Generalized fuzzy Abel Grassmann's Groupoids

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Abstract

Fuzzy semigroup theory concentrates on theoretical aspects, but also includes applications in the areas of fuzzy coding theory, fuzzy finite state machines, and fuzzy languages. In this paper, we introduce the concept of (α,β)-fuzzy bi-ideals in *AG*-groupoids. Using the notion of "belongingness $(\in)$" and "quasi coincidence(q)" of fuzzy points with fuzzy sets, we introduce the concept of an (α,β)-fuzzy bi-ideals of an *AG*-groupoid *G*, where $\alpha,\beta \in \{\in, q, \in\vee q, \in\wedge q\}$ with $\alpha \neq \in\wedge q$. Since the concept of $(\in,\in\vee q)$-fuzzy bi-ideal is an important and useful generalization of ordinary fuzzy bi-deal, we discuss some fundamental aspects of $(\in,\in\vee q)$-fuzzy bi-ideals and $(\overline{\in},\overline{\in}\vee \overline{q})$-fuzzy bi-ideals. A fuzzy subset *F* of an AG-groupoid *G* is an $(\in,\in\vee q)$-fuzzy bi-ideal if and only if F_λ, the level cut of *F* is a bi-ideal of *G*, for all $\lambda\in(0,0.5]$ and *F* is an $(\overline{\in},\overline{\in}\vee \overline{q})$-fuzzy bi-ideal if and only if F_λ is a bi-ideal of *G*, for all $\lambda\in(0,0.5]$. This means that an $(\in,\in\vee q)$-fuzzy bi-ideals and $(\overline{\in},\overline{\in}\vee \overline{q})$-fuzzy bi-ideals are generalizations of the existing concept of fuzzy bi-ideals. We characterize regular *AG*-groupoid in terms of a fuzzy right and a fuzzy left ideal.

***Keywords:* AG-groupoid, regular AG-groupoids, bi-ideals, fuzzy subsets, fuzzy bi-ideals, (α,β)-fuzzy bi-ideals, $(\in,\in\vee q)$-fuzzy bi-ideals, $(\overline{\in},\overline{\in}\vee \overline{q})$-fuzzy bi-ideals.**

1. Introduction

Mordeson et. al. in [25] presented an up to date account of fuzzy sub-semigroups and fuzzy ideals of a semigroup. The book concentrates on theoretical aspects, but also includes applications in the areas of fuzzy coding theory, fuzzy finite state machines, and fuzzy languages. Basic results on fuzzy subsets, semigroups, codes, finite state machines, and languages are reviewed and introduced, as well as certain fuzzy ideals of a semigroup and advanced characterizations and properties of fuzzy semigroups. Kuroki [20] introduced the notion of fuzzy bi-ideals in semigroups. Kehayopulu applied the fuzzy concept in ordered semigroups and studied some properties of fuzzy left (right) ideals and fuzzy filters in ordered semigroups (see [15]). Fuzzy implicative and Boolean filters of R_0-algebra were initiated by Liu and Lee (see [22]).

The idea of a quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [1, 2], played a vital role to generate some different types of fuzzy subgroups. It is worth pointing out that Bhakat and Das (see [1]) gave the concepts of (α,β)-fuzzy subgroups by using the "*belongs to*" relation $(\in)$ and "*quasi-coincident with*" relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in,\in\vee q)$-fuzzy subgroup. In particular, $(\in,\in\vee q)$-fuzzy subgroup is an important and useful generalization of the Rosenfeld's fuzzy subgroup [30]. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. With this objective in view, Ma et. al. in [24], introduced the interval valued $(\in,\in\vee q)$-fuzzy ideals of pseudo-*MV* algebras and gave some important results of pseudo-*MV* algebras. In [5], Davvaz and Mozafar studied $(\in,\in\vee q)$-fuzzy subalgebras and (ideals) of a Lie algebras and provided some basic results of this algebra. Jun and Song (see [12]) discussed general forms of fuzzy interior ideals in semigroups, also see [10]. Kazanci and Yamak introduced the concept of a generalized fuzzy bi-ideal in semigroups [13] and gave some properties of fuzzy bi-ideals in terms of $(\in,\in\vee q)$-fuzzy bi-ideals. Jun et al. (see [10]) gave the concept of a generalized fuzzy bi-ideal in ordered semigroups and characterized regular ordered semigroups in terms of this notion. Davvaz et al. used the idea of generalized fuzzy sets in hyperstructures and introduced different generalized fuzzy subsystems e.g., (see [3, 5, 6, 7, 8, 13, 31-34]). In [23], Ma et al. introduced the concept of a generalized fuzzy filter of R_0-algebra and provided some properties in terms of this notion. Many other researchers used the idea of generalized fuzzy sets and gave several characterizations results in different branches of algebra see references. The con-

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cept of an (α,β)-fuzzy interior ideals in ordered semigroups was first introduced by Khan and Shabir in [16], where some basic properties of (α,β)-fuzzy interior ideals were discussed. On the other hand, an *Abel Grassmann's groupoid*, abbreviated as *AG- groupoid, is a groupoid G* whose elements satisfy the left invertive law: $(ab)c=(cb)a$ for all $a, b, c \in G$. An *AG*-groupoid is the midway structure between a *commutative semigroup* and a groupoid. It is a useful non-associative structure with wide range of applications in the theory of flocks. In an *AG*-groupoid the *medial law*, $(ab)(cd)=(ac)(bd)$ for all $a, b, c, d\in G$ (see [19]) holds: If there exists an element e in an *AG*-groupoid *G* such that $ex=x$ for all $x \in G$ then *G* is called an *AG- groupoid with left identity e*. If an *AG*-groupoid G has the *right identity* then *G* is a *commutative monoid*. If an *AG*-groupoid *G* contains left identity then $(ab)(cd)=(dc)(ba)$ holds for all $a, b, c, d\in G$. Also $a(bc)=b(ac)$ holds for all $a, b, c\in G$.

Using the idea of a quasi-coincidence of a fuzzy point with a fuzzy set, the concept of an (α,β)-fuzzy bi-ideals in an *AG*-groupoid is introduced and some important generalizations are discussed. In this regard, we introduce and study the new sort of fuzzy bi-ideals called (α,β)-fuzzy bi-ideals and to study some interesting characterizations of an *AG*-groupoids in terms of (α,β)-fuzzy bi-ideals. We provide characterizations of *AG*-groupoids in terms of $(\in,\in\vee q)$-fuzzy bi-ideals and $(\overline{\in},\overline{\in}\vee q)$-fuzzy bi-ideals.

2. Preliminaries

For subsets *A*, *B* of *G*, we denote by $AB=\{ab\in G \mid a\in A, b\in B\}$. A nonempty subset *A* of an *AG*-groupoid *G* is called an *AG-subgroupoid* of *G* if $A^2\subseteq A$. *A* is called a *bi-ideal* of *G* if $(AS)A\subseteq A$.

Let *G* be an *AG*-groupoid. By a fuzzy subset *F* of an *AG*-groupoid *G*, we mean a mapping, $F{:}G\to[0,1]$.

For fuzzy subsets F_1 and F_2 of *G*, define

$$F_1\circ F_2: G\to[0,1], a\mapsto$$

$$F_1\circ F_2=\begin{cases}\bigvee\limits_{a=yz}\min\{F_1(y),F_2(z)\} & \text{if } a=yz\\ 0 & \text{if } a\neq yz.\end{cases}$$

We denote by *F*(*G*) the set of all fuzzy subsets of *G*. One can easily see that $(F(G),\circ)$ becomes an *AG*- groupoid as shown in [18]. The order relation "$\subseteq$" on *F*(*G*) is defined as follows:

$$F_1\subseteq F_2 \quad \text{if and only if} \quad F_1(x)\le F_2(x)$$

for all $x\in G$ and for all $F_1,F_2\in F(G)$.

For a nonempty family of fuzzy subsets $\{F_i\}_{i\in I}$, of an *AG*-groupoid *G*, the fuzzy subsets $\bigcup\limits_{i\in I}F_i$ and $\bigcap\limits_{i\in I}F_i$ of *G* are defined as follows:

$$\bigcup_{i\in I}F_i : G\to[0,1], a\mapsto$$

$$\Big(\bigcup_{i\in I}F_i\Big)(a):=\sup\nolimits_{i\in I}\{F_i(a)\}$$

and

$$\bigcap_{i\in I}F_i : G\to[0,1], a\mapsto$$

$$\Big(\bigcap_{i\in I}F_i\Big)(a):=\inf\nolimits_{i\in I}\{F_i(a)\}.$$

If *I* is a finite set, say $I=\{1,2,\ldots,n\}$, then clearly $\bigcup\limits_{i\in I}F_i(a)=\max\{F_1(a),F_2(a),\ldots,F_n(a)\}$ and

$$\bigcap_{i\in I}F_i(a)=\min\{F_1(a),F_2(a),\ldots,F_n(a)\}.$$

A. Definition (cf. [18])

Let *G* be an *AG*-groupoid and *F* a fuzzy subset of *G*. Then *F* is called a *fuzzy bi-ideal* of *G*, if it satisfies the following conditions:

$$(\forall x,y\in S)\big(F(xy)\ge\min\{F(x),F(y)\}\big). \quad (1)$$

$$(\forall x,y,z\in S)\big(F(xy)z\ge\min\{F(x),F(z)\}\big). \quad (2)$$

Let *F* be a fuzzy subset of *G* and $\varnothing\neq A\subseteq G$ then the *characteristic function* F_A of *A* is defined as:

$$F_A: G\to[0,1], a\mapsto$$

$$F_A(a):=\begin{cases}1 \text{ if } a\in A\\ 0 \text{ if } a\notin A\end{cases}.$$

B. Lemma (cf. [18])

Let *G* be an AG-groupoid and *F* a fuzzy subset of *G*. Then *F* is a fuzzy bi-ideal of *G* if and only if F_A is a fuzzy bi-ideal of *G*.

Let *G* be an AG-groupoid and *F* a fuzzy subset of *G*. Then for every $\lambda\in(0,1]$ the set

$$U(F;\lambda):=\{x \mid x\in G \text{ and } F(x)\ge\lambda\}$$

is called a level set of F with support x and value λ.

The proof of the following Theorem is easy and so is omitted.

C. Theorem

Let *G* be an AG-groupoid and *F* a fuzzy subset of *G*. Then *F* is a fuzzy bi-ideal of *G* if and only if $U(F;\lambda)(\neq\varnothing)$ is a bi-ideal of G for every $\lambda\in(0,1]$.

3. (α,β)-Fuzzy Bi-Ideal

In what follows let *G* denote an *AG*-groupoid and α,β denote any one of $\in, q, \in\vee q, \in\wedge q$.

Let *G* be an *AG*-groupoid and *F* a fuzzy subset of *G*, then the set of the form

$$F(y):=\begin{cases}\lambda\neq 0 & \text{if } y=x,\\ 0 & \text{if } y\neq x,\end{cases}$$

is called a *fuzzy point* with support x and value λ and is denoted by x_λ. A fuzzy point x_λ is said to *belong to* (resp. *quasi-coincident*) with a fuzzy set F, written as $x_\lambda \in F(resp.\ x_\lambda qF)$ if $F(x) \geq \lambda(resp. F(x)+\lambda > 1)$. If $x_\lambda \in F$ or $x_\lambda qF$, then we write $x_\lambda \in \vee qF$. The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold.

Every fuzzy bi-ideal of G is an $(\in, \in)$-fuzzy bi-ideal of G as shown in the following Theorem:

A. Theorem

For any fuzzy subset F of G. The conditions (1) and (2) of Definition II-A are equivalent to the following:

$$(\forall x, y \in G)(\forall \lambda_1, \lambda_2 \in (0,1])(x_{\lambda_1} \in F, y_{\lambda_2} \in F) \rightarrow (xy)_{\min\{\lambda_1, \lambda_2\}} \in F) \quad (3)$$

$$(\forall x, z \in G)(\forall \lambda_1, \lambda_2 \in (0,1])(z_{\lambda_2} \in F, y_{\lambda_2} \in F) \rightarrow ((xy)z)_{\min\{\lambda_1, \lambda_2\}} \in F) \quad (4)$$

Proof: (1)$\rightarrow$(3). Let $x, y \in G$ and $\lambda_1, \lambda_2 \in (0,1]$ be such that $x_{\lambda_1} \in F$ and $y_{\lambda_2} \in F$. Then $F(x) \geq \lambda_1$ and $F(x) \geq \lambda_2$. By (1) we have

$$F(xy) \geq \min\{F(x), F(y)\} \geq \min\{\lambda_1, \lambda_2\},$$

and so $(xy)_{\min\{\lambda_1, \lambda_2\}} \in F.$

(3)$\rightarrow$(1). Let $x, y \in G$. Since $x_{F(x)} \in F$ and $y_{F(y)} \in F$. Then by (3), we have $(xy)_{\min\{F(x), F(y)\}} \in F$ and so $F(xy) \geq \min\{F(x), F(y)\}$.

(2)$\rightarrow$(4). Let $x, y, z \in G$ and $\lambda_1, \lambda_2 \in (0,1]$ be such that $x_{\lambda_1} \in F$ and $z_{\lambda_2} \in F$. Then $F(x) \geq \lambda_1$ and $F(z) \geq \lambda_2$. By (2) we have

$$F((xy)z) \geq \min\{F(x), F(z)\} \geq \min\{\lambda_1, \lambda_2\},$$

and so $((xy)z)_{\min\{\lambda_1, \lambda_2\}} \in F.$

(4)$\rightarrow$(2). Let $x, y \in G$. Since $x_{F(x)} \in F$ and $z_{F(z)} \in F$. Then by (3), we have $((xy)z)_{\min\{\lambda_1, \lambda_2\}} \in F$ and $F((xy)z) \geq \min\{F(x), F(z)\}$.

4. $(\in, \in \vee q)$-Fuzzy Bi-Ideals

In [12], Jun et al. introduced the concept of a generalized fuzzy interior ideal of a semigroup. Also in [10], Jun et al. introduced the concept an (α, β)-fuzzy bi-ideal of an ordered semigroup and characterized ordered semigroups in terms of (α, β)-fuzzy bi-ideals. In this section we define the notions of $(\in, \in \vee q)$-fuzzy bi-ideals of an *AG*-groupoid and investigate some of their properties in terms of $(\in, \in \vee q)$-fuzzy bi-ideals.

Let F be a fuzzy subset of G and $F(x) \leq 0.5$ for all $x \in G$. Let $x \in G$ and $\lambda \in (0,1]$ be such that $x_\lambda \in \wedge qF$. Then $x_\lambda \in F$ and $x_\lambda qF$ and so $F(x) \geq \lambda$ and $F(x)+\lambda > 1$. It follows that $1 < F(x)+\lambda \leq F(x)+F(x) = 2F(x)$, and so $F(x) > 0.5$, which is a contradiction. This means that $\{x \in G | x_\lambda \in \wedge qF\} = \varnothing$.

A. Definition

A fuzzy subset F of G is called an (α, β)-*fuzzy bi-ideal* of G, where $\alpha \neq \in \wedge q$, if it satisfies the following conditions:

$$(\forall x, y \in G)(\forall \lambda_1, \lambda_2 \in (0,1])(x_{\lambda_1} \alpha F, y_{\lambda_2} \alpha F \rightarrow (xy)_{\min\{\lambda_1, \lambda_2\}} \beta F). \quad (5)$$

$$(\forall x, y, z \in G)(\forall \lambda_1, \lambda_2 \in (0,1])(x_{\lambda_1} \alpha F, z_{\lambda_2} \alpha F \rightarrow ((xy)z)_{\min\{\lambda_1, \lambda_2\}} \beta F). \quad (6)$$

B. Theorem

A fuzzy subset F of G is an $(\in, \in \vee q)$-fuzzy bi-ideal of G if and only if it satisfies the following conditions:

$$(\forall x, y \in G)(F(xy) \geq \min\{F(x), F(y), 0.5\}). \quad (7)$$

$$(\forall x, y, z \in G)(F((xy)z) \geq \min\{F(x), F(z), 0.5\}). \quad (8)$$

Proof: Let F be an $(\in, \in \vee q)$-fuzzy bi-ideal of G. Let $x, y \in G$, if $F(x) = 0$ or $F(y) = 0$, then $F(xy) \geq \min\{F(x), F(y), 0.5\}$. Let $F(x) \neq 0$ and $F(y) \neq 0$ and assume on contrary that $F(xy) \leq \min\{F(x), F(y), 0.5\}$. Choose $\lambda \in (0,1]$ such that $F(xy) < \lambda \leq \min\{F(x), F(y), 0.5\}$. If $\min\{F(x), F(y)\} < 0.5$, then $F(xy) < \lambda \leq \min\{F(x), F(y)\}$ and so $x_\lambda \in F$, $y_\lambda \in F$ but $(xy)_\lambda \overline{\in \vee q}F$, contradiction. Let $\min\{F(x), F(y)\} \geq 0.5$, then $F(xy) < 0.5$ and $x_{0.5} \in F$, $y_{0.5} \in F$ but $(xy)_{0.5} \overline{\in \vee q}F$, contradiction. Hence $F(xy) \geq \min\{F(x), F(y), 0.5\}$ for all $x, y \in G$. Let $x, y, z \in G$, if $F(x) = 0$ or $F(z) = 0$, then $F((xy)z) \geq \min\{F(x), F(z), 0.5\}$. Let $F(x) \neq 0$ and $F(z) \neq 0$ and assume on contrary that $F((xy)z) \leq \min\{F(x), F(z), 0.5\}$. Choose $\lambda \in (0,1]$ such that $F((xy)z) < \lambda \leq \min\{F(x), F(z), 0.5\}$. If $\min\{F(x), F(z)\} < 0.5a$, then $F((xy)z) < \lambda \leq \min\{F(x), F(z)\}$ and so $x_{\lambda_1} \in F$, $z_{\lambda_2} \in F$ but $((xy)z)_{\lambda_1} \overline{\in \vee q}F$, contradiction. Let $\min\{F(x), F(z)\} \geq 0.5$, then $F((xy)z) < 0.5$ and $x_{0.5} \in F$, $z_{0.5} \in F$ but $((xy)z)_{0.5} \overline{\in \vee q}F$, contradiction. Hence $F(xy) \geq$

$\min\{F(x), F(y), 0.5\}$ for all $x, y \in G$.

Conversely, assume that $x, y \in G$ and $\lambda_1, \lambda_2 \in (0,1]$ be such that $x_{\lambda_1} \in F$, $y_{\lambda_2} \in F$. Then $F(x) \geq \lambda_1$ and $F(y) \geq \lambda_2$ and by (1) we have

$$F(xy) \geq \min\{F(x), F(y), 0.5\}$$
$$\geq \min\{\lambda_1, \lambda_2, 0.5\}.$$

If $\min\{\lambda_1, \lambda_2\} \leq 0.5$, then $F(xy) \geq \min\{\lambda_1, \lambda_2\}$ and so $(xy)_{\min\{\lambda_1, \lambda_2\}} \in F$. If $\min\{\lambda_1, \lambda_2\} > 0.5$, then $F(xy) + \min\{\lambda_1, \lambda_2\} > 0.5 + 0.5 = 1$ and so $(xy)_{\min\{\lambda_1, \lambda_2\}} qF$. Thus $(xy)_{\min\{\lambda_1, \lambda_2\}} \in\vee qF$. Let $x, y, z \in G$ and $\lambda_1, \lambda_2 \in (0,1]$ be such that $x_{\lambda_1} \in F$, $z_{\lambda_2} \in F$. Then $x_{\lambda_1} \in F$, $z_{\lambda_2} \in F$. Then $F(x) \geq \lambda_1$ and $F(z) \geq \lambda_2$ and by (2) we have

$$F((xy)z) \geq \min\{F(x), F(z), 0.5\}$$
$$\geq \min\{\lambda_1, \lambda_2, 0.5\}.$$

If $\min\{\lambda_1, \lambda_2\} \leq 0.5$, then $F((xy)z) \geq \min\{\lambda_1, \lambda_2\}$ and so $((xy)z)_{\min\{\lambda_1, \lambda_2\}} \in F$. If $\min\{\lambda_1, \lambda_2\} > 0.5$, then $F((xy)z) + \min\{\lambda_1, \lambda_2\} > 0.5 + 0.5 = 1$ and so $((xy)z)_{\min\{\lambda_1, \lambda_2\}} qF$. Thus $((xy)z)_{\min\{\lambda_1, \lambda_2\}} \in\vee qF$.

C. Remark

A fuzzy subset F of an AG-groupoid G is an $(\in, \in\vee q)$-fuzzy bi-ideal of G if and only if it satisfies conditions (7), and (8) of the above Theorem.

Using Theorem IV-B, we have the following characterization of $(\in, \in\vee q)$-fuzzy bi-ideal of an AG-groupoid.

D. Proposition

Let G be an AG-groupoid and $\varnothing \neq B \subseteq S$. Then B is a bi-ideal of G if and only if the characteristic function F_B is an $(\in, \in\vee q)$-fuzzy bi-ideal of G.

The converse of Theorem III-A, is not true in general, as shown in the following example.

E. Example

Let $G = \{a, b, c, d, e\}$ be an AG-groupoid with the following multiplication:

•	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	e	c	d
d	a	a	d	e	c
e	a	a	c	d	e

The $(G, \cdot)$ is an *AG*-groupoid (see [25]). The bi-ideals of G are: $\{a\}$ and $\{a, c, d, e\}$. Define a fuzzy subset $F: G \to [0,1]$ by

F(a) = 0.8, F(c) = 0.6, F(d) = 0.4,
F(e) = 0.2, F(b) = 0.1.

Then

$$U(F;\lambda) := \begin{cases} G \text{ if } \lambda \in (0, 0.1] \\ \{a, c, d, e\} \text{ if } \lambda \in (0.1, 0.2] \\ \{a\} \quad \text{if } \lambda \in (0.6, 1] \\ \varnothing \quad \text{if } \lambda \in (0.8, 1] \end{cases}$$

Then obviously, F is an $(\in, \in\vee q)$-fuzzy bi-ideal of G by Theorem C-II. But

(i) F is not an $(\in, \in)$-fuzzy bi-ideal of G, since $d_{0.38} \in F$ but
$$(dd)_{\min\{0.38, 0.38\}} = e_{0.38} \overline{\in} F.$$

(ii) F is not an $(\in, q)$-fuzzy bi-ideal of G, since $d_{0.36} \in F$ but
$$(dd)_{\min\{0.36, 0.36\}} = e_{0.36} \overline{q} F.$$

(iii) F is not a $(q, \in)$-fuzzy bi-ideal of G, since $c_{0.52} qF$ and $e_{0.82} qF$ but
$$(ce)_{\min\{0.52, 0.82\}} = d_{0.52} \overline{\in} F.$$

(iv) F is not a $(q, \in\vee q)$-fuzzy bi-ideal of G, since $c_{0.52} qF$ and $e_{0.82} qF$ but
$$(ce)_{\min\{0.52, 0.82\}} = d_{0.52} \overline{\in\vee q} F.$$

(v) F is not an $(\in\vee q, \in\wedge q)$-fuzzy bi-ideal of G, since $d_{0.38} \in\vee qF$ but
$$(dd)_{\min\{0.38, 0.38\}} = e_{0.38} \overline{\in\wedge q} F.$$

(vi) F is not an $(\in\vee q, q)$-fuzzy bi-ideal of G, since $c_{0.56} \in\vee qF$ and $e_{0.18} \in\vee qF$ but
$$(ce)_{\min\{0.56, 0.18\}} = d_{0.18} \overline{q} F.$$

(vii) F is not an $(\in\vee q, \in)$-fuzzy bi-ideal of G, since $d_{0.38} \in\vee qF$ but
$$(dd)_{\min\{0.38, 0.38\}} = d_{0.38} \overline{\in} F.$$

(viii) F is not $(\in\wedge q, \in)$-fuzzy bi-ideal of G, since $d_{0.38} \in\wedge qF$ but
$$(dd)_{\min\{0.38, 0.38\}} = d_{0.38} \overline{\in} F.$$

(ix) F is not a (q, q)-fuzzy bi-ideal of G, since $c_{0.52} qF$ and $e_{0.82} qF$ but
$$(ce)_{\min\{0.52, 0.82\}} = d_{0.52} \overline{q} F.$$

(x) F is not an $(\in, \in\vee q)$-fuzzy bi-ideal of G, since $c_{0.52} \in F$ and $e_{0.82} \in F$ but

$$(ce)_{\min\{0.52,0.82\}} = d_{0.52}\overline{\in \vee q}F.$$

(xi) F is not an $(\in\vee q, \in\vee q)$-fuzzy bi-ideal of G, since $c_{0.58} \in F$ and $e_{0.86} \in F$ but

$$(ce)_{\min\{0.58,0.86\}} = d_{0.58}\overline{\in \vee q}F.$$

F. Remark

By Remark IV-C, every fuzzy bi-ideal of an AG-groupoid G is an $(\in, \in\vee q)$-fuzzy bi-ideal of G. However, the converse is not true, in general.

G. Example

Consider the AG-groupoid given in Example IV-E, and define a fuzzy subset $F: G \to [0,1]$ by:

F(a)=0.8, F(c)=0.6, F(d)=0.4,
F(e)=0.2, F(b)=0.1.

Clearly F is an $(\in, \in\vee q)$-fuzzy bi-ideal of G. But F is not an (α, β)-fuzzy bi-ideal of G as shown in Example IV-E.

H. Theorem

Every $(\in, \in)$-fuzzy bi-ideal of G is an $(\in, \in\vee q)$-fuzzy bi-ideal of G.

Proof: Straightforward.

I. Theorem

Every $(\in\vee q, \in\vee q)$-fuzzy bi-ideal of G is an $(\in, \in\vee q)$-fuzzy bi-ideal of G.

Proof: Let F be an $(\in\vee q, \in\vee q)$-fuzzy bi-ideal of G. Let $x, y \in G$ and $\lambda_1, \lambda_2 \in (0,1]$ be such that $x_{\lambda_1}, y_{\lambda_2} \in F$. Then $x_{\lambda_1}, y_{\lambda_2} \in\vee qF$, which implies that $(xy)_{\min\{\lambda_1,\lambda_2\}} \vee qF$. Let $x, y, z \in G$ and $\lambda_1, \lambda_2 \in (0,1]$ be such that $x_{\lambda_1}, z_{\lambda_2} \in F$. Then $x_{\lambda_1}, z_{\lambda_2} \in\vee qF$, and we have $((xy)z)_{\min\{\lambda_1,\lambda_2\}} \in\vee qF$.

J. Theorem

Let F be a non-zero (α, β)-fuzzy bi-ideal of G. Then the set $F_0 := \{x \in G \mid F(x) > 0\}$ is a bi-ideal of G.

Proof: Let $x, y \in F_0$. Then $F(x)>0$ and $F(y)>0$. Assume that $F(xy)=0$. If $\alpha \in \{\in, \in\vee q\}$ then $x_{F(x)}\alpha F$ and $y_{F(x)}\alpha F$ but $(xy)_{\min\{F(x),F(y)\}}\overline{\beta}F$ for every $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$, a contradiction. Note that $x_1 qF$ and $y_1 qF$ but $(xy)_{\min\{1,1\}} = (xy)_1\overline{\beta}F$ for every $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$, a contradiction. Hence $F(xy)>0$, that is $xy \in F_0$. Let $x, z \in F_0$ and $y \in G$. Then $F(x)>0$ and $F(z)>0$. Assume that $F((xy)z)=0$. If $\alpha \in \{\in, \in\vee q\}$ then $x_{F(x)}\alpha F$ and $z_{F(z)}\alpha F$ but $(xyz)_{\min\{F(x),F(z)\}}\overline{\beta}F$ for every $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$, a contradiction. Note that $x_1 qF$ and $z_1 qF$ but $((xy)z)_{\min\{1,1\}}((xy)z)_1\overline{\beta}F$ for every $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$, a contradiction. Hence $F((xy)z)>0$, that is $(xy)z \in F_0$. Consequently, F_0 is a bi-ideal of G.

K. Theorem

Let B be a bi-ideal and F a fuzzy subset of G such that

(1) $(\forall_x \in G \setminus B)(F(x) = 0)$

(2) $(\forall_x \in B)(F(x) \geq 0.5)$.

Then

(a) F is a $(q, \in\vee q)$-fuzzy bi-ideal of G.

(b) F is an $(\in, \in\vee q)$-fuzzy bi-ideal of G.

Proof: (a) Let $x, y \in G$ and $\lambda_1, \lambda_2 \in (0,1]$ be such that $x_{\lambda_1} qF$ and $y_{\lambda_2} qF$. Then $x, y \in B$ and we have $xy \in B$. If $\min\{\lambda_1, \lambda_2\} \leq 0.5$, then $F(xy) \geq 0.5 \geq \min\{\lambda_1, \lambda_2\}$ and hence $(xy)_{\min\{\lambda_1,\lambda_2\}} F$. If $\min\{\lambda_1, \lambda_2\} < 0.5$, then

$$F(xy) + \min\{\lambda_1, \lambda_2\} > 0.5 + 0.5 = 1$$

and so $(xy)_{\min\{\lambda_1,\lambda_2\}} qF$. Therefore $(xy)_{\min\{\lambda_1,\lambda_2\}} \in\vee qF$. Let $x, y, z \in G$ and $\lambda_1, \lambda_2 \in (0,1]$ be such that $x_{\lambda_1} qF$ and $z_{\lambda_2} qF$. Then $x, z \in B$ and we have $(xy)z \in (BG)B \subseteq B$. If $\min\{\lambda_1, \lambda_2\} \leq 0.5$, then $F((xy)z) \geq 0.5 \geq \min\{\lambda_1, \lambda_2\}$ and hence $((xy)z)_{\min\{\lambda_1,\lambda_2\}} F$. If $\min\{\lambda_1, \lambda_2\} > 0.5$, then

$$F((xy)z) + \min\{\lambda_1, \lambda_2\} > 0.5 + 0.5 = 1$$

and so $((xy)z)_{\min\{\lambda_1,\lambda_2\}} qF$. Therefore $((xy)z)_{\min\{\lambda_1,\lambda_2\}} \in\vee qF$. Therefore F is a $(q, \in\vee q)$-fuzzy bi-ideal of G.

(b) Let $x, y \in G$ and $\lambda_1, \lambda_2 \in (0,1]$ be such that $x_{\lambda_1} qF$ and $y_{\lambda_2} qF$. Then $x, y \in B$ and we have $xy \in B$. If $\min\{\lambda_1, \lambda_2\} \leq 0.5$, then $F(xy) \geq 0.5 \geq \min\{\lambda_1, \lambda_2\}$ and hence $(xy)_{\min\{\lambda_1,\lambda_2\}} F$. If $\min\{\lambda_1, \lambda_2\} > 0.5$, then

$$F(xy) + \min\{\lambda_1, \lambda_2\} > 0.5 + 0.5 = 1$$

and so $(xy)_{\min\{\lambda_1,\lambda_2\}} qF$. Therefore $(xy)_{\min\{\lambda_1,\lambda_2\}} \in\vee qF$. Now let $x, y, z \in G$ and $\lambda_1, \lambda_2 \in (0,1]$ be such that $x_{\lambda_1} qF$ and $z_{\lambda_2} qF$. Then $x, z \in B$ and we have $xz \in B$. If $\min\{\lambda_1, \lambda_2\} \leq 0.5$, then $F(xy) \geq 0.5 \geq \min\{\lambda_1, \lambda_2\}$ and hence $((xy)z)_{\min\{\lambda_1,\lambda_2\}} F$. If $\min\{\lambda_1, \lambda_2\} > 0.5$, then

$$F((xy)z) + \min\{\lambda_1, \lambda_2\} > 0.5 + 0.5 = 1$$

and so $((xy)z)_{\min\{\lambda_1,\lambda_2\}} qF$. Therefore

$$((xy)z)_{\min\{\lambda_1,\lambda_2\}} \in\vee qF$$

and so *F* is an $(\in, \in\vee q)$-fuzzy bi-ideal of *G*.

From Example *E -IV* , we see that an $(\in, \in\vee q)$-fuzzy bi-ideal is not a $(q, \in\vee q)$-fuzzy bi-ideal (Example *IV- E*, Part iv).

In the following Theorem we give a condition for an $(\in, \in\vee q)$-fuzzy bi-ideal to be an $(\in, \in)$-fuzzy bi-ideal of *G*.

L. Theorem

Let *F* be an $(\in, \in\vee q)$-fuzzy bi-ideal of *G* such that *F*(*x*) < 0.5 for all $x \in G$. Then *F* is an $(\in, \in)$-fuzzy bi-ideal of *G*.

Proof: Let $x, y \in S$ and $\lambda_1, \lambda_2 \in (0,1]$ be such that $x_{\lambda_1}, y_{\lambda_2} \in F$. Then $F(x) \geq \lambda_1$ and $F(y) \geq \lambda_2$ and so

$$F(xy) \geq \min\{F(x), F(y), 0.5\} \geq \min\{\lambda_1, \lambda_2, 0.5\} = \min\{\lambda_1, \lambda_2\}$$

hence $(xy)_{\min\{\lambda_1, \lambda_2\}} \in F$. Now, let $x, y, z \in G$ and $\lambda_1, \lambda_2 \in (0,1]$ be such that $x_{\lambda_1}, z_{\lambda_2} \in F$. Then $F(x) \geq \lambda_1$ and $F(z) \geq \lambda_2$ and we have

$$F((xy)z) \geq \min\{F(x), F(z), 0.5\} \geq \min\{\lambda_1, \lambda_2, 0.5\},$$

consequently, $((xy)z)_{\min\{\lambda_1, \lambda_2\}} \in F$. Therefore *F* is an $(\in, \in)$-fuzzy bi-ideal of *G*.

For any fuzzy subset *F* of an *AG*-groupoid *G* and $\lambda \in (0,1]$, we denote

$$Q(F;\lambda) := \{x \in G | x_\lambda q F\} \text{ and } [F]_\lambda := \{x \in G | x_\lambda \in\vee q F\}.$$

Obviously, $[F]_\lambda = U(F;\lambda) \cup Q(F;\lambda)$.

We call $[F]_\lambda$ an $(\in\vee q)$-level bi-ideal of *F* and $Q(F;\lambda)$ a q-level bi-ideal of *F*.

We have given a characterization of $(\in, \in\vee q)$-fuzzy biideals by using level subsets (see Theorem II-C). Now we provide another characterization of $(\in, \in\vee q)$-fuzzy bi-ideals by using the set $[F]_\lambda$.

M. Theorem

Let *G* be an AG-groupoid and *F* a fuzzy subset of *G*. Then *A* is an $(\in, \in\vee q)$-fuzzy bi-ideal of *G* if and only if $[F]_\lambda$ is a bi-ideal of *G* for all $\lambda \in (0,1]$.

Proof: Let *F* be an $(\in, \in\vee q)$-fuzzy bi-ideal of *G*. Let $x, y \in [F]_\lambda$ and $\lambda \in (0,1]$. Then $x_\lambda \in\vee q F$ and $y_\lambda \in\vee q F$, that is, $F(x) \geq \lambda$ or $F(x) + \lambda > 1$, and $F(y) \geq \lambda$ or $F(y) + \lambda > 1$. Since *F* is an $(\in, \in\vee q)$-fuzzy bi-ideal of *G*, we have,

$$F(xy) \geq \min\{F(x), F(y), 0.5\}.$$

We discuss the following cases:

Case 1: Let $F(x) \geq \lambda$ and $F(y) \geq \lambda$. If $\lambda > 0.5$, then

$$F(xy) \geq \min\{F(x), F(y), 0.5\} = 0.5$$

and hence $(xy)_t qF$. If $\lambda \leq 0.5$, then

$$F(xy) \geq \min\{F(x), F(y), 0.5\} \geq \lambda$$

and so $(xy)_\lambda \in F$. Hence $(xy)_\lambda \in\vee qF$.

Case 2: Let $F(x) \geq \lambda$ and $F(y) + \lambda > 1$. If $\lambda > 0.5$, then

$$F(xy) \geq \min\{F(x), F(y), 0.5\}.$$

If $\lambda > 0.5$, then

$$F(xy) \geq \min\{F(x), F(y), 0.5\}$$
$$= \min\{F(y), 0.5\}$$
$$> \min\{1-\lambda, 0.5\} = 1-\lambda,$$

i.e., $F(xy) + \lambda > 1$ and thus $(xy)_\lambda \in qF$. If $\lambda \leq 0.5$, then

$$F(xy) \geq \min\{F(x), F(y), 0.5\}$$
$$\geq \min\{\lambda, 1-\lambda, 0.5\} = \lambda,$$

and so $(xy)_\lambda \in F$. Hence $(xy)_\lambda \in\vee qF$.

Case 3: Let $F(x) + \lambda > 1$ and $F(y) \geq \lambda$. If $\lambda < 0.5$, then

$$F(xy) \geq \min\{F(x), F(y), 0.5\}$$
$$\geq \min\{F(x), 0.5\}$$
$$\geq \min\{1-\lambda, 0.5\} = 1-\lambda,$$

i.e., $F(xy) + \lambda > 1$ and hence $(xy)_\lambda qF$. If $\lambda < 0.5$, then

$$F(xy) \geq \min\{F(x), F(y), 0.5\}$$
$$\geq \min\{1-\lambda, \lambda, 0.5\} = \lambda$$

and so $(xy)_\lambda \in F$. Hence $(xy)_\lambda \in\vee qF$.

Case 4: Let $F(x) + \lambda > 1$ and $F(y) + \lambda > 1$. If $\lambda > 0.5$, then

$$F(xy) \geq \min\{F(x), F(y), 0.5\}$$
$$> \min\{1-\lambda, 0.5\} = 1-\lambda,$$

i.e., $F(xy) + \lambda > 1$ and thus $(xy)_\lambda qF$. If $\lambda \leq 0.5$, then

$$F(xy) \geq \min\{F(x), F(y), 0.5\}$$
$$\geq \min\{1-\lambda, 0.5\} = 0.5 \geq \lambda,$$

and so $(xy)_\lambda \in F$. Thus in any case, we have $(xy)_\lambda \in\vee qF$.

Therefore $xy \in [F]_\lambda$. Now, let $x, z \in [F]_\lambda$ for $\lambda \in (0,1]$ and $y \in G$. Then $x_\lambda \in\vee qF$ and $z_\lambda \in\vee qF$, that is, $F(x) \geq \lambda$ or $F(x) + \lambda > 1$, and $F(z) \geq \lambda$ or $F(z) + \lambda > 1$. Since *F* is an $(\in, \in\vee q)$-fuzzy bi-ideal of *G*, we have,

$$F((xy)z) \geq \min\{F(x), F(z), 0.5\}.$$

Case 1: Let $F(x) \geq \lambda$ and $F(z) \geq \lambda$. If $\lambda > 0.5$, then

$$F((xy)z) \geq \min\{F(x), F(z), 0.5\} = 0.5$$

and hence $(xyz)_\lambda qF$. If $\lambda \leq 0.5$, then

$$F((xy)z) \geq \min\{F(x), F(z), 0.5\} \geq \lambda,$$

and so $((xy)z)_\lambda \in F$. Hence $((xy)z)_\lambda \in \vee qF$.
Case 2: Let $F(x) \geq \lambda$ and $F(z)+\lambda > 1$. If $\lambda > 0.5$, then

$$F((xy)z) \geq \min\{F(x), F(z), 0.5\}.$$

If $\lambda > 0.5$, then

$$\begin{aligned} F((xy)z) &\geq \min\{F(x), F(z), 0.5\} \\ &= \min\{F(z), 0.5\} \\ &> \min\{1-\lambda, 0.5\} = 1-\lambda, \end{aligned}$$

i.e., $F((xy)z)+\lambda > 1$ and thus $((xy)z)_\lambda qF$. If $\lambda \leq 0.5$, then

$$\begin{aligned} F((xy)z) &\geq \min\{F(x), F(z), 0.5\} \\ &\geq \min\{\lambda, 1-\lambda, 0.5\} = \lambda \end{aligned}$$

and so $((xy)z)_\lambda \in F$. Hence $((xy)z)_\lambda \in \vee qA$.
Case 3: Let $F(x)+\lambda > 1$ and $F(z) \geq \lambda$. If $\lambda < 0.5$, then

$$\begin{aligned} F((xy)z) &\geq \min\{F(x), F(z), 0.5\} \\ &\geq \min\{F(x), 0.5\} \\ &\geq \min\{1-\lambda, 0.5\} = 1-\lambda, \end{aligned}$$

i.e., $F(xyz)+\lambda > 1$ and hence $((xy)z)_\lambda qF$. If $\lambda < 0.5$, then

$$\begin{aligned} F((xy)z) &\geq \min\{F(x), F(z), 0.5\} \\ &\geq \min\{1-\lambda, \lambda, 0.5\} = \lambda \end{aligned}$$

and so $((xy)z)_\lambda \in F$. Hence $((xy)z)_\lambda \in \vee qF$.
Case 4: Let $F(x)+\lambda > 1$ and $F(z)+\lambda > 1$. If $\lambda > 0.5$, then

$$\begin{aligned} F((xy)z) &\geq \min\{F(x), F(z), 0.5\} \\ &> \min\{1-\lambda, 0.5\} = 1-\lambda, \end{aligned}$$

i.e., $F((xy)z)+\lambda > 1$ and thus $((xy)z)_\lambda qF$. If $\lambda \leq 0.5$, then

$$\begin{aligned} F((xy)z) &\geq \min\{F(x), F(z), 0.5\} \\ &\geq \min\{1-\lambda, 0.5\} = 0.5 \geq \lambda, \end{aligned}$$

and so $((xy)z)_\lambda \in F$. Thus in any case, we have $((xy)z)_\lambda \in \vee qF$. Therefore $(xy)z \in [F]_\lambda$.

Conversely, let F be a fuzzy subset of G and let $x, y \in G$ be such that $F(xy) < \lambda < \min\{F(x), F(y), 0.5\}$ for some $\lambda \in (0, 0.5]$. Then $x, y \in U(F;\lambda) \subseteq [F]_\lambda$, it implies that $xy \in [F]_\lambda$. Hence $F(xy) \geq \lambda$ or $F(xy)+\lambda > 1$, a contradiction. Hence $F(xy) \geq \min\{F(x), F(y), 0.5\}$ for all $x, y \in G$. Now let $F((xy)z) < \min\{F(x), F(z), 0.5\}$ for some $x, y, z \in S$. Choose λ such that $F((xy)z) < \lambda < \min\{F(x), F(z), 0.5\}$. Then $x, z \in U(F;\lambda) \subseteq [F]_\lambda$. It follows that $(xy)z \in [F]_\lambda$. This implies that $F((xy)z) \geq \lambda$ or $F(xyz)+\lambda > 1$, a contradiction. Hence $F((xy)z) \geq \min\{F(x), F(z), 0.5\}$ for all $x, y, z \in G$. By Theorem IV-B, it follows that F is an $(\in, \in\vee q)$-fuzzy bi-ideal of G.

$U(F;\lambda)$ and $[F]_\lambda$ are bi-ideals of G for all $\lambda \in (0,1]$, but $Q(F;\lambda)$ is not a bi-ideal of G for all $\lambda \in (0,1]$, in general. As shown in the following Example.

N. Example

Consider the AG-groupoid as given in Example IV-E. Define a fuzzy subset F by

F(a)=0.8, F(c)=0.6, F(d)=0.4
F(e)=0.2, F(b)=0.1.

Then $Q(F;\lambda) = \{a, c, d\}$ for all $0.2 < \lambda \leq 0.4$. Since $c_{0.56} \in F$ and $e_{0.18} \in F$ but $(ce)_{\min\{0.56, 0.18\}} = d_{0.18}\overline{q}F$. Hence $Q(F;\lambda)$ is not a bi-ideal of G for all $\lambda \in (0.2, 0.4]$.

5. $(\overline{\in}, \overline{\in}\vee\overline{q})$-Fuzzy Bi-Ideals

In this section we define $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy bi-ideals in *AG*-groupoids and give some of their related properties.

Let F be a fuzzy subset of G and $J = \{\lambda \mid \lambda \in (0,1] \text{ and } U(F;\lambda) \text{ is an empty set or a bi-ideal of } G\}$. In particular, if $J = (0,1]$ then F is an ordinary fuzzy bi-ideal of G (Theorem II-C) and if $J = (0, 0.5]$ then F is an $(\in, \in\vee q)$-fuzzy bi-ideal of G (Theorem IV-B). Furthermore, if F is a fuzzy subset and $J = \{\lambda \mid \lambda \in (0,1] \text{ and } U(F;\lambda) \text{ is an empty set or a bi-ideal of } G\}$ then we consider the following questions:

(a) if $J = (0.5, 1]$, what type of fuzzy bi-ideals of G will be F?

(b) if $J = (\alpha, \beta], (\alpha, \beta \in (0,1])$ whether F will be a kind of fuzzy bi-ideal of G or not?

In the following we give the answers of these questions.

A. Definition [17]

A fuzzy subset F of G is called an $(\overline{\in}, \overline{\in}\vee\overline{q})$*-fuzzy bi-ideal* of G if it satisfies the following conditions:

$$(\forall_{x,y} \in G)(\forall \lambda_1, \lambda_2 \in (0,1])$$
$$\left((xy)_{\min\{\lambda_1,\lambda_2\}} \overline{\in} F \rightarrow x_{\lambda_1} \overline{\in} \vee\overline{q}F \text{ or } y_{\lambda_2} \overline{\in} \vee\overline{q}F\right) \quad (9)$$
$$(\forall_{x,y,z} \in G)(\forall \lambda_1, \lambda_2 \in (0,1])$$
$$\left(((xy)z)_{\min\{\lambda_1,\lambda_2\}} \overline{\in} F \rightarrow x_{\lambda_1} \overline{\in} \vee\overline{q}F \text{ or } z_{\lambda_2} \overline{\in} \vee\overline{q}F\right). \quad (10)$$

B. Theorem [17]

A fuzzy subset F of G is an $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy bi-ideal of G if and only if it satisfies the following conditions:

$$(\forall x, y \in G)\left(\max\{F(xy), 0.5\} \geq \min\{F(x), F(y)\}\right), \quad (11)$$

$$\left(\forall x,y,z\in G\right)\left(\max\left\{F\left((xy)z\right),0.5\right\}\geq\min\left\{F(x),F(z)\right\}\right). \quad (12)$$

C. Lemma

Let *F* be a fuzzy subset of *G*. Then $U(F;\lambda)$ is a bi-ideal of *G* for all $\lambda\in(0.5,1]$ if and only if *F* satisfies (11) and (12).

Proof: Assume that $U(F;\lambda)$ is a bi-ideal of S for all $\lambda\in(0.5,1]$. If there exist $x,y\in G$ such that

$$\max\left\{F(xy),0.5\right\}<\min\left\{F(x),F(y),0.5\right\}=\lambda_1,$$

then $\lambda_1\in(0.5,1]$, $x,y\in U(F;\lambda_1)$. But $F(xy)<\lambda_1$ implies $x,y\notin U(F;\lambda_1)$, a contradiction. Hence condition (11) is valid.

If there exist $x,y,z\in G$ such that

$$\max\left\{F\left((xy)z\right),0.5\right\}<\min\left\{F(x),F(z),0.5\right\}=\lambda_2,$$

then $\lambda_2\in(0.5,1]$, $x,z\in U(F;\lambda_2)$. But $F\left((xy)z\right)<\lambda_2$ implies $x,y,z\notin U(F;\lambda_2)$, a contradiction. Hence condition (12) is valid.

Conversely, suppose that *F* satisfies conditions (11) and (12). For $x,y\in U\left(F;\lambda\right)$, by (11) we get

$$\begin{aligned}\max\left\{F(xy),0.5\right\}&\geq\min\left\{F(x),F(y)\right\}\\&\geq\lambda>0.5,\end{aligned}$$

and so $F(xy)\geq\lambda$. It follows that $xy\in U\left(F;\lambda\right)$. Let $x,z\in U\left(F;\lambda\right)$, then $F(x)\geq\lambda$ and $F(z)\geq\lambda$. By (12), we get

$$\begin{aligned}\max\left\{F\left((xy)z\right),0.5\right\}&\geq\min\left\{F(x),F(z)\right\}\\&\geq\lambda>0.5,\end{aligned}$$

and so $F\left((xy)z\right)\geq\lambda$. It follows that $(xy)z\in U(F;\lambda)$. Thus $U(F;\lambda)$ is a bi-ideal of *G* for all $\lambda\in(0.5,1]$.

D. Theorem

A fuzzy subset *F* of *G* is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal of *G* if and only if $U(F;\lambda)(\neq\varnothing)$ is a bi-ideal of *G*, for all $\lambda\in(0.5,1]$.

Proof: It is immediate consequence of Theorem V-B and Lemma V-C.

Let *F* be a fuzzy subset of *G* and $J=\left\{\lambda\,|\,\lambda\in(0,1]\text{ and }U(F;\lambda)\text{ is an empty set or a bi-ideal of }G\right\}$. In particular, if $J=(0,1]$ then *F* is an ordinary fuzzy bi-ideal of *G* (Theorem II-C) and if $J=(0,0.5]$ then *F* is an $(\in,\in\vee q)$-fuzzy bi-ideal of *G* (Theorem IV-B); if $J=(0.5,1]$, then *F* is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal of G(Theorem V-D).

Next we extend the above theory to the following:

E. Definition

Let $\lambda_1,\lambda_2\in(0,1]$ and $\lambda_1<\lambda_2$, then a fuzzy subset *F* of *G* is called a *fuzzy bi-ideal with thresholds* $(\lambda_1,\lambda_2]$ of *G* if it satisfies

$$\left(\forall_{x,y}\in G\right)\left(\max\left\{F(xy),\lambda_1\right\}\geq\min\left\{F(x),F(y),\lambda_2\right\}\right), \quad (13)$$

$$\left(\forall_{x,y,z}\in G\right)\left(\max\left\{F\left((xy)z\right),\lambda_1\right\}\geq\min\left\{F(x),F(z),\lambda_2\right\}\right) \quad (14)$$

F. Theorem

A fuzzy subset *F* of *G* is a fuzzy bi-ideal with thresholds $(\lambda_1,\lambda_2]$ of *G* if and only if $U(F;\lambda)(\neq\varnothing)$ is a bi-ideal of *G* for all $\lambda_1<\lambda\leq\lambda_2$.

Proof: The proof is similar to Theorem V-B.

G. Remark

By Definition V-E, we have the following result: if *F* is a fuzzy bi-ideal with thresholds $(\lambda_1,\lambda_2]$ of *G* then we conclude the following:

(i) *F* is an ordinary fuzzy bi-ideal of *G* when $\lambda_1=0$, $\lambda_2=1$;

(ii) *F* is an $(\in,\in\vee q)$-fuzzy bi-ideal of *G* when $\lambda_1=0$, $\lambda_2=0.5$;

(iii) *F* is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal of *G* when $\lambda_1=0.5$, $\lambda_2=1$;

H. Proposition

If $\{F_i\}_{i\in I}$ is a family of $(\in,\in\vee q)$-fuzzy bi-ideals of an AG-groupoid *G*. Then $\bigcap_{i\in I}F_i$ is an $(\in,\in\vee q)$-fuzzy bi-ideals of *G*.

Proof: Let $\{F_i\}_{i\in I}$ be a family of $(\in,\in\vee q)$-fuzzy bi-ideals of *G*. Let $x,y\in G$ Then

$$\begin{aligned}\left(\bigcap_{i\in I}F_i\right)(xy)&=\bigwedge_{i\in I}F_i(xy)\\&\geq\bigwedge_{i\in I}\left(F_i(x)\wedge F_i(y)\right)\\&=\left(\bigwedge_{i\in I}F_i(x)\wedge\bigwedge_{i\in I}F_i(y)\right)\\&=\left(\bigcap_{i\in I}F_i\right)(x)\wedge\left(\bigcap_{i\in I}F_i\right)(y).\end{aligned}$$

Let $x,y,z\in G$. Then

$$\begin{aligned}\left(\bigcap_{i\in I}F_i\right)((xy)z)&=\bigwedge_{i\in I}F_i((xy)z)\\&\geq\bigwedge_{i\in I}\left(F_i(xy)\wedge F_i(z)\right)\\&=\left(\bigwedge_{i\in I}F_i(x)\wedge\bigwedge_{i\in I}F_i(z)\right)\\&=\left(\bigcap_{i\in I}F_i\right)(x)\wedge\left(\bigcap_{i\in I}F_i\right)(z).\end{aligned}$$

Thus $\bigcap_{i\in I}F_i$ is an $(\in,\in\vee q)$-fuzzy bi-ideals of *G*.

The following lemma is obvious and we omit the proof.

I. lemma

Let G be an AG-groupiod and $A, B, \subseteq G$. Then

(1) $A \subseteq B$ *if and only if* $x_A \subseteq x_B$.

(2) $x_A \cap x_B = x_{A \cap B}$.

(3) $x_A \circ x_B = x_{AB}$

An AG- groupiod G is called *regular*, if for every $a \in G$, there exists $x \in G$, such that $a = (ax)a$. Equivalently, $A \subseteq (AG)A$ for all $A \subseteq G$.

J. lemma (cf. [26]).

Let G be an AG-groupiod such that $(xe)G = xG$ for all $x \in G$. Then the following are equivalent:

(1) G is regular.

(2) $A \cap B \subseteq AB$ for every right ideal A and every left ideal B of G.

k. Theorem

Let G be an AG-groupiod such that $(xe)G = xG$ for all $x \in G$. Then the following are equivalent:

(1) G is regular.

(2) $F_1 \cap F_2 \subseteq F_1 \circ F_2$ for every fuzzy right ideal F_1 and every fuzzy left ideal F_2 of G.

Proof: (1)→(2). Let $a \in G$.Since G is regular, there exists $x \in G$ such that $a = (ax)a$. Then

$$(F_1 \circ F_2)(a) = \bigvee_{a=yz} \min\{F_1(y), F_2(z)\}$$
$$\geq \min\{F_1(ax), F_2(a)\}$$
$$\geq \min\{F_1(a), F_2(a)\}.$$

Hence $F_1 \cap F_2 \subseteq F_1 \circ F_2$.

(2)→(1). Let $F_1 \cap F_2 \subseteq F_1 \circ F_2$ for every fuzzy right ideal F_1 and every fuzzy left ideal F_2 of G. Then G is regular. In fact: By Lemma V-J, it is enough to prove that $A \cap B \subseteq AB$ for every right ideal A and every left ideal B of G. Let $x \in A \cap B$, then $x \in A$ and $x \in B$. Then F_A is a fuzzy right ideal and F_B is a fuzzy left ideal of G. By hypothesis, $(F_A \cap F_B)(x) \subseteq (F_A \circ F_B)(x)$. Since $x \in A$ and $x \in B$, we have $F_A(x) = 1 = F_B(x)$. Then $(F_A \circ F_B)(x) = 1$ and by Lemma V-I, $F_A \circ F_B = F_{AB}$ and hence $F_{AB}(x) = 1 \rightarrow x \in AB \rightarrow A \cap B \subseteq AB$. Thus G is regular.

6. Concluding Remarks

In this paper, we study generalized fuzzy bi-ideals and give different characterization theorems of AG- groupoids in terms of this notion. In particular, if $J = \{\lambda \mid \lambda \in (0,1)]$ and $U(F;\lambda)$ is an empty set or a bi-ideal of G}, we give answer of the following questions:

(1) If $J = (0, 0.5]$, what kind of fuzzy bi-ideals of G will be F?

(2) If $J = (0.5, 1]$, what kind of fuzzy bi-ideals of G will be F?

(3) If $J = (\lambda_1, \lambda_2]$, $(\lambda_1, \lambda_2 \in (0,1])$ what kind of fuzzy bi-ideals of G will be F?

In our future work, we want to study those AG- groupoids for which each generalized fuzzy bi-ideal is idempotent. We also want to dene prime (α, β)-fuzzy bi-ideals and establish a generalized fuzzy spectrum of AG-groupoids.

Hopefully, the research along this direction will be continued and our results presented in this paper will constitute a platform for further development of AG-groupoids and their applications in other branches of algebra.

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Errata

Correction to "Global exponential stability of delayed fuzzy cellular neural networks"

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For the above paper, an incorrect list of authors' names was printed in the CONTENTS[1]. The ONLY author of the paper should be "Man-Chun Tan".

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