Fuzzy Roughness in Hyperrings Based on a Complete Residuated Lattice

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Abstract

This paper considers the relations among \( L \)-fuzzy sets, rough sets and hyperring theory. Based on a complete residuated lattice, the concept of (invertible) \( L \)-fuzzy hyperideals of a hyperring is introduced and some related properties are presented. The notions of lower and upper \( L \)-fuzzy rough approximation operators with respect to an \( L \)-fuzzy hyperideal are provided and some significant properties of them are discussed. In addition, a new algebraic structure called (invertible) \( L \)-fuzzy rough hyperideals is defined and investigated.

Keywords: Hyperrings, \( L \)-fuzzy hyperideals, Lower (Upper) \( L \)-fuzzy rough approximations

1. Introduction

The concept of hyperstructure was first introduced by Marty [26] in 1934. Later on, people have observed that hyperstructures have many applications in both pure and applied sciences. A comprehensive review of the theory of hyperstructures can be found in [4]. In a recent book of Corsini and Leoreanu [5], the authors have collected numerous applications of algebraic hyperstructures, especially those from the last fifteen years to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets, rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence, and probabilities.

After introducing the concept of fuzzy sets by Zadeh [40], there are many papers devoted to review various concepts and results from the realm of abstract algebra in the broader framework of fuzzy setting. The study of the fuzzy algebraic structures has started in the pioneering paper of Rosenfeld [31]. Rosenfeld introduced the notion of fuzzy groups and showed that many results in groups can be extended in an elementary manner to develop the theory of fuzzy group. Since then the literature of various fuzzy algebraic concepts has been growing very rapidly. The items when relationships between the fuzzy sets and algebraic hyperstructures have been considered by Corsini, Davvaz, Kehagias, Leoreanu, Vougiouklis, Zhan and others, for example see [1, 2, 9, 10, 19, 23, 27, 32, 38, 39, 41].

Rough set theory, a new mathematical approach to deal with inexact and uncertain knowledge, was originally proposed by Pawlak [28]. It is an expanding research area which stimulates explorations on both real-world applications and on the theory itself. Rough set theory is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. There are at least two approaches for the development of the rough set theory: the constructive and the axiomatic approaches. In constructive methods, lower and upper approximations are constructed from the primitive notions, such as equivalence relations on a universe [28, 33] and neighborhood systems [12, 36]. It is a natural question to ask what happens if we substitute an algebraic system with the universe set. Since Biswas and Nanda [3] applied the notion of rough sets to algebra and introduced the notion of rough subgroups, lots of additional literature on various rough algebraic concepts has been published [7, 11, 16, 20, 21, 29, 37, 38]. Roughness in algebraic hyperstructures was considered by Davvaz and Leoreanu in [6, 8, 22, 24]. In [13], it was pointed out that more efforts should be made to study new algebraic structures induced by fuzzy sets. Therefore, it makes sense to study rough sets on fuzzy algebra structures and to propose new algebraic structures. Jiang et al. [15] studied the product structure of fuzzy rough sets on a group and presented new algebraic structures. Xiao and Zhang [35] studied the rough sets on a semigroup and proposed two new algebraic structures-- rough prime ideals and rough fuzzy prime ideals. Li et al. [25] introduced and investigated \( TL \)-fuzzy rough ideals in a ring. The characterization of rough prime (primary) ideals and rough fuzzy prime (primary) ideals in a commutative ring were introduced by Kazanci and Davvaz [17] and the lower and upper approximations in a quotient hypermodule with respect to fuzzy sets were studied by Kazanci et al. [18].

The initiation and majority of studies on fuzzy rough sets for algebraic structures such as semigroups, groups, rings, and hypermodules have been concentrated on the interval \([0, 1]\) as a basic structure. The interval \([0, 1]\),
However, seems to be restrict the application of the general-
ized rough set model for algebraic sets. To solve this
problem, Radzikowska and Kerre [30] proposed the
concept of $L$-fuzzy rough sets. It differs from fuzzy
rough sets in that it takes a complete residuated lattice $L$
as its basic structure. This is a fairly wide constructive
setting because diverse residuated pairs can be chosen
and, in case $L=[0,1]$, the fuzzy rough set theory follows.
From this point of view, this paper considers the $L$-fuzzy
roughness in hyperrings based on a complete residuated
lattice, which is a generalization of (fuzzy) roughness in
rings and hyperrings.

2. Preliminaries

In this section, we summarize some concepts and re-
results in residuated lattices and hyperrings (see [4, 30, 34])
which will be used throughout the paper. Also we will
define some $L$-fuzzy hyperoperations in a hyperring and
prove some of their properties.

2.1 Residuated lattices and $L$-fuzzy sets

Two binary operations $\otimes$ and $\rightarrow$ on a poset $P$
called an adjoint pair if they satisfy the following
conditions:

(F1) $\otimes: P \times P \rightarrow P$ is monotone increasing;

(F2) $\rightarrow: P \times P \rightarrow P$ is monotone decreasing with re-
spect to the first variable and monotone increasing with
respect to the second variable;

(F3) $r \otimes s \leq t$ if and only if $r \leq s \rightarrow t$ for all
$r, s, t \in P$.

A bounded lattice $L=\langle L; \land, \lor, \otimes, \rightarrow, 0,1 \rangle$ is a called
a residuated lattice if it satisfies the following conditions:

(F4) There is an adjoint pair $\langle \otimes, \rightarrow \rangle$ on $L$;

(F5) $\langle L; \otimes, 1 \rangle$ is a commutative semigroup with an
identity 1, where 1 is the maximal element of $L$.

A residuated lattice $L=\langle L; \land, \lor, \otimes, \rightarrow, 0,1 \rangle$ is said to
be complete if the underlying lattice $\langle L; \land, \lor, 0,1 \rangle$ is
complete.

Based on (F5), for any $r_1, \ldots, r_n \in L(n \geq 1), r_1 \otimes \cdots \otimes r_n$
is well defined and its value is irrelevant to the order of
$r_1, \ldots, r_n$, we write $\otimes_{r_1} r_n = r_1 \otimes \cdots \otimes r_n$ and set $D_0 = \{ r \in L : r \otimes r = r \}$.

From the definition of complete residuated lattices,
the following theorem can be easily derived.

Lemma 2.1 [30, 34]: In each complete residuated lattice
$L=\langle L; \land, \lor, \otimes, \rightarrow, 0,1 \rangle$, the following properties hold for
all $r, s, t, r_i \in L(i \in I)$:

(P1) $r \otimes s \leq r \land s$,

(P2) $1 \rightarrow r = r$ and $0 \otimes r = 0$,

(P3) $r \rightarrow (r \otimes s) \geq s$ and $r \otimes (r \rightarrow s) \leq s$,

(P4) $r \rightarrow (s \rightarrow t) = (r \otimes s) \rightarrow t$,

(P5) $(\lor_{r_i} r_i) \rightarrow s = \lor_{r_i} (r_i \rightarrow s)$,

(P6) $s \rightarrow (\land_{r_i} r_i) = \land_{r_i} (s \rightarrow r_i)$,

where $I$ is an index set.

Let $L$ be a lattice. $L$-fuzzy sets were introduced by
Goguen [14] as a generalization of the notion of Zadeh's
fuzzy sets [40]. Below we provide a more specific definition
of this notion, restricting ourselves to complete residuated
lattices taken as a basic structure. In the sequel, unless otherwise stated, $L$ denotes a complete
residuated lattice. Let $X$ be a non-empty set. An $L$-fuzzy
subset $\mu$ in $X$ is defined as a mapping from $X$ to $L$. The
family of all $L$-fuzzy subsets in $X$ is denoted by $F(X)$.

For any $A \subseteq X$ and $r \in L$, define an $L$-fuzzy subset
$r_\mu$ by $r_\mu(x) = r$ if $x \in A$ and 0 otherwise for all $x \in X$.

In particular, when $r = 1$, $r_\mu$ is said to be the characteristic
function of $A$, denoted by $\chi_A$. When $A = \{ x \}$, the
fuzzy subset $r_\mu$ is said to be an $L$-fuzzy point with sup-
portor $x$ and value $r$ and is denoted by $x_r$. An $L$-
-fuzzy point $x_r$ is said to belong to an $L$-fuzzy subset $\mu$,
written as $x_r \in \mu$, if $\mu(x) \geq r$.

For any $\mu, \nu \in L(X)$, by $\mu \subseteq \nu$ we mean that $\mu(x) \leq
\nu(x)$ for all $x \in X$. And the union, intersection, $\otimes$-
intersection and $\rightarrow$-implication of $\mu$ and $\nu$, denoted by
$\mu \cup \nu$, $\land \cup \nu$, $\mu \land \nu$ and $\mu \rightarrow \nu$, are defined as the $L$-
fuzzy subsets in $X$ by $(\mu \cup \nu)(x) = \mu(x) \lor \nu(x)$,
$(\mu \land \nu)(x) = \mu(x) \land \nu(x)$, $(\mu \land \nu)(x) = \mu(x) \otimes \nu(x)$
and $(\mu \rightarrow \nu)(x) = \mu(x) \rightarrow \nu(x)$ for all $x \in X$. A fuzzy
subset $\mu$ in $X$ is said to have sup-property if for any
$K \subseteq X$, there exists $y \in K$ such that $\mu(y) = \land_{x \in K} \mu(x)$.

2.2 Residuated lattices and $L$-fuzzy sets

Let $H$ be a non-empty set. A hyperoperation on $H$ is a
mapping $+: H \times H \rightarrow H$, written as $(x, y) \rightarrow x + y$. $H$
together with a hyperoperation “+” is called a hypergro-
poid. Let $x \in H$ and $A, B \subseteq H$. Then by $A + B$, $A + x$ and
$x + B$ we mean $A + B = \cup_{x \in A, y \in B} (x + y)$, $A + x = A + \{ x \}$ and $x + B = \{ x \} + B$, respectively. An element 0 of $H$ is called a
defective element (resp., scalar zero element) of $H$ if for any
$x \in H$, we have $x = 0 + x = x + 0$ (resp., $\{ x \} = 0 + x \cap x = 0$).

A hyperring $(H, +)$ is called a hyperring if for any
$x, y, z \in H$, we have $(x + y) + z = x + (y + z)$ and $x + H = H + x = H$. Let $K \subseteq H$. We say that $(K, +)$ is a subhyperr-
roup of \((H, +)\) if for any \(x \in K\), we have \(x + K = K + x = K\). A subhypergroup \(K\) of \(H\) is said to be left invertible if for any \(x, y \in H, x \in K + y\) implies \(y \in K + x\). We say that \(K\) is invertible if it is left and right invertible.

The concept of hypergroups is formulated as follows.

**Definition 2.2:** An algebraic structure \((R, +, \cdot)\) is called a **hyperring** if it satisfies the following conditions:

(i) \((R, +)\) is a commutative hypergroup;

(ii) \((R, \cdot)\) is a semihypergroup;

(iii) The hyperoperation \(\cdot\) is distributive with respect to the hyperoperation \(\cdot\), that is, \(z \cdot (x + y) = z \cdot x + z \cdot y\) and \((x + y) \cdot z = x \cdot z + y \cdot z\) for all \(x, y, z \in R\).

A non-empty subset \(I\) in \(R\) is said to be a **hyperideal of \(R\)** if \((I, +)\) is a subhypergroup of \((R, +, \cdot)\), \(x \cdot y \subseteq I\) and \(y \cdot x \subseteq I\) for all \(x \in I\) and \(y \in R\). A hyperideal \(I\) of \(R\) is said to be invertible if \((i, +)\) is an invertible hypergroup.

In the sequel, unless otherwise stated, let \((R, +, \cdot)\) always denote a hyperring with zero element 0 and we write \(x \cdot y\) instead of \(x \cdot y\) for all \(x, y \in R\).

Next, we define some \(L\)-fuzzy hyperoperations on a hyperring as follows.

**Definition 2.3:** Let \(\mu, \nu \in L(R)\). Define \(\mu \otimes \nu\), \(\mu \maxplus \nu\), \(\mu \otimes \nu\) and \(\nu \otimes \mu\) be \(L\)-fuzzy subsets in \(R\) as follows:

\[
(\mu \otimes \nu)(x) = \bigvee_{y \in R} \mu(y) \otimes \nu(z);
\]

\[
(\mu \maxplus \nu)(x) = \bigwedge_{y \in R} (\mu(y) \to \nu(z));
\]

\[
(\mu \otimes \nu)(x) = \bigvee_{y \in R} \mu(y) \otimes \nu(z) \text{ if } \exists y, z \in R \text{ such that } x \in yz \text{ and } 0 \text{ otherwise};
\]

\[
(\mu \otimes \nu)(x) = \bigwedge_{y \in R} (\mu(y) \to \nu(z)) \text{ if } \exists y, z \in R \text{ such that } x \in yz \text{ and } 0 \text{ otherwise};
\]

for all \(x \in R\). The \(L\)-fuzzy subsets \(\mu \otimes \nu\), \(\mu \maxplus \nu\), \(\mu \otimes \nu\) and \(\nu \otimes \mu\) are called the \(\otimes\)-sum, \(\to\)-sum, \(\otimes\)-product and \(\to\)-product of \(\mu\) and \(\nu\), respectively.

From (F5) and Definition 2.3, it follows that \(\mu \otimes \nu = \nu \otimes \mu\) for all \(\mu, \nu \in L(R)\).

**Lemma 2.4:** Let \(\mu, \nu \in L(R)\). Then

1. \((\mu \otimes \nu)(x) = \bigvee_{y \in R} (\mu(y) \otimes \nu(z))(x)
   \quad = \bigvee_{z \in R} (\mu \otimes \nu)(x) \otimes \nu(z)\).
2. \((\mu \maxplus \nu)(x) = \bigwedge_{y \in R} (\mu(y) \to \nu(z))(x)
   \quad = \bigwedge_{z \in R} ((\mu \otimes \nu)(x) \to \nu(z))\).

**Proof:** It is straightforward by P5 and Definition 2.3.

**Lemma 2.5:** Let \(\mu, \nu, \omega \in L(R)\) be such that \(\text{Im}(\mu) \subseteq D_{\omega}\). Then

1. \((\nu \otimes \omega) = (\mu \otimes \nu) \otimes \omega\),
2. \(\mu \otimes (\nu \otimes \omega) = (\mu \otimes \nu) \otimes \omega\),
3. \(\mu \otimes (\nu \otimes \omega) \subseteq \mu \otimes (\nu \otimes \mu) \otimes \omega\),
4. \((\nu \otimes \omega) \otimes \mu \subseteq \nu \otimes \omega \otimes \mu \otimes \omega\).

**Proof:** We only prove (3). The other properties can be similarly proved. Let \(x \in R\). If \(x \not\in ab\) for all \(a, b \in R\). Then \((\mu \otimes \nu)(\nu \otimes \omega))(x) = 0 \subseteq (\mu \otimes \nu \otimes \mu \otimes \omega)(x)\). Otherwise, we have

\[
(\mu \otimes (\nu \otimes \omega))(x) = \bigvee_{x} \mu(a) \otimes (\nu \otimes \omega)(b))
\]

\[
= \bigvee_{x} \mu(a) \otimes (\nu(c) \otimes (\nu \otimes \omega)(d))
\]

\[
= \bigvee_{x} \mu(a) \otimes (\mu(a) \otimes (\nu \otimes \omega)(d))
\]

\[
\leq \bigvee_{x} \mu(a) \otimes (\nu \otimes \omega)(b')
\]

\[
= (\mu \otimes \nu \otimes \omega)(x).
\]

It follows that \(\mu \otimes (\nu \otimes \omega) \subseteq \mu \otimes \nu \otimes \mu \otimes \omega\).

3. **(Invertible) \(L\)-fuzzy hyperideals of a hyperring**

It is well known that ideal theory plays a fundamental role in the development of hyperstructures. In this section, we define the concept of (invertible) \(L\)-fuzzy ideals of a hyperring and investigate some of their basic properties.

**Definition 3.1:** Let \(\mu \in L(R)\). Then \(\mu\) is called an \(L\)-fuzzy hyperideal of \(R\) if it satisfies the following conditions:

(F1a) \(\mu(0) = 1\);

(F2a) \(x_{\mu(x)} \in \mu\) and \(y_{\nu(y)} \in \nu\) imply \(x_{\mu(x)} \otimes \nu_{\nu(y)} \subseteq \mu\);

(F3a) For any \(y, z \in R\), there exists \(x \in R\) such that \(y_{\mu(y)} \otimes \nu_{\nu(z)} \subseteq x_{\mu(x)} \otimes \nu_{\nu(z)}\);

(F4a) \(x_{\mu(x)} \in \mu\) implies \(y_{\nu(y)} \otimes \mu_{\mu(x)} \subseteq y_{\nu(y)} \otimes \mu_{\mu(x)} \).

An \(L\)-fuzzy hyperideal \(\mu\) of \(R\) is said to be **invertible** if it satisfies:

(F5a) For any \(x, y \in R\) and \(r \in L\), \(x_{r} \in y_{\nu(y)} \otimes \mu_{\mu(x)} \) implies \(y_{r} \in x_{r} \otimes \mu_{\mu(x)}\).

In the sequel, let us first provide some characterizations of \(L\)-fuzzy hyperideals of \(R\).

**Lemma 3.2:** Let \(\mu \in L(R)\). Then (F2a) holds if and only if one of the following conditions holds:

(F2a) For any \(x, y \in \mu_{\mu(x)} \otimes \mu_{\mu(x)}\).
Let \( \mu \in L(R) \). Then (F5a) holds if and only if the following condition holds:

\[
(x \ominus_\mu y)(z) = (y \ominus_\mu x)(z) = 0
\]

and

\[
(x \ominus_\mu y)(x) \leq y \ominus_\mu x \leq x \ominus_\mu y.
\]

Lemma 3.5: Let \( \mu \in L(R) \). Then (F5a) holds if and only if the following condition holds:

\[
(x \ominus_\mu y)(z) = (y \ominus_\mu x)(z) = 0
\]

and

\[
(x \ominus_\mu y)(x) \leq y \ominus_\mu x \leq x \ominus_\mu y.
\]

Lemma 3.6: Let \( (L; +, \cdot, \wedge, \vee, \oplus, \ominus, 0, 1) \) be a complete residuated lattice. For all \( x, y \in L \), we denote \( x + y = \{z \in L | x \wedge y = x \wedge z = y \wedge z\} \) and \( x \cdot y = \{z \in L | x \vee y = x \vee z = y \vee z\} \). Then \( (L, +, \cdot) \) is a hyperring with zero element 1. Now, we define an \( L \)-fuzzy subset \( \mu \) in \( L \) as follows:

\[
\mu(\{x \in R \mid x \geq r\}) = \{z \in L \mid \mu(z) \geq r\}
\]

and

\[
\mu(\{x \in R \mid x \leq r\}) = \{z \in L \mid \mu(z) \leq r\}
\]

Then, from Definition 3.1 and Lemmas 3.2-3.5, it is easy to check that \( \mu \) is an invertible \( L \)-fuzzy hyperideal of \( L \).

For any \( L \)-fuzzy subset \( \mu \) in \( L \) and \( r \in L \), the set \( \mu = \{x \in R \mid x \geq r\} \) is called a level subset of \( \mu \). The next theorem presents the relationships between (invertible) \( L \)-fuzzy hyperideals of \( R \) and crisp (invertible) hyperideals of \( R \).

Theorem 3.7: Let \( \mu \in L(R) \) be such that \( \Im(\mu) \subseteq D_\emptyset \).

Then:

1. \( \mu \) is an \( L \)-fuzzy hyperideal of \( R \) if and only if \( \mu \), \( \mu \neq \emptyset \), is a hyperideal of \( R \) with zero element 0 for all \( r \in L - \{0\} \).

2. \( \mu \) is an (invertible) \( L \)-fuzzy hyperideal of \( R \) if and only if \( \mu \), \( \mu \neq \emptyset \), is an invertible hyperideal of \( R \) with zero element 0 for all \( r \in L - \{0\} \).

Proof: The proof is straightforward.

Now, we will study some properties of invertible \( L \)-fuzzy hyperideals of \( R \).

Lemma 3.8: Let \( \mu \) be an invertible \( L \)-fuzzy hyperideal of \( R \). Then for all \( x, y, z \in R \) such that \( y \in x + z \), we have \( \mu(y) \geq \mu(x) \wedge \mu(z) \).

Proof: Let \( x, y, z \in R \) be such that \( y \in x + z \). Then \( x \ominus_\mu y \leq x \ominus_\mu z \), as required. We have \( \mu(y) \geq \mu(x) \wedge \mu(z) \).

This implies

\[
\mu(y) \geq \mu(x) \wedge \mu(z) \geq \mu(y) \wedge \mu(z).
\]

Thus, \( \mu(y) \geq \mu(x) \wedge \mu(z) \).

This satisfies (F3c) and so (F3a) holds.

Analogous to the proof of Lemmas 3.2 and 3.3, we have the following results.

Lemma 3.4: Let \( \mu \in L(R) \). Then (F4a) holds if and only if one of the following conditions holds:

1. \( \mu(x) \geq \mu(y) \wedge \mu(z) \) for all \( y, z \in R \).

2. \( \mu \wedge_\mu \mu \subseteq \mu \).

Proof: (F2b) \( \wedge_\mu \mu(x) \geq \mu(y) \otimes_\mu \mu(z) \) for all \( y, z \in R \).

(F2c) \( \mu \wedge_\mu \mu \subseteq \mu \).

Proof: (F2b) \( \Rightarrow \) (F2b) This is straightforward.

(F2c) \( \Rightarrow \) (F2c) Let \( x, y, z \in R \) be such that \( x \in y + z \). Then, by (F2b), we have \( \mu(x) \geq \mu(y) \otimes_\mu \mu(z) \). This implies

\[
\wedge_\mu \mu(x) \geq \mu(y) \otimes_\mu \mu(z) \quad \text{and so (F2c) holds.}
\]

Thus, \( \mu \subseteq \mu \).

(F2d) \( \Rightarrow \) (F2a) Let \( x, y, z \in R \). Since \( x \otimes_\mu y \in \mu \), we have \( x \otimes_\mu y \subseteq \mu \), as required.

It is worth noting that \( \mu \wedge_\mu \mu = \mu \) for all \( L \)-fuzzy hyperideal \( \mu \) of \( R \) by Definition 3.1 and Lemma 3.2.

Lemma 3.3: Let \( \mu \in L(R) \) be such that \( \Im(\mu) \subseteq D_\emptyset \).

Then:

1. (F3a) holds if and only if the following condition holds:

(F3c) For any \( y, z \in R \), there exists \( x \in R \) such that \( y \in x + z \) and \( \mu(x) \geq \mu(y) \otimes_\mu \mu(z) \).

2. If (F3a) holds, then the following condition holds:

(F3d) \( \exists_\mu \mu(x) \subseteq \mu \wedge_\mu \mu \subseteq \mu \), as required.

Moreover, (F3d) implies (F3a) if \( \mu \) has the sup-property.

Proof: (F3a) \( \Rightarrow \) (F3c) Let \( y, z \in R \). By (F3a), there exists \( x \in R \) such that \( y \in x + z \) and \( \mu(x) \geq \mu(y) \otimes_\mu \mu(z) \), as required.

(F3c) \( \Rightarrow \) (F3a) Let \( y, z \in R \). By (F3c), there exists \( x \in R \) such that \( y \in x + z \) and \( \mu(x) \geq \mu(y) \otimes_\mu \mu(z) \) and so \( \mu(x) \eta_\mu \mu(z) \geq \mu(y) \otimes_\mu \mu(z) \), as required.

(F3a) \( \Rightarrow \) (F3d) This is straightforward.

In the following, assume that \( \mu \) has the sup-property.

We show (F3d) \( \Rightarrow \) (F3a). Let \( y, z \in R \). If for any \( x \in R \) such that \( y \in x + z \) and \( \mu(x) < \mu(y) \otimes_\mu \mu(z) \), then \( \mu(x) \eta_\mu \mu(z) \geq \mu(y) \otimes_\mu \mu(z) \).

This implies

\[
\mu(y) \otimes_\mu \mu(z) \geq \mu(x) \eta_\mu \mu(z) \geq \mu(y) \otimes_\mu \mu(z).
\]

Thus, \( \mu \subseteq \mu \).

Hence (F3c) is satisfied, and so (F3a) holds.

Analogous to the proof of Lemmas 3.2 and 3.3, we have the following results.

Lemma 3.4: Let \( \mu \in L(R) \). Then (F4a) holds if and only if one of the following conditions holds:

1. \( \mu \subseteq \mu \).

2. \( \mu \wedge_\mu \mu \subseteq \mu \).

3. Let \( x, y, z \in R \) be such that \( y \in x + z \). Then by
Lemma 3.8, we have

\[(μ ⊕ ν)(x)(y) ≥ (μ ⊕ ν)\bigoplus (μ ⊕ ν)(z)\].

(4) Since \(\text{Im}(χ_R) = 1 \in D_0\), by Lemma 2.5, we have

\[χ_R ⊔ (μ ⊕ ν) ⊆ χ_R ⊔ χ_R ⊔ ν ⊆ μ ⊔ ν\]. In a similar way, we have \((μ ⊕ ν) ⊔ χ_R ⊆ μ ⊔ ν\).

(5) Since \(μ\) and \(ν\) are invertible, for any \(x, y, R\) in \(R\), we have

\[(x ⊗ (μ ⊕ ν))(a) = \bigvee_{x,y} (x ⊗ μ)(a) ⊗ ν(b)\]

\[= \bigvee_{a,b} (a ⊗ ν)(x) = \bigwedge_{a,b} (a ⊗ ν)(x)\]

Summing up the above arguments, \(μ ⊕ ν\) is an invertible \(L\)-fuzzy hyperideal of \(R\).

4. The properties of \(L\)-fuzzy approximation operators with respect to an \(L\)-fuzzy hyperideal based on a complete residuated lattice

In this section, we substitute a hyperring as the universe set and give some basic properties of the lower and upper \(L\)-fuzzy approximation operators on a hyperring. We start by introducing the following definition.

**Definition 4.1:** Let \(μ\) be an \(L\)-fuzzy hyperideal of \(R\). The lower and upper \(L\)-fuzzy approximation operators with respect to \(μ\), denoted by \(\underline{Apr}_μ^L\) and \(\overline{Apr}_μ^L\), are defined by \(\underline{Apr}_μ^L = μ ⊓ ν\) and \(\overline{Apr}_μ^L = μ ⊔ ν\) respectively, for all \(ν \in L(R)\). \(\underline{Apr}_μ^L\) (resp., \(\overline{Apr}_μ^L\)) is called the lower (resp., upper) \(L\)-fuzzy approximation of \(ν\) with respect to \(μ\). A pair \((L, U) \in L(R) \times L(R)\) such that \(L = \underline{Apr}_μ^L\) and \(U = \overline{Apr}_μ^L\) for some \(ν \in L(R)\) is called the \(L\)-fuzzy rough set with respect to \(μ\) on \(R\).

Some basic and natural properties of the lower and upper \(L\)-fuzzy approximation operators with respect to an \(L\)-fuzzy hyperideal of \(R\) are provided in the next theorem.

**Theorem 4.2:** Let \(μ\) be an \(L\)-fuzzy hyperideal of \(R\) and \(μ\), \(ν, ω \in L(R)\). Then we have:

\[(R1) \underline{Apr}_μ^L \subseteq χ_R = \underline{Apr}_μ^L \subseteq \overline{Apr}_μ^L \subseteq \overline{Apr}_μ^L \subseteq \overline{Apr}_μ^L,\]

where \(\overline{Apr}_μ^L \subseteq \overline{Apr}_μ^L\) denotes the zero \(L\)-fuzzy set.

\[(R2) \text{If } ν = \overline{Apr}_μ^L \text{, then } \underline{Apr}_μ^L ν \subseteq \overline{Apr}_μ^L ν \text{ and } \overline{Apr}_μ^L ν \subseteq \overline{Apr}_μ^L ν.\]

\[(R3) \underline{Apr}_μ^L (ν \cap ω) = \underline{Apr}_μ^L (ν \cap \overline{Apr}_μ^L ω) \text{ and } \text{Ap}_μ^L (ν \cap ω) = \underline{Apr}_μ^L ν \cap \overline{Apr}_μ^L ω.\]

Proof: It is straightforward.

**Theorem 4.3:** Let \(μ\) be an invertible \(L\)-fuzzy hyperideal of \(R\). Then \(\underline{Apr}_μ^L (x ⊓ μ) = x ⊓ μ = \overline{Apr}_μ^L (x ⊓ μ)\) for all \(x \in R\).

Proof: Let \(x \in R\). We have \(\underline{Apr}_μ^L (x ⊓ μ) = x ⊓ μ\). Now for any \(y, z \in R\), since \((x ⊓ μ)(z) ≤ (y ⊓ μ)(z)\), we have \(z (y ⊓ μ)(z) \rightarrow (y ⊓ μ)(x) = (y ⊓ μ)(x)\).

This implies \(x ⊓ μ \subseteq \underline{Apr}_μ^L (x ⊓ μ)\). On the other hand, by (R5), \(\overline{Apr}_μ^L (x ⊓ μ) \subseteq x ⊓ μ\). Hence \(\underline{Apr}_μ^L (x ⊓ μ) = x ⊓ μ\).

**Theorem 4.4:** Let \(μ\) and \(ν\) be two \(L\)-fuzzy hyperideals of \(R\) such that \(μ ≤ ν\). Then \(ν = \overline{Apr}_μ^L ν\). Moreover, if \(ν\) is invertible, then \(\overline{Apr}_μ^L ν = ν\).

Proof: By Theorem 4.2, \(\overline{Apr}_μ^L ν \subseteq \overline{Apr}_μ^L ν\). Now, by \(μ ≤ ν\), we have \(\overline{Apr}_μ^L ν = ν = \overline{Apr}_μ^L ν\).

Hence \(ν = \overline{Apr}_μ^L ν\). Next let \(ν\) be an invertible \(L\)-fuzzy hyperideal of \(R\) and \(x \in R\). Then we have

\[\overline{Apr}_μ^L ν(x) = (μ ⊓ ν)(x) = \bigvee_{y \in R} ((μ ⊓ ν)(y)(x) \rightarrow ν(y))\]

\[= \bigwedge_{y \in R} ((ν ⊓ ν)(y)(x) \rightarrow (ν ⊓ ν)(y)) = \bigwedge_{y \in R} ((ν ⊓ ν)(y)(x) \rightarrow ν(y))\]

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\[ = \wedge (v \otimes_y x)(y) \rightarrow ((v \otimes_z z)(y) \otimes v(z)) \]
\[ \geq \wedge_{y \in R} (v \otimes_x x)(y) \rightarrow ((v \otimes_x x)(y) \otimes v(x)) \]
\[ \geq \wedge_{y \in R} v(x) \quad \text{(by P3)} = v(x). \]

Thus \( v \subseteq \text{Apr}_{\mu} v \). Hence \( v = \text{Apr}_{\mu}^{-1} v \).

**Theorem 4.5:** Let \( \mu \) be an \( L \)-fuzzy hyperideal of \( R \) and \( v, \omega \in L(R) \). Then \( \text{Apr}_{\mu}^{-1} (v \otimes_\omega \omega) = \text{Apr}_{\mu}^{-1} v \otimes_\omega \text{Apr}_{\mu}^{-1} \omega \).

**Proof:** It is straightforward.

**Theorem 4.6:** Let \( \mu \) be an \( L \)-fuzzy hyperideal of \( R \), \( v, \omega \in L(R) \) and \( \omega = \wedge \). Then \( \text{Apr}_{\mu}^{-1} v \otimes_\omega \text{Apr}_{\mu}^{-1} \omega \subseteq \text{Apr}_{\mu}^{-1} (v \otimes_\omega \omega) \).

**Proof:** This is straightforward by Lemmas 2.5 and 3.4.

**Theorem 4.7:** Let \( K \) be an invertible hyperideal of \( R \) and \( v, \omega \in L(R) \). Then \( \text{Apr}_{\mu}^{-1} v \otimes_\omega \text{Apr}_{\mu}^{-1} \omega \subseteq \text{Apr}_{\mu}^{-1} (v \otimes_\omega \omega) \).

**Proof:** Let \( x \) be any element of \( R \). Then since \( K \) is an invertible hyperideal of \( R \), we have \( \text{Apr}_{\mu}^{-1} (v \otimes_\omega \omega)(x) = \wedge_{y \in K} ((v \otimes_x x)(y) \rightarrow ((v \otimes_x x)(y) \otimes v(y))) \).

Now for any \( z,a,b \in R \) such that \( z \in x+K \) and \( x \in a+b \), we have \( z \in a+b+K = (a+K)+(b+K) \). Hence there exist \( c \in a+K \) and \( d \in b+K \) such that \( z \in c+d \), and \( (v \otimes_\omega \omega)(z) \geq v(c) \otimes o(d) \). This implies \( \wedge_{y \in K} v(c) \otimes v(d) \).

Hence \( \wedge_{y \in K} v(e) \otimes v(f) \).

This implies \( \text{Apr}_{\mu}^{-1} (v \otimes_\omega \omega)(x) = \text{Apr}_{\mu}^{-1} (v \otimes_\omega \omega)(x) \).

The following example shows that the inclusion symbol \( \subseteq \) in Theorem 4.7 may not be replaced by an equal sign.

**Example 4.8:** Let \( (R,+,\cdot) \) be a ring. Define \( x \otimes y = x+y +R \) and \( x \circ y = R \). Then \( (R,\otimes,\circ) \) is a hyperring. Let \( v = v = 0.1 \). Then \( v \otimes_\omega \omega = x_\omega \x_\omega = x \) and so \( \text{Apr}_{\mu}^{-1} (v \otimes_\omega \omega) = \text{Apr}_{\mu}^{-1} x \).

However, if we strengthen the condition, we may obtain the following theorem.

**Theorem 4.9:** Let \( \mu \) be an \( L \)-fuzzy hyperideal of \( R \) and let \( v, \omega \) be invertible \( L \)-fuzzy hyperideals of \( R \) such that \( \mu \subseteq v \) and \( \mu \subseteq \omega \). Then \( \text{Apr}_{\mu}^{-1} v \otimes_\omega \text{Apr}_{\mu}^{-1} \omega = \text{Apr}_{\mu}^{-1} (v \otimes_\omega \omega) \).

**Proof:** By the assumption and Theorem 4.4, we have \( \text{Apr}_{\mu}^{-1} v \otimes_\omega \text{Apr}_{\mu}^{-1} \omega = v \otimes_\omega \omega \subseteq \mu \otimes_\omega \mu = \mu \).

On the other hand, by Theorem 3.9, \( v \otimes_\omega \omega \) is an invertible \( L \)-fuzzy hyperideal of \( R \). Thus, by Theorem 4.4, we have \( \text{Apr}_{\mu}^{-1} v \otimes_\omega \text{Apr}_{\mu}^{-1} \omega = \text{Apr}_{\mu}^{-1} (v \otimes_\omega \omega) \), as required.

**Theorem 4.10:** Let \( \mu \) be an \( L \)-fuzzy hyperideal of \( R \), \( v, \omega \in L(R) \). Then \( \text{Apr}_{\mu}^{-1} (v \otimes_\omega \omega) = \text{Apr}_{\mu}^{-1} (v \otimes_\omega \omega) \).

**Proof:** Let \( x \in R \). Then, by (P4), (P5), (P6), we have \( \text{Apr}_{\mu}^{-1} (v \otimes_\omega \omega)(x) = (\mu \otimes_\omega (v \otimes_\omega \omega))(x) \).

This implies \( \text{Apr}_{\mu}^{-1} (v \otimes_\omega \omega)(x) = \text{Apr}_{\mu}^{-1} (v \otimes_\omega \omega)(x) \).

Up to now, we have studied the properties of the lower and upper \( L \)-fuzzy approximations of the different \( L \)-fuzzy sets with respect to an \( L \)-fuzzy hyperideal of \( R \). Next, we will further investigate the properties of the lower and upper \( L \)-fuzzy approximations of an \( L \)-fuzzy set with respect to different \( L \)-fuzzy hyperideals of \( R \).

**Lemma 4.11:** Let \( \mu \) and \( v \) be \( L \)-fuzzy hyperideals of \( R \) such that \( \mu \subseteq v \) and \( \omega \in L(R) \). Then \( \text{Apr}_{\mu}^{-1} \omega \subseteq \text{Apr}_{\mu}^{-1} \omega \) and \( \text{Apr}_{\mu}^{-1} \omega \subseteq \text{Apr}_{\mu}^{-1} \omega \).

**Lemma 4.12:** Let \( \mu \) and \( v \) be invertible \( L \)-fuzzy hyperideals of \( R \). Then both \( \mu \cap v \) and \( \mu \cap v \) are \( L \)-fuzzy hyperideals of \( R \).

**Proof:** (1) \( \mu \cap v \neq 0 \) and \( \mu \cap v \neq 1 \).

(2) Let \( x, y, z \in R \) be such that \( x \in y + z \). Then \( \mu(x) \geq (\mu(y) \otimes \mu(z)) \) and \( \nu(x) \geq (\nu(y) \otimes \nu(z)) \). Hence \( \mu(x) = (\mu(y) \otimes \mu(z)) \) and \( \nu(x) = (\nu(y) \otimes \nu(z)) \).
(3) Let \( x, y, z \in R \) be such that \( y \in x + z \). Since both \( \mu \) and \( \nu \) are invertible, by Lemma 3.12, \( \mu(x) \geq \mu(y) \odot \mu(z) \) and \( \nu(x) \geq \nu(y) \odot \nu(z) \). Hence \((\mu \cap \nu)(x) = (\mu \cap \nu)(y) \odot (\mu \cap \nu)(z)\).

(4) Let \( x, y, z \in R \) be such that \( y \in xz \). Analogous to the proof of (2), we have
\[
(\mu \cap \nu)(x) = (\mu \cap \nu)(y) \cap (\mu \cap \nu)(z).
\]

Summing up the above arguments, \( \mu \cap \nu \) is an \( L \)-fuzzy hyperideal of \( R \). The case for \( \mu \cap \nu \) can be similarly proved.

**Theorem 4.13:** Let \( \mu \) and \( \nu \) be \( L \)-fuzzy hyperideals of \( R \) and \( \omega \in L(R) \). Then \( \text{Im}(\omega) \subseteq D_\omega \). If \( \text{Im}(\omega) \subseteq D_\omega \), then \( \text{Im}(\omega) \subseteq \text{Im}(\omega) \cap \text{Im}(\omega) \).

**Proof:** By Lemma 4.11, it is clear that \( \text{Im}(\omega) \subseteq \text{Im}(\omega) \).

(1) If \( \text{Im}(\omega) \subseteq D_\omega \) and \( \text{Im}(\omega) \subseteq D_\omega \), then \( \text{Im}(\omega) \subseteq \text{Im}(\omega) \).

(2) If \( \text{Im}(\omega) \subseteq D_\omega \) and \( \text{Im}(\omega) \subseteq D_\omega \), then \( \text{Im}(\omega) \subseteq \text{Im}(\omega) \).

By Theorem 4.14, the result follows.

(2) By Theorem 4.4, the proof is straightforward.

**Theorem 4.15:** Let \( \mu, \nu \) and \( \omega \) be \( L \)-fuzzy hyperideals of \( R \). Then
\[
(1) \text{Im}(\omega) \subseteq \text{Im}(\omega) \cap \text{Im}(\omega) \cap \text{Im}(\omega).
\]

(2) If \( \text{Im}(\omega) \subseteq \text{Im}(\omega) \) and \( \text{Im}(\omega) \subseteq \text{Im}(\omega) \), then \( \text{Im}(\omega) \subseteq \text{Im}(\omega) \).

**Proof:** The proof is similar to that of Theorems 4.5 and 4.9.

5. The (invertible) \( L \)-fuzzy rough hyperideals of a hyperring

In this section, we will introduce a new algebraic structure, called (invertible) \( L \)-fuzzy rough hyperideals, and investigate their properties. Let us begin with introducing the following definition.

**Definition 5.1:** Let \( \mu \) be an \( L \)-fuzzy hyperideal of \( R \) and \( \nu \in F(R) \). Then \( \nu \) is called an (invertible) \( L \)-fuzzy lower (resp., upper) rough hyperideal with respect to \( \mu \) if \( \text{Im}(\nu) \subseteq \text{Im}(\nu) \cap \text{Im}(\nu) \).

\( \text{Im}(\nu) \cap \text{Im}(\nu) \) is an (invertible) \( L \)-fuzzy hyperideal of \( R \) and \( \nu \) is called an (invertible) \( L \)-fuzzy rough hyperideal with respect to \( \mu \) if both \( \text{Im}(\nu) \) and \( \text{Im}(\nu) \) are (invertible) \( L \)-fuzzy hyperideals of \( R \).

The following theorems follow from Theorems 3.9 and 4.4.

**Theorem 5.2:** Let \( \mu \) and \( \nu \) be an \( L \)-fuzzy hyperideal of \( R \) and an invertible \( L \)-fuzzy hyperideal of \( R \), respectively, such that \( \mu \subseteq \nu \). Then \( \nu \) is an invertible \( L \)-fuzzy lower rough hyperideal with respect to \( \mu \) of \( R \).

**Theorem 5.3:** Let \( \mu \) and \( \nu \) be invertible \( L \)-fuzzy hyperideals of \( R \). Then \( \nu \) is an invertible \( L \)-fuzzy upper rough hyperideal with respect to \( \mu \) of \( R \).

**Theorem 5.4:** Let \( \mu \) and \( \nu \) be invertible \( L \)-fuzzy hyperideals of \( R \) such that \( \mu \subseteq \nu \). Then \( \nu \) is an invertible \( L \)-fuzzy rough hyperideal of \( R \).
L-fuzzy rough hyperideal with respect to μ of R.

The following example shows that ν does not need to be an (invertible) L-fuzzy hyperideal even if ν is an (invertible) L-fuzzy rough hyperideal with respect to an invertible L-fuzzy hyperideal μ of R.

**Example 5.5:** Let R = {a, b, c, d} be a set with hyperoperations “+” and “•” as follows:

<table>
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<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>a</td>
<td>{a}</td>
<td>{0,a}</td>
<td>{0,a}</td>
<td>{0,a}</td>
<td>{0,a}</td>
</tr>
<tr>
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<td>{b}</td>
<td>{b}</td>
<td>{0,a}</td>
<td>{d}</td>
<td>{c}</td>
</tr>
<tr>
<td>c</td>
<td>{c}</td>
<td>{c}</td>
<td>{d}</td>
<td>{0,a}</td>
<td>{b}</td>
</tr>
<tr>
<td>d</td>
<td>{d}</td>
<td>{d}</td>
<td>{c}</td>
<td>{b}</td>
<td>{0,a}</td>
</tr>
</tbody>
</table>

Then (R, +, •) is a hyperring. Define μ and ν be L-fuzzy subsets in R by μ = \( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \) and ν = \( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \), respectively. Then μ is an invertible L-fuzzy hyperideal of R. Routine calculation gives \( \text{Apr}_L^L \nu = \frac{1}{0} + \frac{1}{a} + \frac{1}{b} \) and \( \text{Apr}_L^L \nu = \frac{1}{0} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \). It is clear that both \( \text{Apr}_L^L \nu \) and \( \text{Apr}_L^L \nu \) are invertible L-fuzzy hyperideals of R except for ν.

The following theorem follows from Theorem 4.2.

**Theorem 5.6:** Let μ be an L-fuzzy hyperideal of R and ν ∈ L(R). If ν is an (invertible) L-fuzzy lower (resp., upper) rough hyperideal with respect to μ of R. Then \( \text{Apr}_L^L \nu \) (resp., \( \text{Apr}_L^L \nu \)) is an (invertible) L-fuzzy rough hyperideal with respect to μ of R.

The following theorem follows from Theorems 3.9 and 4.5.

**Theorem 5.7:** Let μ be an invertible L-fuzzy hyperideal of R and ν, ω ∈ L(R). If ν and ω are invertible L-fuzzy upper rough hyperideals with respect to μ of R, then so is ν ⊕ ω.

In the following theorems we will establish the lattice structures of the set of all the L-fuzzy hyperideals of R.

**Theorem 5.8:** Let μ be an invertible L-fuzzy hyperideal of R and let LFLRI(R) be the set of all the L-fuzzy lower rough hyperideals with respect to μ of R. Then (LFLRI(R), ⊓, ⊔) is a complete lattice with maximal element \( \chi_R \), where

\[
ν ⊓ ω = \bigcap_{i \in I} \{ λ_i ∈ LFLRI(R) \mid ν ⊆ λ_i, ω ⊆ λ_i \}
\]

for all ν, ω ∈ LFLRI(R).

**Proof:** Let ν, ω ∈ LFLRI(R). Then both \( \text{Apr}_L^L \nu \) and \( \text{Apr}_L^L ω \) are L-fuzzy hyperideals of R, and it is easy to see that \( \text{Apr}_L^L (ν ⊔ ω) = \text{Apr}_L^L ν ⊔ \text{Apr}_L^L ω \) is also an L-fuzzy hyperideal of R. Hence ν ⊓ ω = ν ⊔ ω ∈ LFLRI(R). Similarly, it is obvious that ν ⊔ ω = ν ⊓ ω ∈ LFLRI(R). Thus, combing that \( \chi_R \) is also an L-fuzzy lower rough hyperideal with respect to μ of R, we can conclude that (LFLRI(R), ⊓, ⊔) is a complete lattice with maximal element \( \chi_R \).

**Theorem 5.9:** Let μ be an L-fuzzy hyperideal of R and let LFURI(R) be the set of all the L-fuzzy upper rough hyperideals with respect to μ of R such that ν ⊕ ω ⊆ ν and ν(0) = 1 for all ν ∈ LFURI(R). Then (LFURI(R), ⊕, ∈) is a lattice, where

\[
ν ⊕ ω = (∈)_ν \bigoplus (∈)_ω \{ λ_i ∈ LFURI(R) \mid ν ⊆ λ_i, ω ⊆ λ_i \}
\]

for all ν, ω ∈ LFURI(R).

**Proof:** Let ν, ω ∈ LFURI(R). Then both \( \text{Apr}_L^L \nu \) and \( \text{Apr}_L^L ω \) are L-fuzzy hyperideals of R, and it follows from Theorems 3.9 and 4.5 that \( \text{Apr}_L^L ν ⊕ \text{Apr}_L^L ω = \text{Apr}_L^L (ν ⊕ ω) \) is an L-fuzzy hyperideal of R. Now it is easy to see that ν ⊕ ω ∈ LFURI(R). Next, we prove ν ⊔ ω = ν ⊕ ω. Since ν(0) = ω(0) = 1, it is clear that ν ⊆ ν ⊕ ω and ω ⊆ ν ⊕ ω. Now, let η ∈ LFURI(R) be such that ν ⊆ η and ω ⊆ η. Then ν ⊕ ω ⊆ η ⊕ ω. Hence ν ⊔ ω = ν ⊕ ω. Similarly, we may prove that ν ⊓ ω = ν ⊓ ω. The result follows.

**6. Conclusions**

Hyperrings owe their importance to the fact that so many models arising in the solutions of specific problems turn out to be hyperrings. For this reason the basic concepts introduced here have exhibited some universal-
ity and are applicable in so many diverse contexts. These concepts are important and effective tools in algebraic systems and physics.

As a generalization of fuzzy rough set, Radzikowska and Kerre [30] introduced the concept of $L$-fuzzy rough sets which takes a complete residuated lattice $L$ as its basic structure. So, in this paper we considered the $L$-fuzzy roughness in hyperrings based on a complete residuated lattice $L$ instead of the interval $[0, 1]$. We introduced and investigated (invertible) $L$-fuzzy hyperideals of a hyperring based on a complete residuated lattice. We considered a hyperring as a universal set and studied the lower and upper $L$-fuzzy rough approximation operators with respect to an $L$-fuzzy hyperideal. We also provided and studied a new algebraic structure, called $L$-fuzzy rough hyperideals. The results obtained here are significant as compared to the prior work on roughness in (hyper) algebras and can also be applied to other hyperstructures. Our further work on this topic will focus on the properties of $L$-fuzzy roughness in other algebraic structures such as $n$-ary hyperstructures.

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