here, we omit it.

Definition 13: Let \( v_o \in G_o(U) \), the fuzzy core
\[ FC_o(U, v_o, \Gamma_f) \] of \((U, v_o, \Gamma_f)\) is written as
\[ FC_o(U, v_o, \Gamma_f) = \{ \gamma \in \text{Supp}(U \setminus \Gamma_f) \mid \sum_{i \in \text{Supp}(U \setminus \Gamma_f)} \gamma_i = \sum_{i \in \text{Supp}(S \setminus \Gamma_f)} \Pi(i) \Pi(1 - U(i)) v_0(T_i), \sum_{i \in \text{Supp}(S \setminus \Gamma_f)} \gamma_i \geq \sum_{i \in \text{Supp}(S \setminus \Gamma_f)} \Pi(i) \Pi(1 - U(i)) v_0(T_i), \forall S \in L(U, \Gamma_f) \}. \]

where \( \Gamma_f^S \) is a fuzzy coalition structure in \( S \) with respect to \( \Gamma_f \), and \( \text{Supp}(S, \Gamma_f) \) denotes the support of \( L(S, \Gamma_f) \).
Example 3 (cf. Example 2): Let the player set \(N=\{1,2,3,4\} \), and \(U=\{U(i)\}_{i\in\mathbb{N}} \) be a fuzzy coalition in \(L(N)\) as defined in Example 2, i.e., \(U(i) > 0\) for any \(i \in N\), and \(\Gamma = \{B_1, B_2\}\) is a fuzzy coalition structure in \(U\) as defined in Example 2, i.e., \(B_1 = \{U(1), U(2)\}\) and \(B_2 = \{U(3), U(4)\}\), then \(\text{Supp}(U, \Gamma_F) = \{\emptyset, \{i\}|i \in N, \{1,2\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, N\} \). If \(S = \{U(1), U(2), U(3)\} \subseteq U\), then \(L(S, \Gamma_F) = \{\emptyset, \{U(i)\}|i \in \{1,2,3\}, \{U(1), U(2), U(2)\}, S\} \) and \(\text{Supp}(L(S, \Gamma_F)) = \{\emptyset, \{i\}|i \in \{1,2,3\}, \{1,2\}, \{1,2,3\}\} \).

**Theorem 8:** Let \(v_0 \in G_0(U)\), the function \(\beta^O: (U, v_0, \Gamma_F) \rightarrow \mathbb{R}^{L(U)}\) is defined by

\[
\beta^O(U, v_0, \Gamma_F) = \sum_{R \in L(U)} \sum_{i \in \text{Supp}(U, \Gamma_F)} \delta^O_i \sum_{j \in R} v_0(\Gamma_F) (j)
\]

for any \(i \in \text{Supp} U\), where \(L(R)\) denotes the power set of \(R\). \(\Gamma_F^{\text{Supp}B_p}\) is a fuzzy coalition structure in \(S^{\text{Supp}B_p} B_p\) with respect to \(\Gamma_F\), and \(\text{Supp}(L(S^{\text{Supp}B_p} B_p, \Gamma_F^{\text{Supp}B_p}))\) denotes the support of \(L(S^{\text{Supp}B_p} B_p, \Gamma_F^{\text{Supp}B_p})\). \(\delta^O_m\) and \(\lambda^O_{\beta_m}\) as given in Eq.(2). Then \(\beta^O\) is the unique Owen value in \((U, \Gamma_F)\) for \(v_0 \in G_0(U)\).

**Proof:** From Theorem 4, we get the existence. In the following, we shall show the uniqueness. Since any game \(v_0 \in G_0(U)\) can be uniquely expressed by \(v_0 = \sum_{i \in \text{Supp} U} c_i u_i\), where \(c_i = \sum_{j \in R} (-1)^{\text{Supp} T = \text{Supp} S} \sum_{H_i \subseteq \text{Supp} L(T, U, \Gamma_F)} (1 - U(i)) v_0(H_i)\), \(u_i(S) = 1\) for any \(T \subseteq S\), and otherwise, \(u_i(S) = 0\).

From axiom 3 in Definition 12, it is sufficient to show the uniqueness of Eq.(6) for the unanimity games. From above analysis, we give the following proof.

For given \(\Gamma_F\) in \(U\), we will prove Eq.(6) is unique in \(T \in L(U, \Gamma_F) (\emptyset \neq T)\) for unanimity game \(u_T\). Let \(M^* = \{p \in M | \text{Supp} B_p \cap \text{Supp} T \neq \emptyset\}\) and \(\text{Supp} B_p = \text{Supp} B_p \cap \text{Supp} T\).

The unanimity quotient game \(u_T^\beta\) for \(u_T\) is defined by

\[
u_T^\beta(R) = \begin{cases} 1 & M^* \subseteq R \\ 0 & M^* \not\subseteq R \end{cases}
\]

where \(R \subseteq M\).

Let \(F\) be a solution in \((U, u_T, \Gamma_F)\) that satisfies the mentioned axioms in Definition 12. From efficiency, we have \(F_i(U, u_T^\beta, \Gamma_F) = 0\) for any \(k \not\in M^*\). When \(k \in M^*\), from symmetry in fuzzy quotient game, fuzzy null player and efficiency, it is derived that

\[
\sum_{i \in \text{Supp} B_p} F_i(U, u_T, \Gamma_F) = \begin{cases} 0 & k \not\in M^* \\ \frac{1}{m^*} & k \in M^* \end{cases}
\]

where \(m^*\) denotes the cardinality of \(M^*\).

Furthermore, it is apparent that \(F_i(U, u_T, \Gamma_F) = 0\) for any \(i \not\in \text{Supp} T\). For any \(i \in \text{Supp} B_p\), from fuzzy null player, efficiency and symmetry within fuzzy coalitions, we obtain

\[
F_i(U, u_T, \Gamma_F) = \begin{cases} 0 & i \not\in \text{Supp} T \\ \frac{1}{m^*} & i \in \text{Supp} B_p \end{cases}
\]

From Eq. (6), we have

\[
\beta^O(U, u_T, \Gamma_F) = \sum_{M^* \subseteq \text{Supp} B_p \subseteq \text{Supp} S} \delta^O_m \times |\text{Supp} S||(|\text{Supp} B_p | - |\text{Supp} S| - 1) |
\]

Since

\[
\sum_{M^* \subseteq \text{Supp} B_p \subseteq \text{Supp} S} \delta^O_m = \frac{(m^* - 1)(m^* - m)!}{m!} + C^1_{m-m} \frac{m^!}{m!}(m^* - m - 1)! + \cdots
\]

and

\[
\frac{C^1_{m^* - m}}{m^* - m} \frac{(m^* - 1)!}{m^* - m + 1} + \cdots + \frac{(m-1)!}{(m-m)!}
\]

From mathematical induction, we have

\[
\sum_{M^* \subseteq \text{Supp} B_p \subseteq \text{Supp} S} \delta^O_m = \frac{1}{m^*}
\]

For the same reason, we get

\[
\sum_{\text{Supp} B_p \subseteq \text{Supp} S} |\text{Supp} S|(|\text{Supp} B_p | - |\text{Supp} S| - 1) |
\]
Thus, \( \beta^0(U,u_r,\Gamma_r) = \frac{1}{m |\text{Supp}B_r|} \).

From above discussion, we have \( \beta^0(U,u_r,\Gamma_r) = F(U,u_r,\Gamma_r) \), and the proof is finished.

**Corollary 2:** Let \( v_o \in G_o(U) \) be fuzzy convex in \( (U,\Gamma_r) \), then \( \left( \beta^0(U,v_o,\Gamma_r) \right)_{i \in \text{Supp}U} \in FC_o(U,v_o,\Gamma_r) \), where \( \beta^0(U,v_o,\Gamma_r) \) as shown in Eq.(6).

**Corollary 3:** Let \( T \) be a fuzzy carrier in \( U \) for \( v_o \in G_o(U) \), then \( \beta^0(T,v_o,\Gamma_r) = \beta^0(T,v_o,\Gamma_r) \) for any \( i \in \text{Supp}U \), where \( \Gamma_r \) is a fuzzy coalition structure in \( T \) with respect to \( \Gamma_r \), and \( v_o \) is the restriction in \( T \) for \( v_o \).

**Corollary 4:** Let \( v_o \in G_o(U) \) be fuzzy convex in \( (U,\Gamma_r) \), then \( \beta^0(U,v_o,\Gamma_r) \geq v_o(U(i)) \) for any \( i \in \text{Supp}U \).

**Remark 4:** When a fuzzy coalition structure \( (U,\Gamma_r) \) has only one fuzzy coalition \( U \) or all fuzzy coalitions in \( (U,\Gamma_r) \) have only one player, then the Owen value in \( (U,\Gamma_r) \) degenerates to be the Shapley value given by Meng and Zhang [13].

**Example 4** (cf. Example 2): Let the player set \( N=\{1,2,3,4\} \), and \( U=|U(i)| \forall i \in N \) be a fuzzy coalition in \( L(N) \) as defined in Example 2, i.e., \( U(i)>0 \) for any \( i \in N \). \( \Gamma=\{B_1,B_2\} \) is a fuzzy coalition structure in \( U \) as defined in Example 2, i.e., \( B_1=|U(1),U(2)| \) and \( B_2=|U(3),U(4)| \). The coalition values of the associated crisp game \( v_o \in G_o(U) \) of \( v_o \in G_o(U) \) are given by Table 1.

<table>
<thead>
<tr>
<th>( S_i )</th>
<th>( v_o(S_i) )</th>
<th>( S_j )</th>
<th>( v_o(S_j) )</th>
<th>( S_k )</th>
<th>( v_o(S_k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1]</td>
<td>1</td>
<td>[1]</td>
<td>3</td>
<td>[1]</td>
<td>5</td>
</tr>
<tr>
<td>[2]</td>
<td>1</td>
<td>[3]</td>
<td>4</td>
<td>[2]</td>
<td>8</td>
</tr>
<tr>
<td>[3]</td>
<td>[1]</td>
<td>[2,3]</td>
<td>5</td>
<td>[1]</td>
<td>10</td>
</tr>
<tr>
<td>[4]</td>
<td>1</td>
<td>[1,2,3]</td>
<td>6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

If rates of players' participation in fuzzy coalition \( U \) are given by \( U(1)=0.5 \), \( U(2)=0.6 \), \( U(3)=0.4 \) and \( U(4)=0.8 \). From Eq. (5), we have \( v_o(U) = 3.468 \). It is apparent that \( v_o \in G_o(U) \) is fuzzy convex, and the fuzzy core \( FC_o(U,v_o,\Gamma_r) = \left\{(x_1,x_2,x_3,x_4) \mid \sum_{i=1}^{4} x_i = 3.468, x_i \in [0.024,2.296], x_2 \in [0.036,2.756], x_3 \in [0.016,2.34], x_4 \in [0.096,3.164], x_1 + x_2 \in [0.168,3.1], x_3 + x_4 \in [0.368,3.3]\right\} \).

From Eq.(6), we obtain \( \beta^0(U,v_o,\Gamma_r) = 0.699 \), \( \beta^0(U,v_o,\Gamma_r) = 0.935 \).

\( \beta^0(U,v_o,\Gamma_r) = 0.699 \), \( \beta^0(U,v_o,\Gamma_r) = 1.143 \).

It is obvious that \( \left( \beta^0(U,v_o,\Gamma_r) \right)_{i \in \{1,2,3,4\}} \in FC_o(U,v_o,\Gamma_r) \).

**B. The Owen Value for Fuzzy Games with Choquet Integral Form and a Coalition Structure**

The value of fuzzy coalition for fuzzy games with Choquet integral form is given as (see [22])

\[ v_c(U) = \sum_{i=1}^{\#(U)} v_i(U)_i(h_i - h_{i,i}) \]

for any \( U \in L(N) \), where \( Q(U) = |U(i)| \forall i \in N \) and \( [U]_i = \{i \in \text{Supp}U | U(i) \geq h_i \} \). \( q(U) \) denotes the cardinality of \( Q(U) \). The elements in \( Q(U) \) are written in the increasing order as \( 0 = h_0 \leq h_1 \leq \ldots \leq h_{q(U)} \).

Let \( G_c(U) \) denote the set of this kind of fuzzy games in \( U \in L(N) \). For \( v_c \in G_c(U) \), the value of \( S \subseteq U \) with respect to \( (U,\Gamma_r) \) is written as

\[ v_c(S) = \sum_{1 \leq i_1 < i_2 < \ldots < i_q \leq \#(U)} v_i(U)_i(h_i - h_{i,i}) \]

where \( r(U)_i \) is a crisp coalition structure in \( U \) with respect to \( \Gamma_r \), and \( P([U]_i) \cdot r(U)_i \) denotes the set of all feasible coalitions in \( ([U]_i) \cdot r(U)_i \). When the fuzzy game \( v \in G(N) \) is restricted in the setting of \( v_c \in G_c(U) \), from Definition 12, we obtain the definition of the Owen value for \( v_c \in G_c(U) \). Here, we omit it.

**Definition 14:** Let \( v_c \in G_c(U) \), the fuzzy core \( FC_c(U,v_c,\Gamma_r) \) of \( (U,v_c,\Gamma_r) \) is written as

\[ FC_c(U,v_c,\Gamma_r) = \left\{ y \in \mathbb{R}^{\#(U)} \mid \sum_{i=1}^{\#(U)} y_i = \sum_{i \in \text{Supp}U} v_i(U)_i \right\} \]

\[ \times (h_i - h_{i,i}) \right\} \], \( \forall S \subseteq L(U,\Gamma_r) \).

**Theorem 9:** Let \( v_c \in G_c(U) \), the function \( \beta^C(U,v_c,\Gamma_r) \rightarrow \mathbb{R}^{\#(U)} \) is defined by

\[ \beta^C(U,v_c,\Gamma_r) = \sum_{i=1}^{\#(U)} \beta_i([U]_i,v_i,\Gamma_r(U)_i) (h_i - h_{i,i}) \]

for any \( i \in \text{Supp}U \), where \( \beta_i([U]_i,v_i,\Gamma_r(U)_i) \) denotes the Owen value in \( ([U]_i) \cdot \Gamma_r(U)_i \) as given in Eq.(1). Then \( \beta^C \) is the unique Owen value in \( (U,\Gamma_r) \) for \( v_c \in G_c(U) \).

**Proof:** From Theorem 4, we know the existence holds. In the following, we shall show the uniqueness. Since any \( v_c \in G_c(U) \) can be expressed by \( v_c = \sum_{o \in \text{Supp}U} c_o u_r \), where \( c_o = \ldots \)
Let \( F \) be a solution in \((U, u_F, \Gamma_F)\) that satisfies the mentioned axioms in Definition 12, then it is not difficult to get \( F \) is a solution in \((U_h, u_{\Gamma_h h}, \Gamma^{U_h}_F)\), which satisfies the mentioned axioms in Definition 12, when \( v \in G(N) \) is restricted in the setting of \( u_{\Gamma_h h} \).

From efficiency, we have \( F_i(M_h, u_{\Gamma_h h}, \Gamma^{U_h}_F) = 0 \) for any \( k \in M' \). When \( k \in M' \), from symmetry in quotient game, null player and efficiency, it is derived that

\[
\sum_{i \in [T]} \left\{ \begin{array}{ll}
F_i((U)_h, u_{\Gamma h}, \Gamma^{U}_F) = 0 & k \not\in M' h \\
1/m_h & k \in M' h
\end{array} \right.
\]

where \( m_h \) denotes the cardinality of \( M' h \).

Furthermore, it is apparent that \( F_i((U)_h, u_{\Gamma h}, \Gamma^{U}_F) = 0 \) for any \( i \not\in [T] \). When \( i \in [B_h] \), from null player, efficiency and symmetry within coalitions, we obtain

\[
F_i((U)_h, u_{\Gamma h}, \Gamma^{U}_F) = \left\{ \begin{array}{ll}
0 & i \not\in [T] \\
1/m_h |[B_h]| & i \in [B_h]
\end{array} \right.
\]

where \(|[B_h]|\) denotes the cardinality of \([B_h] \).

On the other hand, for any \( i \not\in [T] \), it is apparent \( \beta_i((U)_h, u_{\Gamma h}, \Gamma^{U}_F) = 0 \). When \( i \in [T] \), without loss of generality, suppose \( i \in [B_h] = [B_h] \cap [T] \), then

\[
u_{\Gamma h}((Q_0 \cup S_0 \cup i) - u_{\Gamma h}((Q_0 \cup S_0) = 1,
\]

if and only if \( Q_0 = \bigcup_{p \in p}(B_p)_h \), where \( M_h \setminus k \subseteq R \subseteq M_h \setminus k \)

and \( [B_i] \setminus i \subseteq [B_i] \setminus i \).

From Eq. (1), we have

\[
\beta_i((U)_h, u_{\Gamma h}, \Gamma^{U}_F) = \sum_{i \in [T]} \sum_{l \in [T] \setminus i} \left( m_h - r - 1 \right) m_h ! \frac{1}{m_h ^{[B_h]}} \frac{1}{|[B_h]|}.
\]

where \( m_h \) denotes the cardinality of \( M_h \).

Thus, \( \beta((U)_h, u_{\Gamma h}, \Gamma^{U}_F) = F((U)_h, u_{\Gamma h}, \Gamma^{U}_F) \) for any \( T \subseteq L(U, \Gamma_F) \setminus \{\emptyset\} \) and any \( i \in \{1,2, \ldots, q(T)\} \).

Corollary 5: Let \( v_C \in G_C(U) \) be fuzzy convex in \((U, \Gamma_F)\), then

\[
(\beta^C_i(U, v_C, \Gamma_F) = FC_C(U, v_C, \Gamma_F), \quad \beta^C(U, v_C, \Gamma_F) \text{ as given in Eq. (9).}
\]

Theorem 10: If the associated crisp game \( v_G \in G_0(N) \) of \( v_C \in G_C(U) \) is convex in \((U)_h, \Gamma^{U}_F) \) for any \( l \in \{1,2, \ldots, q(U)\} \), then \( FC_C(U, v_C, \Gamma_F) \neq \emptyset \).

Proof: Let \((\beta_i((U)_h, v_0, \Gamma^{U}_F))_{i \in [Q_0]} \) be the Owen value in \((U)_h, \Gamma^{U}_F) \) as given in Eq. (1). From the convexity of \( v_0 \in G_0(N) \), we have

\[
(\beta_i((U)_h, v_0, \Gamma^{U}_F))_{i \in [Q_0]} \in C((U)_h, v_0, \Gamma^{U}_F),
\]

where \( C((U)_h, v_0, \Gamma^{U}_F) \) denotes the core of \( v_0 \in G_0(N) \) in \((U)_h, \Gamma^{U}_F) \). In the following, we show

\[
(\beta^C_i(U, v_0, \Gamma_F))_{i \in [Q_0]} \in FC_C(U, v_0, \Gamma_F).
\]

From Eq. (9), we have

\[
\sum_{i \in [Q_0]} \beta_i(U, v_0, \Gamma_F) = \sum_{i \in [Q_0]} \sum_{l \in [T] \setminus i} \beta_i((U)_h, v_0, \Gamma^{U}_F) (h_i - h_{i-1})
\]

\[
= \sum_{l \in [T]} \sum_{i \in [Q_0]} \beta_i((U)_h, v_0, \Gamma^{U}_F) (h_i - h_{i-1})
\]

\[
= \sum_{i \in [Q_0]} v_0((U)_h) (h_i - h_{i-1})
\]

\[
v_0(U).
\]

From \((\beta_i((U)_h, v_0, \Gamma^{U}_F))_{i \in [Q_0]} \in C((U)_h, v_0, \Gamma^{U}_F)) \), we get

\[
\sum_{i \in [Q_0]} \beta_i((U)_h, v_0, \Gamma^{U}_F) \geq v_0((S)_h)
\]

for any \( S \subseteq L(U, \Gamma_F) \) and any \( l \in \{1,2, \ldots, q(U)\} \). Thus,
\[
\sum_{i \in \text{Supp}} \beta_i^c(U, v_c, \Gamma_F) = \sum_{i \in \text{Supp}} \sum_{j \in [1]} \beta_i([U]_h, v_0, \Gamma^{[U]_h}_F)(h_j - h_{-1}) \\
= \sum_{i \in \text{Supp}} \sum_{j \in [1]} \beta_i([U]_h, v_0, \Gamma^{[U]_h}_F)(h_j - h_{-1}) \\
\geq \sum_{i \in \text{Supp}} v_0([S]_h)(h_j - h_{-1}) \\
= \sum_{i \in \text{Supp}} v_0([S]_h)(h_j - h_{-1}) \\
= v_0(S). (S).
\]

Namely, \( \beta_i^c(U, v_c, \Gamma_F) \) \( i \in \text{Supp} \) \( v_c \in G_c(N) \) \( v_c \in G_c(U) \) is convex in \( ([U]_h, \Gamma^{[U]_h}_F) \) for any \( l \in \{1, 2, \ldots, q(U)\} \), then \( \beta_i^c(U, v_c, \Gamma_F) \geq v_c(U(i)) \) for any \( i \in \text{Supp} \).

**Proof:** From the convexity of \( v_c \in G_c(N) \), we have
\[
\beta_i([U]_h, v_0, \Gamma^{[U]_h}_F) \geq \sum_{r \subseteq M^{T}_h} \sum_{s \subseteq [B_{l_i}]_h} r(|m_r - r| - 1)! \frac{m_r!}{|m_r|!} v_0(i) \\
= \sum_{r \subseteq M^{T}_h} \frac{r(|m_r - r| - 1)!}{|m_r|!} v_0(i)
\]
and
\[
\beta_i^c(U, v_c, \Gamma_F) = \sum_{i \in \text{Supp}} \beta_i([U]_h, v_0, \Gamma^{[U]_h}_F)(h_j - h_{-1}) \\
\geq \sum_{i \in \text{Supp}} v_0(i)(h_j - h_{-1}) \\
= v_0(i)U(i) \\
= v_c(U(i))
\]
where \( m_r \) and \( [B_{l_i}]_h \) denote the cardinalities of \( M^{T}_h \) and \( [B_{l_i}]_h \), respectively.

**Theorem 11:** If the associated crisp game \( v_0 \in G_0(N) \) for \( v_c \in G_c(U) \) is convex in \( ([U]_h, \Gamma^{[U]_h}_F) \) for any \( l \in \{1, 2, \ldots, q(U)\} \), then \( \beta_i^c(U, v_c, \Gamma_F) \geq v_c(U(i)) \) for any \( i \in \text{Supp} \).

**Proof:** From Theorem 4 in [22], we know \( T \) is a fuzzy carrier in \( U \) for \( v_c \in G_c(U) \), then \( [T]_h \) is a carrier in \( [U]_h \) for the associated crisp game \( v_0 \in G_0(N) \), where \( l \in \{1, 2, \ldots, q(U)\} \). Hence, we have \( v_0(S_0 \cap [T]_h) = v_0(S_0) \) and \( v_0(S_0 \cup i) = v_0(S_0) \) for any \( S_0 \subseteq [U]_h \) and any \( i \in [T]_h \).

From Eq. (1), we get
\[
\beta_i([U]_h, v_0, \Gamma^{[U]_h}_F) = \beta_i([T]_h, v_0^{[T]_h}, \Gamma^{[T]_h}_F) = 0 \quad \forall i \in [T]_h,
\]
where \( \Gamma^{[T]_h} = \{[B_{l_1}^{[T]_h}], [B_{l_2}^{[T]_h}], \ldots, [B_{l_q}^{[T]_h}]\} \) denotes the crisp coalition structure in \( [T]_h \) with respect to \( \Gamma_F \), and \( v_0^{[T]_h} \) denotes the restriction in \( [T]_h \) for the associated crisp game \( v_0 \in G_0(N) \).

When \( i \in [T]_h \), we get
\[
\beta_i([U]_h, v_0, \Gamma^{[U]_h}_F) = \sum_{r \subseteq M^{T}_h} \sum_{s \subseteq [B_{l_i}]_h} r(|m_r - r| - 1)! \frac{m_r!}{|m_r|!} v_0(i) \\
\times s([B_{l_i}]_h, s, [B_{l_i}]_h) \left( v_0(S_0 \cup i) - v_0(S_0) \right) \\
= \sum_{r \subseteq M^{T}_h} \sum_{s \subseteq [B_{l_i}]_h} r(|m_r - r| - 1)! \frac{m_r!}{|m_r|!} v_0(i) \\
\times s([B_{l_i}]_h, s, [B_{l_i}]_h) \left( v_0(S_0 \cup i) - v_0(S_0) \right) \\
= \sum_{r \subseteq M^{T}_h} \sum_{s \subseteq [B_{l_i}]_h} r(|m_r - r| - 1)! \frac{m_r!}{|m_r|!} v_0(i) \\
\times s([B_{l_i}]_h, s, [B_{l_i}]_h) \left( v_0(S_0 \cup i) - v_0(S_0) \right) \\
= \beta_i([T]_h, v_0^{[T]_h}, \Gamma^{[T]_h}_F),
\]
where \( m_r \) and \( |B_{l_i}^{[T]_h}| \) denote the cardinalities of \( M^{T}_h \) and \( |B_{l_i}^{[T]_h}| \) respectively.

**Remark 5:** When a fuzzy coalition structure \( (U, \Gamma_F) \) has only one fuzzy coalition \( U \) or all fuzzy coalitions in \( \Gamma_F \) have only one player, then the Owen value in \( (U, \Gamma_F) \) for \( v_c \in G_c(U) \) degenerates to be the Shapley value proposed by Tsurumi et al [22].

**Example 5:** Let the player set \( N = \{1, 2, 3, 4, 5\} \), and
$U = \{U(i)\}_{i \in N}$ be a fuzzy coalition in $L(N)$ such that $U(i) > 0$ for any $i \in N$. $\Gamma = \{B_1, B_2\}$ is a fuzzy coalitional structure in $U$ such that $B_1 = \{U(1), U(2), U(3)\}$ and $B_2 = \{U(4), U(5)\}$. The coalition values of the associated crisp game $v_0 \in G_c(N)$ of $v_c \in G_c(U)$ are given by Table 2.

If rates of players’ participation in fuzzy coalition $U$ are given by $U(1) = 0.3$, $U(2) = 0.5$, $U(3) = 0.6$, $U(4) = 0.5$ and $U(5) = 0.8$.

From Eq. (8), we have $v_c(U) = 7.4$ in $(U, \Gamma_f)$ for $v_c \in G_c(U)$. It is apparent that $v_c \in G_c(U)$ is fuzzy convex, so we get the fuzzy core

$$FC_c(U, v_c, \Gamma_f) = \{ (x_1, x_2, \ldots, x_i, \ldots, x_5) : x_i \in [0.3, 0.9],$$

$$x_2 \in [0.5, 2.7], x_3 \in [0.6, 3.1], x_4 \in [0.5, 2.7], x_5 \in [0.8, 2.1],$$

$$x_i + x_5 \in [1.1, 3.9], x_i + x_5 \in [1.5, 4.6], x_3 + x_5 \in [1.6, 3.9],$$

$$x_4 + x_5 \in [2.3, 4.9] \}.$$  

From Eq. (9), we obtain

$$\beta_f^c(U, v_c, \Gamma_f) = \beta_i([U]_{h_3}, v_0, [\Gamma_f]_{h_3}) 0.3 = 0.9.$$  

Similarly, we have

$$\beta_f^c(U, v_c, \Gamma_f) = 1.25, \quad \beta_f^c(U, v_c, \Gamma_f) = 1.65, \quad \beta_f^c(U, v_c, \Gamma_f) = 1.875, \quad \beta_f^c(U, v_c, \Gamma_f) = 1.725.$$  

It is obvious that $(\beta_f^c(U, v_c, \Gamma_f))_{i \in N} \in FC_c(U, v_c, \Gamma_f).$

The Owen Value for Fuzzy Games with Proportional Value and A Coalition Structure

The value of fuzzy coalition for fuzzy games with Choquet integral form is expressed by (see [7])

$$v_g(U) = \sum_{i=1}^{q(U)} v_i([U]_{h_i}) h_i$$

for any $U \in L(N)$, where $[U]_{h_i} = \{ i \in \text{Supp} U | U(i) = h_i \}, Q(U)$ and $q(U)$ as given in Eq.(7).

Let $G_c(U)$ denote the set of this kind of fuzzy games in $U \in L(N)$. For $v_g \in G_c(U)$, the value of $S \subseteq U$ with respect to $(U, \Gamma_f)$ is written as

$$v_g(S) = \sum_{i \in \text{Supp}(S), [S]_{h_i} = \{ i \in \text{Supp} U | U(i) = h_i \} \subseteq [U]_{h_i}} v_i([S]_{h_i}) h_i,$$

where $[U]_{h_i}$ is a crisp coalition structure in $[U]_{h_i} = \{ i \in \text{Supp} U | U(i) = h_i \}$ with respect to $\Gamma_f$, and $P([U]_{h_i}, [\Gamma_f]_{h_i})$ denotes the set of all feasible coalitions in $([U]_{h_i}, [\Gamma_f]_{h_i})$.

When the fuzzy game $v_g \in G_c(U)$ is restricted in the setting of $v_g \in G_c(U)$, from Definition 12, we get the definition of the Owen value for $v_g \in G_c(U)$. Here, we omit it.

**Definition 15:** Let $v_g \in G_c(U)$, the fuzzy core $FC_g(U, v_g, \Gamma_f)$ of $(U, v_g, \Gamma_f)$ is written as

$$FC_g(U, v_g, \Gamma_f) = \left\{ y \in [\mathfrak{R}]^{\text{Supp}(U)} \mid \sum_{i \in \text{Supp}(U)} y_i = \sum_{i \in \text{Supp}(U)} \sum_{i \in \text{Supp}(U)} v_i([U]_{h_i}) h_i, \forall \gamma \in [\mathfrak{R}]^{\text{Supp}(U)} \right\}$$

**Theorem 13:** Let $v_g \in G_c(U)$, the function $\beta_f^g : (U, v_g, \Gamma_f) \rightarrow [\mathfrak{R}]^{\text{Supp}(U)}$ is uniquely defined by

$$\beta_f^g(U, v_g, \Gamma_f) = \beta_i([U]_{h_i}, v_0, [\Gamma_f]_{h_i}) \gamma(i)$$

for any $i \in \text{Supp} U$, where $\beta_i([U]_{h_i}, v_0, [\Gamma_f]_{h_i})$ denotes the Owen value in $([U]_{h_i}, [\Gamma_f]_{h_i})$ as given in Eq.(1), $\gamma(i)$ as shown in Eq.(11). Then $\beta_f^g$ is the unique Owen value in $(U, \Gamma_f)$ for $v_g \in G_c(U)$.

**Proof:** From Theorem 4, we know the existence holds. In the following, we shall show the uniqueness. Since

$$\beta_i([U]_{h_i}, v_0, [\Gamma_f]_{h_i}) \gamma(i) \gamma(i)$$

where $[U]_h = \{ i \in \text{Supp} U | U(i) = h_i \}.$

Similarly to Theorem 9, we only need to show the uniqueness of Eq.(12) for unanimity game $u_{\Gamma_f}$, where

$$u_{\Gamma_f}(S) = \begin{cases} 1 & \text{if} \gamma(i), T \in L(U, \Gamma_f) \setminus \{ \emptyset \} (\{\emptyset\} \neq \emptyset) \\ 0 & \text{otherwise} \end{cases},$$

and $l \in \{1, 2, \ldots, q(U)\}$.

Let $M_h = \{ p \in M | [B_p]_{h_i} \neq \emptyset \}, [B_p]_{h_i} = \{ B_p \} \cap [T]_{h_i}\ (p \in M), \ [\Gamma_f]_{h_i} = \{ [B_1]_{h_i}, [B_2]_{h_i}, \ldots, [B_n]_{h_i} \}$ and $M' = \{ p \in M_h | [B_1]_{h_i} \cap [T]_{h_i} \neq \emptyset \}$. From the uniqueness proof of Eq.(9), we have Eq.(12) can be uniquely expressed by

$$\beta_i([U]_{h_i}, u_{\Gamma_f}, [\Gamma_f]_{h_i}) = \begin{cases} 0 & \text{if} i \notin [T]_{h_i} \\ \frac{1}{m_i \gamma([B_i]_{h_i})} & \text{if} i \in [T]_{h_i}, \end{cases}$$

where $m_i$ and $\gamma([B_i]_{h_i})$ denote the cardinalities of $M_h$ and $[B_i]_{h_i}$, respectively.

**Corollary 6:** Let $v_g \in G_c(U)$ be fuzzy convex in $(U, \Gamma_f)$,
then \( (\beta^p(U, v_b, \Gamma_F) \mid_{\text{Supp}} \in FC_p(U, v_b, \Gamma_F), \) \)
where \( \beta^p(U, v_b, \Gamma_F) \) as shown in Eq. (12).

**Theorem 14:** If the associated crisp game \( v_b \in G_C(N) \) of \( v_b \in G_p(U) \) is convex in \((U)_b, \Gamma^{U_b}_F) \) for any \( l \in \{1, 2, \ldots, q(U)\} \), then \( FC_p(U, v_b, \Gamma_F) \neq \emptyset \).

**Proof:** The proof of Theorem 14 is similar to that of Theorem 10.

**Theorem 15:** If the associated crisp game \( v_b \in G_C(N) \) of \( v_b \in G_p(U) \) is convex in \((U)_b, \Gamma^{U_b}_F) \) for any \( l \in \{1, 2, \ldots, q(U)\} \), then \( FC_p(U, v_b, \Gamma_F) \neq \emptyset \).

**Proof:** The proof of Theorem 15 is similar to that of Theorem 11.

**Theorem 16:** Let \( T \) be a fuzzy carrier in \( U \) for \( v_b \in G_p(U) \), then \( \beta^p(U, v_b, \Gamma_F) = \beta^p(T, v_b, \Gamma_F) \) for any \( i \in \text{Supp} U \), where \( \Gamma^F_F \) denotes the fuzzy coalition structure in \( T \) with respect to \( \Gamma_F \), and \( v_b \) denotes the restriction in \( T \) for \( v_b \).

**Proof:** The proof of Theorem 16 is similar to that of Theorem 12.

**Remark 6:** When a fuzzy coalition structure \((U, \Gamma_F)\) has only one fuzzy coalition \( U \) or all fuzzy coalitions in \( \Gamma_F \) have only one player, then the Owen value in \((U, \Gamma_F)\) for \( v_b \in G_p(U) \) degenerates to be the Shapley value proposed by Butnariu [7].

**D. The Owen value for fuzzy games with weighted function and a coalition structure**

The value of fuzzy coalition for fuzzy games with weighted function is denoted by (see [8])

\[
v_p(U) = \sum_{i \in \{0, 1\}} \phi(t) v_i([U]),
\]

for any \( U \in L(N) \), where \([U] = \{i \in \text{Supp} U \mid U(i) = t\} \). \( \phi(t) \) is a weighted function such that \( \phi(t) = 0 \) if and only if \( t = 0 \) and \( \phi(1) = 1 \).

Let \( G_p(U) \) denote the set of this kind of fuzzy games in \( U \in L(N) \). For \( v_p \in G_p(U) \), the value of \( S \subseteq U \) with respect to \((U, \Gamma_F)\) is written as

\[
v_p(S) = \sum_{i \in \{0, 1\}, i \in P(U), i \in \Gamma_F^{U_S}} \phi(t) v_i([S]),
\]

where \( \Gamma_F^{U_S} \) is a crisp coalition structure in \([U] = \{i \in \text{Supp} U \mid U(i) = t\} \) with respect to \( \Gamma_F \), and \( P([U], \Gamma_F^{U_S}) \) denotes the set of all feasible coalitions in \(([U], \Gamma_F^{U_S}) \).

When the fuzzy game \( v \in G(N) \) is restricted in the setting of \( v_b \in G_p(U) \), from Definition 12, we obtain the definition of the Owen value for \( v_p \in G_p(U) \). Here, we omit it.

**Definition 16:** Let \( v_p \in G_p(U) \), the fuzzy core \( FC_p(U, v_p, \Gamma_F) \) of \((U, v_p, \Gamma_F) \) is written as

\[
FC_p(U, v_p, \Gamma_F) = \left\{ y \in \mathbb{R}^{|\text{Supp}|} \mid \sum_{i \in \text{Supp}} y_i = \sum_{(i_0, i) \in P(U), i \in \Gamma_F^{U_i}} \phi(t) v_i([S]), \forall S \in L(U, \Gamma_F) \right\}.
\]

**Theorem 17:** Let \( v_p \in G_p(U) \), the function \( \beta^p(U, v_p, \Gamma_F) \) is convex in \((U)_b, \Gamma^{U_b}_F) \) for any \( i \in \text{Supp} U \), where \( \beta^p(U, v_p, \Gamma_F) \) as given in Eq. (12). Then \( \beta^p(U, v_p, \Gamma_F) \) is the unique Owen value in \((U, \Gamma_F)\) for \( v_p \in G_p(U) \).

**Proof:** The proof of Theorem 17 is similar to that of Theorem 13.

From Eq. (15), we have \( \beta^p(U, v_p, \Gamma_F) = \beta^p(U, v_b, \Gamma_F) \) when \( \phi(t) = t \).

**Corollary 7:** Let \( v_p \in G_p(U) \) be fuzzy convex in \((U, \Gamma_F)\), then \( \beta^p(U, v_p, \Gamma_F) \) is convex in \((U)_b, \Gamma^{U_b}_F) \) for any \( i \in \{0, 1\} \), then \( FC_p(U, v_p, \Gamma_F) \neq \emptyset \).

**Proof:** The proof of Theorem 18 is similar to that of Theorem 10.

**Theorem 19:** If the associated crisp game \( v_b \in G_C(N) \) of \( v_p \in G_p(U) \) is convex in \((U)_b, \Gamma^{U_b}_F) \) for any \( t \in \{0, 1\} \), then \( \beta^p(U, v_p, \Gamma_F) \) as shown in Eq. (15).

**Proof:** The proof of Theorem 19 is similar to that of Theorem 11.

**Theorem 20:** Let \( T \) be a fuzzy carrier in \( U \) for \( v_p \in G_p(U) \), then \( \beta^p(U, v_p, \Gamma_F) = \beta^p(T, v_p, \Gamma_F) \) for any \( i \in \text{Supp} U \), where \( \Gamma^T_F \) denotes the fuzzy coalition structure in \( T \) with respect to \( \Gamma_F \), and \( v_p \) denotes the restriction in \( T \) for \( v_p \).

**Proof:** The proof of Theorem 20 is similar to that of Theorem 12.

**Remark 7:** When a fuzzy coalitional structure \((U, \Gamma_F)\) has only one coalition \( U \) or all fuzzy coalitions in \( \Gamma_F \) have only one fuzzy coalition \( U \), the Owen value in \((U, \Gamma_F)\) for \( v_p \in G_p(U) \) degenerates to be the Shapley value proposed by Butnariu and Kroupa [8].

**Example 6:** Let the player set \( N = \{1, 2, 3, 4, 5\} \), and
If rates of players' participation in fuzzy coalition $U$ are given by $U(i) = 0.5(i = 1, 2, 3, 4)$ and $U(5) = 0.6$.

From Eq. (14), we have $v_s(U) = 12\varphi(0.5) + \varphi(0.6)$. It is apparent that $v_s \in G_s(U)$ is fuzzy convex, the fuzzy core $FC_s(U, v_s, \Gamma_f) = \left\{ (x_1, x_2, x_3, x_4, x_5) \mid \sum_{i=1}^{5} x_i = 12\varphi(0.5) + \varphi(0.6), x_1 \in [\varphi(0.5), 3\varphi(0.5)], x_2 \in [\varphi(0.5), 6\varphi(0.5)], x_3 \in [\varphi(0.5), 4\varphi(0.5)], x_4 \in [\varphi(0.5), 6\varphi(0.5)], x_5 = \varphi(0.6) \right\}$.

From Eq. (15), we obtain
\[
\beta^s(U, v_s, \Gamma_f) = \varphi(0.5) \beta^1([U]_{3,5}, v_s, [\Gamma_f]_{3,5}) = 3\varphi(0.5).
\]

Similarly, we get
\[
\beta^w(U, v_w, \Gamma_f) = 2.5\varphi(0.5), \beta^w(U, v_w, \Gamma_f) = 3\varphi(0.5), \\
\beta^s(U, v_s, \Gamma_f) = 3.5\varphi(0.5), \beta^s(U, v_s, \Gamma_f) = \varphi(0.6).
\]

It is apparent that $\left\{ \beta^w(U, v_w, \Gamma_f) \right\}_{i=1}^{5} \in FC_w(U, v_w, \Gamma_f)$.

When $\varphi(t) = t$, we get the value $v_s(U) = v_w(U) = 6.6$, and the Owen value for players are
\[
\beta^s(U, v_s, \Gamma_f) = \beta^w(U, v_w, \Gamma_f) = \beta^w(U, v_w, \Gamma_f) = 1.5, \\
\beta^w(U, v_w, \Gamma_f) = 1.25, \\
\beta^w(U, v_w, \Gamma_f) = 1.75, \\
\beta^w(U, v_w, \Gamma_f) = 0.6.
\]

5. Conclusions

We have discussed the Owen value for fuzzy games with a coalition structure. The concepts related to the Owen value have been extended to the case of fuzzy games. The relationship between the Owen value and the fuzzy core of fuzzy games with a coalition structure is studied. In order to better understand the Owen value, we have studied four special classes of fuzzy games with a coalition structure. The existence and uniqueness of the Owen value for these four kinds of fuzzy games with a coalition structure are shown. Like the crisp case, we can have different axiomatic systems to define the Owen value for fuzzy games with a coalition structure, and give the corresponding proof.

However, we mainly study the Owen value for cooperative fuzzy games with a coalition structure, and it will be interesting to discuss other payoff indices for fuzzy games with a coalition structure, such as the Owen-Banzhaf value and the symmetric Banzhaf coalitional value.

In this paper, we only learn games with fuzzy coalitions and crisp values. We will further study games with fuzzy payoffs which combine the operator of fuzzy numbers [9, 10, 24, 26] and the discussed payoff indices.

Acknowledgments

The authors first gratefully thank the Associate Editor and two anonymous referees for their valuable comments, which have much improved the paper. This work was supported by the National Natural Science Foundation of China (No. 70771010, 70801064 and 71071018).

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