Fuzzy Optimal Solution for Unbalanced Fully Fuzzy Minimal Cost Flow Problems

Manjot Kaur and Amit Kumar

Abstract

To the best of our knowledge, till now there is no method in the literature to find the fuzzy optimal solution of unbalanced fully fuzzy minimal cost flow (FFMCF) problems. In this paper, a new method is proposed to find the fuzzy optimal solution of these problems. The proposed method is illustrated with the help of a numerical example. Also, the proposed method is used to find the fuzzy optimal solution of an unbalanced real life FFMCF problem.

Keywords: FFMCF problems, Fuzzy linear programming, LR flat fuzzy number.

1. Introduction

Minimal cost flow (MCF) problem is a general form of the network flow problem whose aim is to find the least cost of the shipment of a commodity through a capacitated network in order to satisfy demands at certain nodes from available supplies at other nodes. Because it represents a general form of the network flow, the results from the study of the MCF problem can be applied to many other network problems such as transportation, maximum flow, assignment, shortest path, and transshipment problems.

In actual practice, the costs and the capacities of the network are generally vague or uncertain. Fuzzy set theory [16] appears to be ideally suited to solve such vague aspects. Shih and Lee [15] proposed a fuzzy version of MCF problems by using multi-level linear programming problem.


In this paper the shortcomings and limitations of the existing method [4] are pointed out and to overcome these shortcomings and limitations, a new method is proposed to find the fuzzy optimal solution of FFMCF problems.

This paper is organized as follows: In Section 2, some basic concepts of fuzzy sets are presented. In Section 3, linear programming formulations of balanced crisp and fully fuzzy MCF problems are presented. In Section 4, shortcomings of existing fuzzy linear programming formulation of balanced FFMCF problems are pointed out. In Section 5, limitations of an existing method are pointed out. To overcome the shortcomings of the existing linear programming formulation a modified representation of linear programming formulation of crisp and FFMCF problems is presented in Section 6. In Section 7, a new method is proposed to find the solution of FFMCF problems. In Section 8, advantages of the proposed method over existing method are discussed. In Section 9 an existing real life unbalanced FFMCF problem is solved by using the proposed method. The results and conclusions are discussed in Section 10 and Section 11 respectively.

2. Preliminaries

In this section, some basic definitions and arithmetic operations of LR flat fuzzy numbers are presented.

A. Basic Definitions

In this section, some basic definitions are presented [2].

Definition 1: A function \( L : [0, \infty) \to [0,1] \) (or \( R : [0, \infty) \to [0,1] \)) is said to be reference function of fuzzy numbers if and only if

(i) \( L(x) = L(-x) \) (or \( R(x) = R(-x) \))

(ii) \( L(0) = 1 \) (or \( R(0) = 1 \))

(iii) \( L \) (or \( R \)) is non-increasing on \([0, \infty)\).

Definition 2: A fuzzy number \( \tilde{A} = (x, \bar{x}, x^L, x^R) \) is said to be an LR flat fuzzy number if the membership function \( \mu_\tilde{A}(x) \) is given by
If $\bar{x} = x = \bar{x}$ then $\tilde{A} = (x, \bar{x}, x^L, x^R)_{LR}$ will be converted into $\tilde{A} = (x, x^L, x^R)_{LR}$ and is said to be an LR fuzzy number.

**Definition 3:** Let $\tilde{A} = (x, \bar{x}, x^L, x^R)_{LR}$ be an LR flat fuzzy number and $\lambda$ be a real number in the interval [0,1] then the crisp set $A_\lambda = \{x \in X : \mu_\lambda(x) \geq \lambda\} = [x - x^L L^{-1}(\lambda), x^R + x^R R^{-1}(\lambda)]$ is said to be $\lambda$-cut of $\tilde{A}$.

**Definition 4:** Two LR flat fuzzy numbers $\tilde{A}_1 = (x_1, \bar{x}_1, x_1^L, x_1^R)_{LR}$ and $\tilde{A}_2 = (x_2, \bar{x}_2, x_2^L, x_2^R)_{LR}$ are said to be equal i.e., $A_1 = A_2$ if and only if $x_1 = x_2$, $\bar{x}_1 = \bar{x}_2$, $x_1^L = x_2^L$ and $x_1^R = x_2^R$.

**Definition 5:** An LR flat fuzzy number $\tilde{A} = (x, \bar{x}, x^L, x^R)_{LR}$ is said to be non-negative (non-positive) LR flat fuzzy number if $x - x^L \geq 0$, $(x - x^R \leq 0)$, $x^L, x^R \geq 0$.

**B. Arithmetic Operations**

In this section, some arithmetic operations of LR flat fuzzy numbers, are presented [2]. Let $\tilde{A}_1 = (x_1, \bar{x}_1, x_1^L, x_1^R)_{LR}$, $\tilde{A}_2 = (x_2, \bar{x}_2, x_2^L, x_2^R)_{LR}$ be two LR flat fuzzy numbers and $\tilde{A}_3 = (x_3, \bar{x}_3, x_3^L, x_3^R)_{RL}$ be a RL flat fuzzy number. Then

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = (x_1 + x_2, \bar{x}_1 + \bar{x}_2, x_1^L + x_2^L, x_1^R + x_2^R)_{LR}$

(ii) $\tilde{A}_1 \Theta \tilde{A}_2 = (x_1 - x_2, \bar{x}_1 - \bar{x}_2, x_1^L + x_2^R, x_1^R + x_2^L)_{LR}$

(iii) $\tilde{A}_1 \otimes \tilde{A}_2 = (x_1 x_2, \bar{x}_1 \bar{x}_2, x_1 x_2^L - x_1^L x_2^R, x_1 x_2^R - x_1^R x_2^L)_{LR}$ for $x_1 - x_2 \geq 0$ and $x_1 - x_2^R \geq 0$.

**3. Existing Linear Programming Formulation of Balanced Crisp and FFMCF Problems**

In this section, existing linear programming formulation of balanced crisp and FFMCF problems are presented.

**A. Classification of Nodes**

The nodes used in MCF problems can be categorized as follows:

- **Purely source node:** A node $S$ is said to be a purely source node if there exist at least one node $S'$ such that the product may be supplied from $S$ to $S'$ but there does not exist any node $S''$ such that product may be supplied from $S'$ to $S$. The set of all such nodes is represented by $N_{PS}$.

- **Purely destination node:** A node $D$ is said to be a purely destination node if there does not exist any node $D'$ such that the product may be supplied from $D$ to $D'$ but there exist at least one node $D''$ such that product may be supplied from $D''$ to $D$. The set of all such nodes is represented by $N_{PD}$.

- **Intermediate node:** The following nodes in the network are said to be intermediate nodes:

  - A node $S$ at which some quantity of the product is available to transship at other nodes and also there exist some nodes such that some quantity of the product is supplied from those nodes to node $S$. All such intermediate nodes are said to be source nodes and the set of all such nodes is represented by $N_S$.

  - A node $D$ at which some quantity of the product is required and also there exist some nodes such that the product is supplied from node $D$ to those nodes. All such nodes $D$ are said to be destination nodes and the set of all such intermediate nodes is represented by $N_D$.

  - A node $T$ at which neither any quantity of the product is available to transship at other nodes nor any quantity of the product is required but there exist some nodes such that some quantity of the product is supplied from those nodes to node $T$ and the same quantity of the product is supplied from $T$ to some other nodes. All such nodes $T$ are said to be transition nodes and the set of all such intermediate nodes is represented by $N_T$.

**B. Existing Linear Programming Formulation of Balanced Crisp MCF Problems**

Any balanced crisp MCF problem (total supply = total demand) can be formulated into the crisp linear programming problem $(P_1)$ [1]:

Minimize $\sum_{(i,j) \in A} c_{ij}x_{ij}$

subject to $\sum_{j:(i,j) \in A} x_{ij} = a_i \forall i \in N_{PS},$

$\sum_{j:(i,j) \in A} x_{ij} - \sum_{j:(j,i) \in A} x_{ji} = e_i \forall i \in N_S,$

$\sum_{i:(i,j) \in A} x_{ij} = b_j \forall j \in N_{PD}, (P_1)$
\[
\sum_{i,j \in A} x_{ij} - \sum_{i,j \in A} x_{ji} = d_j \quad \forall j \in N_D, \\
\sum_{j(i,j) \in A} x_{ij} = \sum_{j(j,i) \in A} x_{ji} \quad \forall i \in N_T, \\
l_j \leq x_{ij} \leq u_j, \quad x_{ij} \geq 0; \quad \forall (i,j) \in A
\]

where, \( A \): The set of arcs \((i,j), x_{ij}\): Decision variable denoting the flow through arc \((i,j), c_{ij}\): Cost per unit flow through arc \((i,j), a_i\): Supply of the product at \(i\)th purely source node, \(b_j\): Demand of the product at \(j\)th purely destination node, \(d_j\): Demand of the product at \(j\)th destination node, \(u_j\): Capacity for arc \((i,j), l_j\): Minimum flow required for arc \((i,j)\).

### C. Existing Linear Programming Formulation of Balanced FFMCF Problems

Replacing the parameters \(x_{ij}, c_{ij}, a_i, e_i, b_j, d_j\) and \(u_j\) by \(\tilde{x}_{ij}, \tilde{e}_{ij}, \tilde{a}_i, \tilde{e}_i, \tilde{b}_j, \tilde{d}_j\) and \(\tilde{u}_j\) respectively, the crisp balanced MCF problem \((P)\) can be converted into the fuzzy linear programming problem \((P_2)\): [4]:

Minimize 
\[
\sum_{(i,j) \in A} \tilde{c}_{ij} \otimes \tilde{x}_{ij} 
\]

Subject to
\[
\sum_{j(i,j) \in A} \tilde{x}_{ij} = \tilde{a}_i; \quad \forall i \in N_P, \\
\sum_{j(j,i) \in A} \tilde{x}_{ij} = \tilde{e}_i; \quad \forall i \in N_S, \\
\sum_{i,j \in A} \tilde{x}_{ij} = \tilde{b}_j; \quad \forall j \in N_P, (P_2) \\
\sum_{i,j \in A} \tilde{x}_{ij} \Theta H \sum_{j(i,j) \in A} \tilde{x}_{ji} = \tilde{a}_j; \quad \forall j \in N_D, \\
\sum_{j(i,j) \in A} \tilde{x}_{ij} = \sum_{j(j,i) \in A} \tilde{x}_{ji}; \quad \forall i \in N_T, \\
\tilde{0} \leq \tilde{x}_{ij} \leq \tilde{u}_j; \quad \forall (i,j) \in A
\]

\(\tilde{x}_{ij}\) is a non-negative LR fuzzy number \(\forall (i,j) \in A\)

### D. Existing Method to Convert Fuzzy Restrictions into Crisp Restrictions

To solve the fuzzy linear programming problem \((P_2)\), Ghatee and Hashemi [4] pointed out that the average, left and right spreads of a feasible flow have to be less than the maximal value of the corresponding quantities. Hence, Ghatee and Hashemi [4] have replaced the fuzzy restriction \(0 \leq \tilde{x}_{ij} \leq \tilde{u}_j\) i.e., \((0,0,0)_{LR} \leq (x_{ij}, x_{ij}^R)_{LR}\) by the following crisp restrictions:

\[0 \leq x_{ij} \leq u_j, \quad 0 \leq x_{ij}^L \leq u_j^L, \quad 0 \leq x_{ij}^R \leq u_j^R.\]

### 4. Shortcoming of Existing Linear Programming Formulation

In this section, the shortcoming of the existing formulation of balanced FFMCF problem, presented in Section 3, is pointed out.

In the existing method [4] the formulation \((P_2)\) is used to find the fuzzy optimal solution of FFMCF problems and in the formulation \((P_2)\) the following type of equation is used:

\[\tilde{A} \Theta H \tilde{B} = \tilde{C},\]

where \(\tilde{A}, \tilde{B}\) and \(\tilde{C}\) are LR fuzzy numbers i.e., in the existing formulation, it is assumed that Hukuhara’s difference of two LR flat fuzzy numbers \(\tilde{A}\) and \(\tilde{B}\) will also be an LR flat fuzzy number \(\tilde{C}\). However, the Hukuhara’s difference of two LR flat fuzzy numbers \(\tilde{A}\) and \(\tilde{B}\) is not necessarily an LR flat fuzzy number e.g., for the LR flat fuzzy numbers \(\tilde{A} = (2,5,3,4)_{LR}\) and \(\tilde{B} = (1,6,1,2)_{LR}\), \(\tilde{C} = \tilde{A} \Theta H \tilde{B} = (1,-,1,2,2)_{LR}\) is not an LR flat fuzzy number.

### 5. Limitations of the Existing Method

In this section, the limitations of the existing method [4] are pointed out.

The existing method [4] can be used for solving such balanced FFMCF problems for which \(\tilde{I}_y = (0,0,0)_{LR}\) or \(\tilde{I}_y = (0,0,0,0)_{LR}\) \forall (i,j) \in A\) but the existing method [4] can’t be used for solving such balanced FFMCF problems for which \(\tilde{I}_y = (0,0,0)_{LR}\) or \(\tilde{I}_y = (0,0,0,0)_{LR}\) \forall (i,j) \in A\) and such balanced FFMCF problems for which \(\tilde{I}_y = (0,0,0)_{LR}\) or \(\tilde{I}_y = (0,0,0,0)_{LR}\) for some \((i,j) \in A\) e.g., the balanced FFMCF problem, chosen in Example 5.1, can’t be solved by using the existing method [4].

![Figure 1. A network with 4 nodes and 5 arcs.](image)
Example 5.1: Consider a network, shown in Figure 1, with four nodes including one purely source node 2, one purely destination node 4, one destination node 3 and one transition node 1. Let the fuzzy supply of the product \( \tilde{a}_2 \) at purely source node 2, fuzzy demand of the product \( \tilde{b}_4 \) at purely destination node 4, fuzzy demand of the product \( \tilde{a}_3 \) at destination node 3, be represented by \( LR \) flat fuzzy numbers \((150,250,100,50)_{LR}, (50,100,20,0)_{LR} \) and \((100,150,80,50)_{LR} \) respectively and all the fuzzy transportation cost \((\tilde{c}_{ij})\), fuzzy capacity \((\tilde{u}_{ij})\) and fuzzy minimum flow \((\tilde{l}_{ij})\) required for each arc \((i,j)\) of the network be represented by \( LR \) flat fuzzy numbers \((12,15,2,5)_{LR}, (9,11,2,19)_{LR}, (10,12,3,7)_{LR}, (10,15,5,5)_{LR}, (70,80,10,10)_{LR}, (251,260,1,2)_{LR}, (30,2,5,2)_{LR}, (90,100,10,20)_{LR}, (60,70,10,10)_{LR}, (10,0,0,0)_{LR}, (10,15,2,3)_{LR}, (15,17,3,10)_{LR}, (0,0,0,0)_{LR}, (0,0,0,0)_{LR}, (0,0,0,0)_{LR}, (0,0,0,0)_{LR}, (0,0,0,0)_{LR}, (0,0,0,0)_{LR}, (0,0,0,0)_{LR} \) respectively.

6. Modified Representation of Linear Programming Formulation of Balanced Crisp and FFMCF Problems

In this section, to overcome all the shortcomings, pointed out in Section 4, the existing linear programming formulation of crisp and FFMCF problems, presented in Section 3, is represented in such a modified manner so that the physical meaning of existing and modified linear programming formulation of crisp MCF problems are same while the modified representation of FFMCF problem represents the FFMCF problems in a more realistic manner as compared to the existing linear programming formulation of FFMCF problems.

A. Modified Representation of Linear Programming Formulation of Balanced Crisp MCF Problems

The crisp linear programming formulation \((P_1)\) of balanced crisp MCF problem can be converted into crisp linear programming problem \((P_3)\):

Minimize\( \sum_{(i,j) \in A} c_{ij}x_{ij} \)

subject to

\[ \sum_{j \in A(i)} x_{ij} = a_i; \forall i \in N_{PS}, \]
\[ \sum_{j \in A(i)} x_{ji} = \sum_{j \in A(i)} x_{ij} + e_i; \forall i \in N_{S}, \]
\[ \sum_{i \in A(j)} x_{ij} = b_j; \forall j \in N_{PP}, (P_3) \]
\[ \sum_{i \in A(j)} x_{ij} = \sum_{i \in A(j)} x_{ji}; \forall i \in N_D, \]
\[ \sum_{j \in A(i)} x_{ij} = \sum_{j \in A(i)} x_{ji}; \forall i \in N_T, \]
\[ l_{ij} \leq x_{ij} \leq u_{ij}; \forall (i,j) \in A, \]

B. Modified Representation of Linear Programming Formulation of Balanced FFMCF Problems

The linear programming formulation of balanced FFMCF problems, presented in Section 3 (C), is obtained by using the linear programming formulation of
balanced crisp MCF problems, presented in Section 3 (B). On the same direction the modified crisp linear programming problem of balanced crisp MCF problems \( P_1 \), can be converted into the following fuzzy linear programming problem \( P_2 \) :

Minimize \( \sum_{(i,j) \in A} \bar{c}_{ij} \otimes \bar{x}_{ij} \)

subject to

\[
\sum_{j \in A} \bar{x}_{ij} = \bar{a}_i; \quad \forall i \in N_{PS},
\]

\[
\sum_{i \in A} \bar{x}_{ij} = \bar{b}_j; \quad \forall j \in N_{PD},
\]

\[
\sum_{i \in A} \bar{x}_{ij} = \bar{\tilde{d}}_j; \quad \forall j \in N_D,
\]

\[
\bar{x}_{ij} \leq \bar{\tilde{x}}_{ij}; \quad \forall (i,j) \in A
\]

\( \bar{x}_{ij} \) is a non-negative LR flat fuzzy number \( \forall (i,j) \in A \).

### 7. Proposed Method

In this section, to overcome all the limitations of the existing method, discussed in Section 5, a new method is proposed for finding the exact fuzzy optimal solution of balanced and unbalanced FFMC problems by representing all the parameters by LR flat fuzzy numbers. If the supply of product at \( \text{ith} \) purely source node and \( \text{jth} \) source node are \( \bar{a}_i = (a_i, a_i^L, a_i^R) \) LR and \( \bar{a}_j = (a_j, a_j^L, a_j^R) \) LR respectively and the demand of the product at \( \text{jth} \) purely destination node and \( \text{jth} \) destination node are \( \bar{b}_j = (b_j, b_j^L, b_j^R) \) LR and \( \bar{d}_j = (d_j, d_j^L, d_j^R) \) LR respectively then the exact fuzzy optimal solution of FFMC problems can be obtained by using the following steps:

**Step 1:** Find the total fuzzy supply \( \sum_{i \in N_{PS}} \bar{a}_i \otimes \sum_{i \in N_{S}} \bar{c}_i \) and total fuzzy demand \( \sum_{j \in N_{PD}} \bar{b}_j \otimes \sum_{j \in N_{D}} \bar{d}_j \). Let \( \sum_{i \in N_{PS}} \bar{a}_i \otimes \sum_{i \in N_{S}} \bar{c}_i = (m, m^L, m^R) \) LR and \( \sum_{j \in N_{PD}} \bar{b}_j \otimes \sum_{j \in N_{D}} \bar{d}_j = (n, n^L, n^R) \) LR. Examine that the problem is balanced or not, i.e.,

\[
\sum_{i \in N_{PS}} \bar{a}_i \otimes \sum_{i \in N_{S}} \bar{c}_i = \sum_{j \in N_{PD}} \bar{b}_j \otimes \sum_{j \in N_{D}} \bar{d}_j \quad \text{or} \quad \sum_{i \in N_{PS}} \bar{a}_i \otimes \sum_{i \in N_{S}} \bar{c}_i \neq \sum_{j \in N_{PD}} \bar{b}_j \otimes \sum_{j \in N_{D}} \bar{d}_j.
\]

**Case (i)** If the problem is balanced i.e.,

\[
\sum_{i \in N_{PS}} \bar{a}_i \otimes \sum_{i \in N_{S}} \bar{c}_i = \sum_{j \in N_{PD}} \bar{b}_j \otimes \sum_{j \in N_{D}} \bar{d}_j \quad \text{then Go to Step 2.}
\]

**Case (ii)** If the problem is unbalanced i.e.,

\[
\sum_{i \in N_{PS}} \bar{a}_i \otimes \sum_{i \in N_{S}} \bar{c}_i \neq \sum_{j \in N_{PD}} \bar{b}_j \otimes \sum_{j \in N_{D}} \bar{d}_j \quad \text{then convert the unbalanced problem into balanced problem as follows:}
\]

**Case (a)** If \( m^L - m^R \leq n^L - n^R, m^L \leq n^L, \bar{m} - m \geq \bar{n} - n \) and \( m^R \leq n^R \) then introduce a dummy purely source node with fuzzy supply \( (m - m, \bar{m} - m, m^L - n^L, n^R - m^R) \) LR. Assume the fuzzy transportation cost for the one unit quantity of the product from the introduced dummy purely source node to all purely destination nodes and all intermediate nodes as zero LR flat fuzzy number then Go to Step 2.

**Case (b)** If \( m^L - m^R \geq n^L - n^R, m^L \geq n^L, \bar{m} - m \geq \bar{n} - n \) and \( m^R \geq n^R \) then introduce a dummy purely destination node with fuzzy demand \( (m - n, \bar{m} - n, m^L - n^L, m^R - n^R) \) LR. Assume the fuzzy transportation cost for the one unit quantity of the product from all purely source nodes and intermediate nodes to the introduced dummy purely destination node as zero LR flat fuzzy number then Go to Step 2.

**Case (c)** If neither Case (a) nor Case (b) is satisfied then introduce a dummy purely source node with fuzzy supply

\[
(\max\{0, (\bar{n} - n^L) - (m - m^L)\} + \max\{0, (n^L - m^L)\}, \max\{0, (n - n^L) - (m - m^L)\} + \max\{0, (n^L - m^L)\}, \max\{0, (\bar{m} - n^R) - (m - m^R)\}, \max\{0, (n^R - m^R)\})_L .
\]

and also introduce a dummy purely destination node with fuzzy demand

\[
(\max\{0, (m - m^R) - (n - n^R)\} + \max\{0, (m^R - m^L)\}, \max\{0, (m - m^R) - (n - n^R)\} + \max\{0, (m^L - n^L)\}, \max\{0, (\bar{m} - m) - (n - n^L)\}, \max\{0, (m^L - n^L)\}, \max\{0, (\bar{m} - m) - (n - n^L)\}, \max\{0, (m^L - n^L)\}, \max\{0, (\bar{m} - m) - (n - n^L)\}, \max\{0, (m^L - n^L)\}, \max\{0, (\bar{m} - m) - (n - n^L)\})_L .
\]

Assume the fuzzy transportation cost for the one unit quantity of the product from the introduced dummy purely source node to all intermediate nodes, existing purely destination nodes and introduced dummy purely destination node as zero LR flat fuzzy number. Similarly assume the fuzzy transportation cost for the one unit quantity of the product from all intermediate nodes, existing purely source nodes and introduced dummy purely source node to the introduced dummy purely destination node as zero LR flat fuzzy number then Go to Step 2.
Step 2: Assuming $\tilde{c}_{ij} = (c_{ij}, \xi_{ij}, c_{ij}^L, c_{ij}^R)_{LR}$, $\tilde{x}_{ij} = (x_{ij}, \phi_{ij}, x_{ij}^L, x_{ij}^R)_{LR}$, $\tilde{a}_i = (a_i, \alpha_i, a_i^L, a_i^R)_{LR}$, $\tilde{b}_j = (b_j, \beta_j, b_j^L, b_j^R)_{LR}$, $\tilde{d}_j = (d_j, \eta_j, d_j^L, d_j^R)_{LR}$, $\tilde{w}_R = (w_R, \zeta_R, w_R^L, w_R^R)_{LR}$, and $\tilde{w}_P = (w_P, \zeta_P, w_P^L, w_P^R)_{LR}$ using the arithmetic operations of LR flat fuzzy numbers, defined in Section 2 (B) and using the fuzzy linear programming formulation (P_2) of the balanced FFMC problem, the fuzzy linear programming formulation of balanced FFMC problem, obtained in Step 1, can be written as:

Minimize $\sum_{(i,j) \in A} (c_{ij} \tilde{x}_{ij} \tilde{a}_i \tilde{b}_j$, $\tilde{x}_{ij}^L \tilde{c}_{ij}^L - x_{ij}^L c_{ij}^L, \tilde{c}_{ij}^R + \tilde{x}_{ij}^R)$}_{LR}$

subject to

$\sum_{j \in \{j \mid (j,i) \in A\}} x_{ij}^L \geq a_i^L, \sum_{j \in \{j \mid (j,i) \in A\}} x_{ij}^R = a_i^R$: (C1)

$\sum_{i \in N \setminus PS} \sum_{j \in \{j \mid (j,i) \in A\}} x_{ij}^L + \sum_{j \in \{j \mid (j,i) \in A\}} x_{ij}^R = \sum_{j \in \{j \mid (j,i) \in A\}} x_{ij}^L + e_i$: (C2)

Step 4: Suppose the fuzzy linear programming problem (P_2) have f feasible solutions and $\{(\tilde{x}_{ij})^w, (\tilde{c}_{ij})^w, (\tilde{b}_j)^w, (\tilde{d}_j)^w\}$ be the wth feasible solution then our goal is to find such a feasible solution corresponding to which the value of the objective function is minimum i.e., minimum$\sum_{i \in N \setminus PS} \sum_{j \in \{j \mid (j,i) \in A\}} (c_{ij} \tilde{x}_{ij} \tilde{a}_i \tilde{b}_j$, $\tilde{x}_{ij}^L \tilde{c}_{ij}^L - x_{ij}^L c_{ij}^L, \tilde{c}_{ij}^R + \tilde{x}_{ij}^R)$}_{LR}$

$\sum_{(i,j) \in A} (c_{ij} \tilde{x}_{ij} \tilde{a}_i \tilde{b}_j$, $\tilde{x}_{ij}^L \tilde{c}_{ij}^L - x_{ij}^L c_{ij}^L, \tilde{c}_{ij}^R + \tilde{x}_{ij}^R)$}_{LR}$ is a non-negative LR flat fuzzy number.

Step 3: Using Definition 4 and Definition 5, the fuzzy linear programming problem, obtained in Step 2, can be written as:

Minimize $\sum_{(i,j) \in A} (c_{ij} \tilde{x}_{ij} \tilde{a}_i \tilde{b}_j$, $\tilde{x}_{ij}^L \tilde{c}_{ij}^L - x_{ij}^L c_{ij}^L, \tilde{c}_{ij}^R + \tilde{x}_{ij}^R)$}_{LR}$

subject to

$\sum_{j \in \{j \mid (j,i) \in A\}} x_{ij}^L \geq a_i^L, \sum_{j \in \{j \mid (j,i) \in A\}} x_{ij}^R = a_i^R$: (C1)

$\sum_{i \in N \setminus PS} \sum_{j \in \{j \mid (j,i) \in A\}} x_{ij}^L + \sum_{j \in \{j \mid (j,i) \in A\}} x_{ij}^R = \sum_{j \in \{j \mid (j,i) \in A\}} x_{ij}^L + e_i$: (C2)

$\sum_{(i,j) \in A} (c_{ij} \tilde{x}_{ij} \tilde{a}_i \tilde{b}_j$, $\tilde{x}_{ij}^L \tilde{c}_{ij}^L - x_{ij}^L c_{ij}^L, \tilde{c}_{ij}^R + \tilde{x}_{ij}^R)$}_{LR}$
where, \( R(p, \overline{p}, p^f, p^R)_{LR} = \frac{1}{2} \left( (p - p^R)\mathcal{L}^1(\lambda) \right) d\lambda + \int_0^1 (p - p^f)\mathcal{L}^1(\lambda) d\lambda \)

\( (\overline{p} + p^R\mathcal{L}^{-1}(\lambda)) \) represents the Liou and Wang ranking index [10] of an LR flat fuzzy number. In other words, by using the existing method [3, 8, 9, 11-14], the fuzzy optimal solution of the fuzzy linear programming problem, obtained in Step 3, can be obtained by solving the following crisp linear programming problem:

Minimize \( \sum_{(i,j)\in A} R(C_{ij}, \overline{c}_{ij}, \overline{x}_{ij}, c_{ij}^L, x_{ij}^L, c_{ij}^R, x_{ij}^R)_{LR} \)

subject to constraints \((C_i)\) of problem \((P_3)\)

with \( R(C_{ij}, \overline{c}_{ij}, \overline{x}_{ij}, c_{ij}^L, x_{ij}^L, c_{ij}^R, x_{ij}^R)_{LR} \leq R(u_{ij}, \overline{u}_{ij}, u_{ij}^L, u_{ij}^R)_{LR} \)

Step 5: Solve the crisp linear programming problem, obtained in Step 4, to find the optimal solution \( \{x_{ij}, \overline{x}_{ij}, c_{ij}^L, x_{ij}^L, c_{ij}^R, x_{ij}^R\} \).

Step 6: Put the values of \( x_{ij}, \overline{x}_{ij}, c_{ij}, c_{ij}^L, c_{ij}^R, x_{ij}^L, x_{ij}^R \) in \( \overline{x}_{ij} = (x_{ij}, \overline{x}_{ij}, c_{ij}, c_{ij}^L, x_{ij}^L, x_{ij}^R)_{LR} \) to obtain the fuzzy optimal solution \( \{\overline{x}_{ij}\} \) of FFMCF problem.

Step 7: Put the values of \( \overline{x}_{ij} \), obtained from Step 6, in \( \sum_{(i,j)\in A} ((C_{ij}, \overline{c}_{ij}, c_{ij}^L, c_{ij}^R)_{LR} \otimes (x_{ij}, \overline{x}_{ij}, c_{ij}^L, x_{ij}^L, c_{ij}^R, x_{ij}^R)_{LR}) \) to find the minimum total fuzzy transportation cost.

8. Advantages of the Proposed Method over Exising Method

In this section, advantages of the proposed method over existing method [4] are discussed.

- The existing method [4] can be used to find the fuzzy optimal solution of such balanced FFMCF problems in which \( \overline{t}_{ij} = (0,0,0,0)_{LR} \) \( \forall (i,j) \in A \) but can’t be used to find the fuzzy optimal solution of similar type of unbalanced FFMCF problems. However, the proposed method can be used to find the fuzzy optimal solution of similar type of balanced and unbalanced FFMCF problems.

- The existing method [4] can’t be used to find the fuzzy optimal solution of such balanced and unbalanced FFMCF problems in which \( \overline{t}_{ij} \neq (0,0,0,0)_{LR} \) for some \((i,j)\in A\) or \( \overline{t}_{ij} \neq (0,0,0,0)_{LR} \) \( \forall (i,j) \in A \) or \( \overline{t}_{ij} = (0,0,0,0)_{LR} \) \( \forall (i,j) \in A \). However, the proposed method can be used to find the fuzzy optimal solution of these problems.

- Since, in the proposed method instead of fuzzy linear programming formulation \((P_4)\) the modified fuzzy linear programming formulation \((P_4)\) of FFMCF problems is used. So, all the shortcomings, pointed out in Section 4, occurring due to Hukuhara’s difference are resolved.

9. Case Study

Ghatee and Hashemi [6, Definition 5.2, pp. 804] have claimed that if \( \overline{a} \) and \( \overline{b} \) are two non-negative fuzzy numbers such that \( \overline{a} \neq \overline{b} \), then, it can be converted into \( \overline{a} = \overline{b} \) by the following manner:

Find \( \overline{e} = \overline{a} \Theta \overline{b} \) and check that \( \overline{e} \) is negative or positive.

Case (i) If \( \overline{e} \) is positive then \( \overline{b} \Theta \overline{e} = \overline{a} \).

Case (ii) If \( \overline{e} \) is negative then \( \overline{a} \Theta \overline{e} = \overline{b} \), where \( \overline{e} = \Theta \overline{b} \).

However, it is not always possible to convert \( \overline{a} \neq \overline{b} \) into \( \overline{a} = \overline{b} \) by using the described method due to the following reasons:

If \( \overline{a} \) and \( \overline{b} \) are two fuzzy numbers such that \( \overline{a} \neq \overline{b} \) then \( \overline{e} = \overline{a} \Theta \overline{b} \) may be neither negative nor positive. i.e., neither \( \overline{b} \Theta \overline{e} = \overline{a} \) nor \( \overline{a} \Theta \overline{e} = \overline{b} \). e.g., in the existing real life problem [6], described in Section 9 (A), total fuzzy supply \( \overline{a} = (1580,49,100)_{LR} \) is not equal to the total fuzzy demand \( \overline{b} = (1498,9,64,59)_{LR} \) so it is an unbalanced FFMCF problem. However, \( \overline{e} = \overline{a} \Theta \overline{b} = (81,1,108,164) \) is neither negative nor positive fuzzy number and so neither \( \overline{b} \Theta \overline{e} = \overline{a} \) nor \( \overline{a} \Theta \overline{e} = \overline{b} \).

Since, Ghatee and Hashemi [6] have used the linear programming formulation of balanced FFMCF problem to obtain the fuzzy optimal solution of this unbalanced real life FFMCF problem without converting it into balanced FFMCF problem. So the results of this real life problem, obtained by Ghatee and Hashemi [6], are not genuine.

A. Description of Problem

A simplified network of petroleum industry distribution system Iran, shown in Figure 2, which is operated to transport crude oil from production units and import terminals to refineries, export terminals and storage tanks, and from their to destinations with minimal cost is depicted in Figure 3.
Figure 2. General petroleum supply chain, including the suppliers and demanders of petroleum.

Figure 3. A simple diagram of a pilot in Iranian petroleum industry.

The fuzzy supply of the crude oil at different sources and destinations are shown in Table 1 and the fuzzy cost and the fuzzy capacity for transporting one unit quantity of the crude oil from different sources to different destinations are shown in Table 2 with 

\[ L(x) = R(x) = \max(0,1 - x^2) \] 

\[ L(x) = R(x) = \max(0,1 - x^2) \] 

Table 1. The fuzzy supply and fuzzy demand.

<table>
<thead>
<tr>
<th>Node</th>
<th>Fuzzy supply</th>
<th>Fuzzy demand</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0, 0, 0) (L), R</td>
<td>(588, 10, 5) (L), R</td>
</tr>
<tr>
<td>2</td>
<td>(200, 10, 10) (L), R</td>
<td>(0, 0, 0) (L), R</td>
</tr>
<tr>
<td>3</td>
<td>(80, 4, 7) (L), R</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>(0, 0, 0) (L), R</td>
<td>(0, 0, 0) (L), R</td>
</tr>
<tr>
<td>5</td>
<td>(220, 4, 12) (L), R</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td>(0, 0, 0) (L), R</td>
<td>(0, 0, 0) (L), R</td>
</tr>
<tr>
<td>7</td>
<td>(220, 5, 16) (L), R</td>
<td>-</td>
</tr>
<tr>
<td>8</td>
<td>(70, 2, 6) (L), R</td>
<td>-</td>
</tr>
<tr>
<td>9</td>
<td>(150, 5, 9) (L), R</td>
<td>-</td>
</tr>
<tr>
<td>10</td>
<td>(0, 0, 0) (L), R</td>
<td>(0, 0, 0) (L), R</td>
</tr>
<tr>
<td>11</td>
<td>(0, 0, 0) (L), R</td>
<td>(0, 0, 0) (L), R</td>
</tr>
<tr>
<td>12</td>
<td>(200, 15, 16) (L), R</td>
<td>-</td>
</tr>
<tr>
<td>13</td>
<td>(0, 0, 0) (L), R</td>
<td>(0, 0, 0) (L), R</td>
</tr>
<tr>
<td>14</td>
<td>(0, 0, 0) (L), R</td>
<td>(0, 0, 0) (L), R</td>
</tr>
<tr>
<td>15</td>
<td>(0, 0, 0) (L), R</td>
<td>(0, 0, 0) (L), R</td>
</tr>
<tr>
<td>16</td>
<td>(0, 0, 0) (L), R</td>
<td>(0, 0, 0) (L), R</td>
</tr>
<tr>
<td>17</td>
<td>(400, 17, 20) (L), R</td>
<td>(185, 9, 3, 15) (L), R</td>
</tr>
<tr>
<td>18</td>
<td>(0, 0, 0) (L), R</td>
<td>(0, 0, 0) (L), R</td>
</tr>
<tr>
<td>19</td>
<td>(500, 15, 15) (L), R</td>
<td>(500, 15, 15) (L), R</td>
</tr>
<tr>
<td>20</td>
<td>-</td>
<td>(352, 34, 19) (L), R</td>
</tr>
<tr>
<td>21</td>
<td>-</td>
<td>(352, 34, 19) (L), R</td>
</tr>
</tbody>
</table>

Table 2. The fuzzy cost and fuzzy capacity.

<table>
<thead>
<tr>
<th>Node ( \rightarrow ) Node</th>
<th>Fuzzy cost</th>
<th>Fuzzy capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 ( \rightarrow ) 1</td>
<td>(450, 220, 68, 1410) (L), R</td>
<td>(10, 0.78, 1.75) (L), R</td>
</tr>
<tr>
<td>3 ( \rightarrow ) 2</td>
<td>(450, 220, 68, 1410) (L), R</td>
<td>(15, 1.365, 1.515) (L), R</td>
</tr>
<tr>
<td>4 ( \rightarrow ) 3</td>
<td>(5000, 169, 94) (L), R</td>
<td>(200, 0.94, 2.22) (L), R</td>
</tr>
<tr>
<td>5 ( \rightarrow ) 4</td>
<td>(10000, 595, 2960) (L), R</td>
<td>(30, 1.577, 5.34) (L), R</td>
</tr>
<tr>
<td>6 ( \rightarrow ) 5</td>
<td>(10000, 330, 440) (L), R</td>
<td>(40, 0.44, 7.08) (L), R</td>
</tr>
<tr>
<td>7 ( \rightarrow ) 6</td>
<td>(10000, 525, 80) (L), R</td>
<td>(60, 1.44, 7.08) (L), R</td>
</tr>
<tr>
<td>8 ( \rightarrow ) 7</td>
<td>(10000, 610, 960) (L), R</td>
<td>(70, 1.97, 7.09) (L), R</td>
</tr>
<tr>
<td>9 ( \rightarrow ) 8</td>
<td>(10000, 440, 290) (L), R</td>
<td>(80, 1.84, 8.89) (L), R</td>
</tr>
<tr>
<td>10 ( \rightarrow ) 9</td>
<td>(2500, 76, 25, 940) (L), R</td>
<td>(90, 8.46, 13.95) (L), R</td>
</tr>
<tr>
<td>11 ( \rightarrow ) 10</td>
<td>(10000, 625, 2280) (L), R</td>
<td>(100, 5.7, 18.3) (L), R</td>
</tr>
<tr>
<td>12 ( \rightarrow ) 11</td>
<td>(10000, 590, 1920) (L), R</td>
<td>(110, 5.28, 18.26) (L), R</td>
</tr>
<tr>
<td>13 ( \rightarrow ) 12</td>
<td>(10000, 430, 1940) (L), R</td>
<td>(120, 5.12, 13.44) (L), R</td>
</tr>
<tr>
<td>14 ( \rightarrow ) 13</td>
<td>(10000, 515, 960) (L), R</td>
<td>(130, 5.12, 13.44) (L), R</td>
</tr>
<tr>
<td>15 ( \rightarrow ) 14</td>
<td>(10000, 770, 240) (L), R</td>
<td>(140, 0.84, 22.4) (L), R</td>
</tr>
<tr>
<td>16 ( \rightarrow ) 15</td>
<td>(10000, 395, 400) (L), R</td>
<td>(150, 1.5, 28.35) (L), R</td>
</tr>
<tr>
<td>17 ( \rightarrow ) 16</td>
<td>(2000, 131, 368) (L), R</td>
<td>(160, 7.36, 28.32) (L), R</td>
</tr>
<tr>
<td>18 ( \rightarrow ) 17</td>
<td>(5000, 127, 5460) (L), R</td>
<td>(390, 12.41, 21.68) (L), R</td>
</tr>
<tr>
<td>19 ( \rightarrow ) 18</td>
<td>(5000, 227, 51520) (L), R</td>
<td>(390, 13.68, 30.06) (L), R</td>
</tr>
<tr>
<td>20 ( \rightarrow ) 19</td>
<td>(5000, 167, 51520) (L), R</td>
<td>(390, 14.44, 27.47) (L), R</td>
</tr>
<tr>
<td>21 ( \rightarrow ) 20</td>
<td>(1600, 80, 6, 96) (L), R</td>
<td>(200, 3.25, 4) (L), R</td>
</tr>
<tr>
<td>22 ( \rightarrow ) 21</td>
<td>(2500, 145, 75, 790) (L), R</td>
<td>(410, 14.7, 30.69) (L), R</td>
</tr>
<tr>
<td>23 ( \rightarrow ) 22</td>
<td>(1000, 10, 32, 80) (L), R</td>
<td>(588, 10, 31, 46) (L), R</td>
</tr>
<tr>
<td>24 ( \rightarrow ) 23</td>
<td>(30000, 885, 7890) (L), R</td>
<td>(230, 14, 95, 23.52) (L), R</td>
</tr>
</tbody>
</table>

Table 3. The fuzzy optimal flow between each couple of node in the simplified pilot in Iran.

<table>
<thead>
<tr>
<th>Node ( \rightarrow ) Node</th>
<th>Fuzzy flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 ( \rightarrow ) 1</td>
<td>(0, 0, 0) (L), R</td>
</tr>
<tr>
<td>3 ( \rightarrow ) 2</td>
<td>(0, 0, 0) (L), R</td>
</tr>
<tr>
<td>4 ( \rightarrow ) 3</td>
<td>(290, 10, 0) (L), R</td>
</tr>
<tr>
<td>5 ( \rightarrow ) 4</td>
<td>(0, 0, 0) (L), R</td>
</tr>
<tr>
<td>6 ( \rightarrow ) 5</td>
<td>(280, 14, 7) (L), R</td>
</tr>
<tr>
<td>7 ( \rightarrow ) 6</td>
<td>(0, 0, 0) (L), R</td>
</tr>
<tr>
<td>8 ( \rightarrow ) 7</td>
<td>(290, 14, 7) (L), R</td>
</tr>
<tr>
<td>9 ( \rightarrow ) 8</td>
<td>(250, 7, 44) (L), R</td>
</tr>
<tr>
<td>10 ( \rightarrow ) 11</td>
<td>(50, 18, 0, 7) (L), R</td>
</tr>
</tbody>
</table>

B. Fuzzy Optimal Solution of Chosen Real Life Problem

In MCF problems if the product can not be supplied from a source node to a destination node then in the optimal solution the quantity of the product that should be transported from source node to destination node should be zero.

Since, it is obvious from Figure 3 that node 5, node 7 and node 14 are not connected to node 21, node 21 and node 18 respectively so in the fuzzy optimal solution the fuzzy quantity of the crude oil that should be transported from node 5 \( \rightarrow \) 21, 7 \( \rightarrow \) 21 and 14 \( \rightarrow \) 18 should be zero fuzzy number. However, on solving the chosen real life problem by using the proposed method the fuzzy quantity of the crude oil that should be transported from
node 5 → 21, 7 → 21 and 14 → 18 are non zero fuzzy numbers so the obtained fuzzy optimal solution of the chosen real life problem is a pseudo fuzzy optimal solution.

The fuzzy quantity of crude oil that should be transported from one node to another node, obtained by using the proposed method, are shown in Table 3. Since, the fuzzy cost for transporting one unit quantity of the crude oil from 5 → 21, 7 → 21 and 14 → 18 is not given in the existing data [6]. So, assuming this cost as $M$ the obtained minimum total fuzzy transportation cost is

\[(12429794 + 575.33M, 781344.46 + 28.44M, 3775337.62 + 24M)_{LR}\]

### 10. Results and Discussion

To show the advantages of proposed method over existing method [4], the results of an existing balanced FFMCF problem and the results of FFMCF problems, chosen in Examples 5.1 and 5.2, obtained by using the existing method [4] and the proposed method, are shown in Table 4.

**Table 4. Comparative study.**

<table>
<thead>
<tr>
<th>Example</th>
<th>Existing method [4]</th>
<th>Proposed method</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.5</td>
<td>1924000, 19033100, 7299800, (\text{LR} = (12429794, 19033100, 7299800)_{LR})</td>
<td>Feasible</td>
</tr>
<tr>
<td>5.1</td>
<td>Infeasible</td>
<td>(1650, 3550, 1270, 4750)_{LR}</td>
</tr>
<tr>
<td>5.2</td>
<td>Not applicable</td>
<td>(1650, 2000, 1270, 4550)</td>
</tr>
</tbody>
</table>

The results, presented in Table 4, can be explained as follows:

- Since, the existing problem [5, Example 3.5, pp. 2498] is a balanced FFMCF problem for which \(\vec{L}_{ij} = (0,0,0,0)_{LR}\). So, as discussed in Section 8, it can be solved by using the existing as well as proposed method.

- Since, the problem, chosen in Example 5.1, \(\vec{L}_{ij} \neq (0,0,0,0)_{LR}\) for some \((i, j) \in A\) so due to the limitations of existing method [4], pointed out in first point of Section 5 (A), the chosen problem has infeasible solution. While, on solving the same problem by using the proposed method a fuzzy optimal solution is obtained.

- Since, the problem, chosen in Example 5.2, is an unbalanced FFMCF problem so, due to the limitations of existing method [4], pointed in second point of Section 5, it can’t be solved by using the existing method [4]. However, the same problem can be solved by the proposed method.

### 11. Conclusions

On the basis of presented study, it can be concluded that all the FFMCF problems which can be solved by the existing method [4] can also be solved by the proposed method. However, there exist several FFMCF problems which can be solved by using the proposed method but can’t be solved by using the existing method [4]. Hence, it is better to use the proposed method as compared to the existing method [4] for solving FFMCF problems.

### Acknowledgement

The authors would like to thank to Editor in Chief, Associate editor and anonymous referees for various suggestions which have led to an improvement in both the quality and clarity of the paper. I, Dr. Amit Kumar, want to acknowledge the adolescent inner blessings of Mehar. I believe that Mehar is an angel for me and without Mehar's blessing it was not possible to think the idea proposed in this paper. Mehar is a lovely daughter of Parmpreet Kaur (Research Scholar under my supervision). The authors also acknowledge the financial support given by the University Grant Commission, Govt. of India for completing the Major Research. Project (39-40/2010(SR)).

### References


Manjot Kaur is Ph.D student under the supervision of Dr. Amit Kumar. She has completed MSc (Mathematics and Computing) from Thapar University, Patiala (Punjab), India. She has published 8 papers in journals and conference proceedings. Her area of research is Fuzzy Optimization.

Amit Kumar is an Assistant Professor in School of Mathematics and Computer Applications, Thapar University, Patiala (Punjab), India. He has published more than 60 papers in journals and conference proceedings. His current area of research is Fuzzy Optimization, Fuzzy Reliability Analysis and Vague Set Theory.