# Operations and Properties of Fuzzy Logic Systems

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## Abstract

**Consider several unknown non-linear functions, which are approximated respectively by several known fuzzy logic systems (FLSs) with fuzzy if-then rules which may be proposed by several experts based on intuitiveness or the heuristic estimation from sample data. This paper discusses the problem for how to use the several known fuzzy logic systems to generate a new fuzzy logic system which can be utilized to construct a new approximate model for a new nonlinear function composed of the several unknown nonlinear functions. The example shows the validity of the methods in this paper.**


***Keywords: Fuzzy logic system, approximation, operation, nonlinear function.***

## 1. Introduction

It is well-known that FLSs have been successfully applied to a wide variety of practical control problems in the last few years [1-5]. When a system is too complex or poorly understood to be described in precise mathematical term, the FLS models provide the ability to linguistically specify approximate relationships between the input and the desired output. In other words, the expert experience and heuristic estimation have been used to produce fuzzy rules. However, the ability to produce fuzzy rules by the expert experience and heuristic estimation becomes more challenging as the relationships within the system increases in number and complexity. The difficulties encountered in heuristic rule construction have led to the development of alternative approaches for producing fuzzy models [6-8].

It is an important method to produce a FLS model from intuitiveness and sample data based on the universal approximate theorems, such as Mamdani model and Takagi-Sugeno model [1, 6]. However, the universal approximate theorems declare only the existence of a FLS in certain structure form. How to choose the fuzzy rule number and the fuzzy membership functions in the antecedent and the consequent may not be resulted directly from the universal approximate theorems.

Generally speaking, the basic configuration of a FLS is composed of four principal elements: fuzzification interface, fuzzy knowledge base, fuzzy inference machine and defuzzification interface. The fuzzy knowledge base is generally represented by a set of fuzzy if-then rules in which the antecedent is an approximate representation of the state of the underlying system and the consequent provides a range of potential responses. Approximation quality of a FLS depends on its basic configuration, especially on its fuzzy knowledge base. Approximation quality also depends on the characteristics of the function to be approximated.

There is often a practical case that several FLSs had been given by several experts based on intuitiveness or the heuristic estimation from sample data, which are utilized to approximate several unknown nonlinear functions, respectively. If there is a new nonlinear function to be synthesized by the mathematic operations of the several old nonlinear functions, then an open problem is how to utilize the several known FLSs to generate a new FLS model of the new nonlinear function. The similar cases also happen in the intelligent systems (or robots) composed of autonomous fuzzy agents, where every fuzzy agent can perform locally qualitative uncertainty reasoning with incomplete and fuzzy knowledge in an integral part of the entirety environment that contains linguistic variables. Two main methods have been proposed in the literature: the first one in the distributed approach while the second one is the centralized approach. That is, the monolith intelligence of the whole system is decomposed into autonomous agents' intelligence, and then all autonomous agents' intelligence is centralized and synthesized to generate entirety intelligence for completing the specified task [9-11].

The above-mentioned cases may mean how to set up the mathematic operation between two FLSs such as to synthesize a new FLS according to the minimal error model. It is also noted that if the basic elements fuzzification interface, fuzzy inference machine and defuzzification interface are fixed, the mathematic operation is only considered between the two fuzzy knowledge bases. In other word, in this condition, the

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key problem is how to construct a new fuzzy rule bases by utilizing the old fuzzy rule bases such that the new proposed FLS can be represented as an approximation model of this new nonlinear function formed by the old nonlinear functions that are approximated by the old FLSs.

This paper discusses mainly the mathematic operations and properties of the several given FLSs, which is employed to generate new FLSs. The paper is organized as follows. In the next section we introduce the main definitions and results for fuzzy sets, which are needed subsequently. In Section 3 we introduce several definitions and representation for FLSs, which is the basic to the description of Section 4. Section 4 defines briefly the mathematic operations and then proves the output properties for FLSs. In Section 5 the example shows the validity of the mathematic operations and properties in this paper. Finally, in Section 6, the conclusion is given.

## 2. Basic Concepts for Fuzzy Sets

Let $A$ be a fuzzy set on the universe of discourse $U \subseteq R^n$ with membership function $A(x)$.

*Definition 1[12] (Singleton):* A real point $a \in U$ may be referred to a singleton fuzzy set $\{a\}$ with the fuzzy membership function

$$\mu_a(x) = \begin{cases} 1, & x = a \\ 0, & otherwise \end{cases}. \quad (1)$$

*Definition 2[12]:* Let $Core(A)$ denote the set $Core(A)$ = $\{x | A(x) = 1\}$.

*Definition 3:* The fuzzy set $A$ is known as a fuzzy point when $Core(A)$ is a singleton set. Especially, a fuzzy point defined on the real-line is called a fuzzy number.

*Definition 4:* The tensor product of the fuzzy sets $A_1$ and $A_2$ is defined as $A_1 \otimes A_2$ with the following membership function

$$[A_1 \otimes A_2](x) = A_1(x) A_2(x) \quad (2)$$

*Definition 5:* The center product of the fuzzy sets $A_1$ and $A_2$ is defined as $A_1 \odot A_2$ with the following membership function

$$[A_1 \odot A_2](x) = \sup_{\substack{x=yz \\ y,z \in U}} \{A_1(y) A_2(z)\} \quad (3)$$

It follows therefore that if $A_1$ and $A_2$ are two fuzzy numbers, then $A_1 \odot A_2$ is still a fuzzy number with $[A_1 \odot A_2](x_1 x_2) = 1$, where $A_1(x_1) = 1$, $A_2(x_2) = 1$.

*Definition 6:* The center product of a real number $a \in U$ and a fuzzy set $A$ is defined as

$$a \odot A = \{a\} \odot A \quad (4)$$

where $\{a\}$ is shown in Definition 1.

From Definitions 1, 5, it is seen that the membership function of the center product $a \odot A$ can be obtained by

$$[a \odot A](x) = \sup_{\substack{x=yz \\ y,z \in U}} \{\mu_a(y) A(z)\} \quad (5)$$

It follows therefore that if $A$ is a fuzzy number with $A(x_0) = 1$, then $[a \odot A](a x_0) = 1$.

From Definitions 3 and 5, we can easily obtain the following result.

*Theorem 1:* If $A_1$ and $A_2$ are two fuzzy numbers, then

$$Core(A_1 \odot A_2) = \{a_1 a_2 | a_i \in Core(A_i), i = 1, 2\} \quad (6)$$

*Definition 7:* Let $A_i$ ($i$=1,2,…, $N$) be $N$ fuzzy sets on $U \subseteq R^n$, with the membership functions $A_i(x)$, respectively. Then the $\Lambda = (A_1, \cdots, A_N)^T$ is called a $N$-dimensional fuzzy vector with the vector membership function

$$\Lambda(x) = (A_1(x), \cdots, A_N(x))^T \quad (7)$$

For convenience sake, this fuzzy vector and its vector membership function are abbreviated to $\Lambda = (A_i)_{N\times 1}$ and $\Lambda(x) = (A_i(x))_{N\times 1}$, respectively.

*Definition 8:* Let $\Lambda_j = (A_{ij})_{N\times 1}$, $j$=1,2, be two N-dimensional fuzzy vectors on $U_j$, $j$=1,2, respectively, with vector membership functions $\Lambda_j(x_j) = (A_{ij}(x_j))_{N\times 1}$, $j$=1,2, respectively. Then the Hadamard product of $\Lambda_1$ and $\Lambda_2$ is defined as a new fuzzy vector $pp(\Lambda_1, \Lambda_2)$ on $U_1 \times U_2$, with the vector membership function

$$pp(\Lambda_1, \Lambda_2)(x_1, x_2) = (A_{i1}(x_1) A_{i2}(x_2))_{N\times 1} \quad (8)$$

Similarly, the Hadamard product of $k$ fuzzy vectors $\Lambda_j$ ($j$=1, 2,…, $k$) on $U_j$ can be recurrently defined as the following fuzzy set on $U_1 \times U_2 \times \cdots \times U_k$

$$pp(\Lambda_1, \cdots, \Lambda_k) = pp(pp(\Lambda_1, \cdots, \Lambda_{k-1}), \Lambda_k) \quad (9)$$

## 3. Basic Concepts for FLSs

This paper discusses the FLS with if-then rules in the following form

$$L_i: \text{If } x_1 \text{ is } A_{i1} \text{ and } x_2 \text{ is } A_{i2} \text{ and } \ldots \text{ and } x_n \text{ is } A_{in} \text{ then y is } B_i, \ i=1,\ 2,\ldots,\ N. \quad (10)$$

where $x_j$ ($j$=1,2,…,n) is the input variable on $U_j \subseteq R$, $y$ is the output variable in $R$, $A_{ij}$ and $B_i$ are the fuzzy sets with membership functions $A_{ij}(x_j)$ and $B_i(y)$, respectively.

*Assumption 1:* $B_i$ s are fuzzy numbers for $i$=1, 2,…,$N$.

If the singleton fuzzifier, product inference machine and centroid defuzzifier are adopted, the output of the FLS with fuzzy rules (10) can be calculated by [1]

$$F(x)=\frac{\sum_{i=1}^{N} y^i(\prod_{j=1}^{n} A_{ij}(x_j))}{\sum_{i=1}^{N}(\prod_{j=1}^{n} A_{ij}(x_j))} \quad (11)$$

where $x=\begin{pmatrix} x_1 & x_2 & \cdots & x_n\end{pmatrix}^T$ $y^i \in Core(B_i)$.

*Assumption 2:* All of FLSs considered in this paper possess the rules in the form of (10) and the outputs in the form of (11).

For convenience sake, the FLS (10) with output (11) can be denoted as follows

$$F:\begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_N \end{pmatrix}:\begin{Bmatrix} A_{11} & A_{12} & \cdots & A_{1n} & B_1 \\ A_{21} & A_{22} & \cdots & A_{2n} & B_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{Nn} & B_N \end{Bmatrix} \quad (12)$$

Consider the following fuzzy vectors in (12).

$$\Lambda_j=\left(A_{ij}\right)_{N\times 1},\ \ \Upsilon=\left(B_i\right)_{N\times 1},\ j=1,\ldots,\ n. \quad (13)$$

*Definition 9:* The fuzzy vector $\Lambda_j$ in (13) is called the fuzzy vector interrelated to the variable $x_j$, abbreviated to $\Lambda_j = If[x_j;F]$. Similarly, the fuzzy vector $\Upsilon$ in (13) is called the fuzzy vector interrelated to the variable $y$, abbreviated to $\Upsilon = Then(y;F)$.

Let $Core(Then(y;F)) = \{Y | Y=\begin{pmatrix} y^1 & \cdots & y^N\end{pmatrix}^T,\ \ y^i \in Core(B_i),1\le i\le N\}$. By using Definition 9, FLS (12) can be represented as

$$F = L(N;n):\{If[x_1;F],\cdots,If[x_n;F],Then(y;F)\} \quad (14)$$

*Definition 10:* Consider a fuzzy vector $\xi=\left(\xi_1,\cdots,\xi_{n_1}\right)^T$ on $U_1 \subseteq R^{n_1}$, the function $Sum(\xi;x)$ is defined as $Sum(\xi;x)=\sum_{k=1}^{n_1}\xi_k(x)$.

*Definition 11:* Let $\xi=\left(\xi_1,\cdots,\xi_{n_1}\right)^T$ and $\eta=\left(\eta_1,\cdots,\eta_{n_2}\right)^T$ be two fuzzy vectors on $U_1 \subseteq R^{n_1}$ and $U_2 \subseteq R^{n_2}$, respectively. (1) The fuzzy vector $\xi \cup \eta \triangleq \left(\xi_1,\cdots,\xi_{n_1},\eta_1,\cdots,\eta_{n_2}\right)^T$, it is called the union of the fuzzy vectors; (2) The fuzzy vector $\xi\otimes\eta$ is called tensor product of $\xi$ and $\eta$ such that

$$\begin{aligned}\xi\otimes\eta=(&\xi_1\otimes\eta_1,\xi_1\otimes\eta_2,\cdots,\xi_1\otimes\eta_{n_2},\\ &\xi_2\otimes\eta_1,\xi_2\otimes\eta_2,\cdots\xi_2\otimes\eta_{n_2},\cdots,\\ &\xi_{n_1}\otimes\eta_1,\xi_{n_1}\otimes\eta_2,\cdots,\xi_{n_1}\otimes\eta_{n_2})^T\end{aligned}$$

(3) The fuzzy vector $\xi\odot\eta$ is called center-tensor product of $\xi$ and $\eta$ such that

$$\begin{aligned}\xi\odot\eta=(&\xi_1\odot\eta_1,\xi_1\odot\eta_2,\cdots,\xi_1\odot\eta_{n_2},\\ &\xi_2\odot\eta_1,\xi_2\odot\eta_2,\cdots,\xi_2\odot\eta_{n_2},\cdots,\\ &\xi_{n_1}\odot\eta_1,\xi_{n_1}\odot\eta_2,\cdots,\xi_{n_1}\odot\eta_{n_2})^T.\end{aligned}$$

*Definition 12:* Define the sum of rule bases of the FLS (14) as

$$Sum(F;x)=Sum\{pp\{If[F]\};x\} \quad (15)$$

By using Definitions 8~10, it follows therefore that

$$Sum(F;x)=\sum_{i=1}^{N}(\prod_{j-1}^{n}A_{ij}(x_j)) \quad (16)$$

Let $pp\{If[F]\} \triangleq pp\{If[x_1;F],\cdots,If[x_n;F]\}$. So, the output of the FLS (14) can represented as

$$F(x)=\frac{< pp\{If[F]\}(x),Y >}{Sum(F;x)} \quad (17)$$

where $Y \in Core[Then(y,F)]$, $<\cdot>$ denotes the inner product of the vectors in Euclidean space.

## 4. The Operations for FLSs

Let us introduce the following special FLSs.

*Definition 13:*

(1) The following FLS is called the *unity* of the FLS F.

$$I_F = L(N;n):\{If[x_1;F],\cdots,\ If[x_n;F],Then(y;I_F)\} \quad (18)$$

where $Then(y;I_F)$ satisfies

$$Core\{Then(y;I_F)\}=\left\{Y=\underbrace{\begin{pmatrix}1 & \cdots & 1\end{pmatrix}}_{N}^T\right\}.$$

(2) The following FLS is called the Zero of the FLS F.

$$O_F = L(N;n):\{If[x_1;F],\cdots,\ If[x_n;F],Then(y;O_F)\} \quad (19)$$

where $Then(y;O_F)$ satisfies

$$Core\{Then(y;O_F)\}=\left\{Y=\underbrace{\begin{pmatrix}0 & \cdots & 0\end{pmatrix}}_{N}^T\right\}.$$

*Definition 14:* The product operation between a real number $\lambda$ and the FLS F in (14) is defined as

$$\lambda F = L(N;n):\{If[x_1;F],\cdots,If[x_n;F],Then(y;\lambda\odot F)\} \quad (20)$$

where $Then\{y,\lambda\odot F\}$ is defined as

$$Then\{y,\lambda\odot F\}=\left(\lambda\odot B_1,\cdots,\lambda\odot B_N\right)^T \quad (21)$$

From Definitions 13 and 14, the following theorem is easily proved.

*Theorem 2:* Consider the FLS (14). Then the following outputs of the special FLSs are true.

(1). $I_F(x)=1$;

(2). $O_F(x)=0$;
(3). $[\lambda F](x)=\lambda F(x)$;
(4). $[\lambda I_F](x)=\lambda$;
(5). $[\lambda O_F](x)=0$.

Consider two FLSs $F_i$ ($i$=1, 2) as follows

$$F_i = L^i(N_i;n):\{If[x_1;F_i],\cdots,If[x_n;F_i],Then(y;F_i)\} \quad (22)$$

with outputs $F_i(x)$, $i$=1,2, respectively.

*Definition 15:* Define the 'union' operation for FLSs $F_i$ ($i$=1,2) as

$$F_1\cup F_2 = L(N_1+N_2;n):\{If[x_1;F_1\cup F_2],\cdots, If[x_n;F_1\cup F_2],Then(y;F_1\cup F_2)\}. \quad (23)$$

where

$$If[x_j;F_1\cup F_2]=If[x_j;F_1]\cup If[x_j;F_2],$$
$$Then(y;F_1\cup F_2)=Then(y;F_1)\cup Then(y;F_2).$$

From Definitions 8, 11, 12 and 15, the following theorem is easily proved.

*Theorem 3:* Consider FLSs $F_i$ ($i$=1, 2), the following equalities are true.

$$pp\{If[x_j;F_1\cup F_2]\}=pp\{If[x_j;F_1]\}\cup pp\{If[x_j;F_2]\} \quad (24)$$

$$Sum\{F_1\cup F_2;x\}=Sum\{F_1;x\}+Sum\{F_2;x\} \quad (25)$$

$$<pp\{F_1\cup F_2\}(x),Z>= <pp\{F_1\}(x),Y_1>+<pp\{F_2\}(x),Y_2> \quad (26)$$

$$Sum\{\lambda F_1;x\}=Sum\{F_1;x\}. \quad (27)$$

where

$$Y_i\in Center\{Then(y;F_i)\},\ i=1,2.$$
$$Z=Y_1\cup Y_2\in Center\{Then(y;F_1)\cup Then(y;F_2)\}.$$

By using (17), Theorem 3 and Definitions 13, 15, the following theorem is easily obtained.

*Theorem 4:* Consider FLSs $F_i$ ($i$=1, 2), the following equalities are true.

$$[F_1\cup F_2](x)=F_1(x)[O_{F_2}\cup I_{F_1}](x)+F_2(x)[O_{F_1}\cup I_{F_2}](x) \quad (28)$$

*Definition 16:* Define the 'tensor product' operation between two FLSs $F_i$ ($i$=1, 2) as

$$F_1\otimes F_2=L(N_1N_2;n):\{If[x_1;F_1\otimes F_2],\cdots, If[x_n;F_1\otimes F_2],Then(y;F_1\otimes F_2)\} \quad (29)$$

where

$$If[x_j;F_1\otimes F_2]=If[x_j;F_1]\otimes If[x_j;F_2],$$
$$Then(y;F_1\otimes F_2)=Then(y;F_1)\odot Then(y;F_2).$$

That is, the FLS (29) has $N_1N_2$ fuzzy rules with the following form

$$F_1\otimes F_2:\begin{pmatrix}L_1^{\otimes}\\ L_2^{\otimes}\\ \vdots\\ L_{N_1N_2}^{\otimes}\end{pmatrix}:\Xi \quad (30)$$

where

$$\Xi=\left\{\begin{matrix}
A_{11}^1\otimes A_{11}^2 & A_{12}^1\otimes A_{12}^2 & \cdots & A_{1n}^1\otimes A_{1n}^2 & B_1^1\odot B_1^2\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
A_{11}^1\otimes A_{N_21}^2 & A_{12}^1\otimes A_{N_22}^2 & \cdots & A_{1n}^1\otimes A_{N_2n}^2 & B_1^1\odot B_{N_2}^2\\
A_{21}^1\otimes A_{11}^2 & A_{22}^1\otimes A_{12}^2 & \cdots & A_{2n}^1\otimes A_{1n}^2 & B_2^1\odot B_1^2\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
A_{21}^1\otimes A_{N_21}^2 & A_{22}^1\otimes A_{N_22}^2 & \cdots & A_{2n}^1\otimes A_{N_2n}^2 & B_2^1\odot B_{N_2}^2\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
A_{N_11}^1\otimes A_{11}^2 & A_{N_12}^1\otimes A_{12}^2 & \cdots & A_{N_1n}^1\otimes A_{1n}^2 & B_{N_1}^1\odot B_1^2\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
A_{N_11}^1\otimes A_{N_21}^2 & A_{N_12}^1\otimes A_{N_22}^2 & \cdots & A_{N_1n}^1\otimes A_{N_2n}^2 & B_{N_1}^1\odot B_{N_2}^2
\end{matrix}\right\}$$

By using Definitions 4, 5, 8-12, Theorem1 and (17), (30), the following results are obtained

$$pp\{If[F_1\otimes F_2]\}(x)= \left(\underbrace{\prod_{j=1}^{n}\{[A_{1j}^1\otimes A_{kj}^2](x_j)\}}_{k=1,\cdots,N_2},\underbrace{\prod_{j=1}^{n}\{[A_{2j}^1\otimes A_{kj}^2](x_j)\}}_{k=1,\cdots,N_2}, \cdots,\underbrace{\prod_{j=1}^{n}\{[A_{N_1j}^1\otimes A_{kj}^2](x_j)\}}_{k=1,\cdots,N_2}\right)^T_{N_1N_2\times 1} \quad (31)$$

$$Sum(F_1\otimes F_2;x)=Sum\{pp\{If[F_1\otimes F_2]\};x\} =Sum\{If(F_1);x\}\cdot Sum\{If(F_2);x\} \quad (32)$$

$$center\{Then(y;F_1\otimes F_2)\} =\{Y_1\otimes Y_2 | Y_i\in Core(Then(y;F_i)),i=1,2\} \quad (33)$$

where

$$Y_1\otimes Y_2=\begin{pmatrix}\underbrace{y_1^1y_k^2}_{k=1,\cdots N_2} & \underbrace{y_2^1y_k^2}_{k=1,\cdots N_2} & \cdots & \underbrace{y_{N_1}^1y_k^2}_{k=1,\cdots N_2}\end{pmatrix}^T_{N_1N_2\times 1}$$

$$Y_l=\begin{pmatrix}y_1^l & y_2^l & \cdots & y_{N_l}^l\end{pmatrix}^T,\ l=1,2.$$

$$<pp\{If(F_1\otimes F_2\}(x),Y> =<pp\{If(F_1\}(x),Y_1>\cdot<pp\{If(F_2\}(x),Y_2> \quad (34)$$

Therefore, from (17), (32)-(34), the following theorem is obtained.

*Theorem 5:* The output of the FLS (29) satisfies

$$[F_1\otimes F_2](x)=F_1(x)F_2(x) \quad (35)$$

*Definition 17:* Define the 'add' operation between two FLSs $F_i$ ($i$=1,2) as

$$F_1+F_2=L(2N_1N_2;n):\{If[x_1;(F_1\otimes 2I_{F_2})\bigcup(F_2\otimes 2I_{F_1})],\cdots,\; Then(y;(F_1\otimes 2I_{F_2})\bigcup(F_2\otimes 2I_{F_1})\} \quad (36)$$

*Definition 18:* Define the 'subtraction' operation between two FLSs $F_i$ ($i$=1,2) as

$$F_1-F_2=F_1+(-1)F_2 \quad (37)$$

Consider $m$ FLSs $F_i$ in the form of (14), $i$=1, 2,…, $m$. By using Definition17, $\sum_{i=1}^{m}F_i$ can be recurrently defined as

$$\sum_{i=1}^{m}F_i=(\sum_{i=1}^{m-1}F_i)+F_m \quad (38)$$

By using Definition 15, Definition 17 and Theorem3, Theorem4, it is obtained that

$$[F_1+F_2](x)=\{(F_1\otimes 2I_{F_2})\bigcup(F_2\otimes 2I_{F_1})\}(x)$$
$$=2F_1(x)\frac{Sum(I_{F_1\otimes 2I_{F_2}};x)}{Sum(O_{F_2\otimes 2I_{F_1}};x)+Sum(I_{F_1\otimes 2I_{F_2}};x)}+$$
$$2F_2(x)\frac{Sum(I_{F_2\otimes 2I_{F_1}};x)}{Sum(O_{F_1\otimes 2I_{F_2}};x)+Sum(I_{F_2\otimes 2I_{F_1}};x)} \quad (39)$$

By using Definitions 12, 13 and Theorem 2, it is easily seen that

$$Sum(O_{F_2\otimes 2I_{F_1}};x)=Sum(I_{F_1\otimes 2I_{F_2}};x)=Sum(F_1\otimes F_2;x) \quad (40)$$

Therefore, from (39) and (40), the following theorems can be verified.

*Theorem 6:* The output of the FLS (36) satisfies

$$[F_1+F_2](x)=F_1(x)+F_2(x) \quad (41)$$

*Theorem 7:* The output of the FLS (37) satisfies

$$[F_1-F_2](x)=F_1(x)-F_2(x) \quad (42)$$

Furthermore, the output of the FLS (38) can be recurrently obtained by

$$[\sum_{i=1}^{m}F_i](x)=\sum_{i=1}^{m}F_i(x) \quad (43)$$

Let $\varepsilon_i$ ($i$=1,2,…) or $\varepsilon$ denote non-negative real numbers, and $f_i(x)$ ($i$=1,2,…) or $f(x)$ denote continuous functions on the compact set $U\subseteq R^n$. By using the above operations of FLSs and the following equality

$$f_1(x)f_2(x)-F_1(x)F_2(x)=f_1(x)[f_2(x)-F_2(x)]+$$
$$[F_2(x)-f_2(x)][f_1(x)-F_1(x)]+f_2(x)[f_1(x)-F_1(x)] \quad (44)$$

the following results for function approximation errors are easily obtained.

*Theorem 8:* Consider $m$ FLSs $F_i$ in the form of (14), $i$=1,2,…,$m$. If $\sup_{x\in U}\left|f_i(x)-F_i(x)\right|\le\varepsilon_i$, then the following results are true.

(1). $\sup_{x\in U}\left|\sum_{i=1}^{m}\lambda_i f_i(x)-[\sum_{i=1}^{m}(\lambda_i F_i)](x)\right|\le\sum_{i=1}^{m}\left|\lambda_i\right|\varepsilon_i$ , for real numbers $\lambda_i$, $i$=1,2,…,$m$.

(2). $\sup_{x\in U}\left|f_1(x)f_2(x)-[F_1\otimes F_2](x)\right|\le\bar{\varepsilon}(\bar{\varepsilon}+2M)$ , where $\bar{\varepsilon}=Max(\varepsilon_1,\varepsilon_2)$, $M=Max(M_1,M_2)$, $M_j=\underset{x\in U}{Max}\left|f_j(x)\right|$, $j$=1, 2.

*Theorem 9:* Consider two FLSs $F_i$ ($i$=1,2). If $\sup_{x\in U}\left|f(x)-F_i(x)\right|\le\varepsilon_i$, then it follows that.

$$\sup_{x\in U}\left|f(x)-[F_1\bigcup F_2](x)\right|\le Max(\varepsilon_1,\varepsilon_2) \quad (45)$$

*Corollary 1:* Consider the FLS (14). If $\sup_{x\in U}\left|f(x)-F(x)\right|\le\varepsilon$, then it follows that

(1). $\sup_{x\in U}\left|\lambda f(x)-[\lambda F](x)\right|\le\left|\lambda\right|\varepsilon$, for real number $\lambda$;

(2). $\sup_{x\in U}\left|f^2(x)-[F\otimes F](x)\right|\le\varepsilon(\varepsilon+2M)$ , where $M=\underset{x\in U}{Max}\left|f(x)\right|$.

*Corollary 2:* Consider two FLSs $F_i$ ($i$=1,2). If $\sup_{x\in U}\left|f_i(x)-F_i(x)\right|\le\varepsilon_i$, then it follows that.

$$\sup_{x\in U}\left|f_1(x)+f_2(x)-[F_1+F_2](x)\right|\le\varepsilon_1+\varepsilon_2 \quad (46)$$

## 5. Explanative Example

*Example 1:* Consider the continuous function $f(x)=\frac{\sin x}{x}$ on the interval [-3, 3]. In [1, 13], the FLS, with triangle-shaped membership functions and 25 fuzzy rules, is constructed to approximate the function at the approximation accuracy 0.02.

Now if we want to approximate the two-variate continuous function $f(x_1,x_2)=\frac{\sin x_1\sin x_2}{x_1x_2}$ on the compact domain [-3, 3]×[-3, 3], how to construct the whole fuzzy approximator by using the known sub-approximators instead of a second partition on [-3, 3]× [-3, 3]? What is the approximation accuracy?

A solution to the above problem can be obtained by using the methods in this paper.

Let $F_k$, $k$=1,2, denote the fuzzy approximators with the approximation accuracy 0.02 for $f_k(x_1)=\frac{\sin x_k}{x_k}$, $k$=1, 2, respectively, then from [1, 13], the fuzzy rules can be obtained as follows, respectively,

$$F_k: \text{If } x_1 \text{ is } A_j^k \text{ Then } y \text{ is } y_j^k,\; k=1,2 \quad (47)$$

where $A_j^k$ ( $k$=1,2; $j$=1,2,…25;) have triangle-shaped membership functions generated by using the partition points $e_j = -3 + (j-1)h$ , $h$=0.25. $y_j^k = f_k(e_j)$ . (See the details in [1, 13])

Now Let our attention into the function $f(x_1, x_2) = \frac{\sin x_1 \sin x_2}{x_1 x_2}$ on [-3, 3]× [-3, 3]. Obviously, $f(x_1, x_2) = f_1(x_1) f_2(x_2)$ , and the fuzzy rules (47) can be represented by the following form, respectively.

$$F_1: \text{If } x_1 \text{ is } A_j^1 \text{ and } x_2 \text{ is '1' Then } y \text{ is } y_j^1 \quad (48)$$

$$F_2: \text{If } x_1 \text{ is '1' and } x_2 \text{ is } A_j^2 \text{ Then } y \text{ is } y_j^2 \quad (49)$$

where, $j$=1,2, …25; the '1's denote the fuzzy sets defined by membership functions with all values equal to 1.

Therefore, we can obtain the fuzzy logic system $F_1 \otimes F_2$ by using (48), (49) and Theorem 5 with the following fuzzy rules.

$F_1 \otimes F_2$:

$$\text{If } x_1 \text{ is } A_{j_r}^1 \text{ and } x_2 \text{ is } A_r^2 \text{ Then } y \text{ is } y_{j_r}^1 y_r^2 \quad (50)$$

where $j_r$ , $r$=1,2, …25.

It is seen that the fuzzy logic system $F_1 \otimes F_2$ have 625 rules. From Theorem 8 in this paper, the approximation accuracy for $f(x_1, x_2) = \frac{\sin x_1 \sin x_2}{x_1 x_2}$ on [-3, 3]× [-3, 3] is no more than 0.0404.

The similar method can be applied for general case $f(x_1, \cdots, x_n) = \frac{\prod_{j=1}^{n} [\sin x_j]^{k_j}}{\prod_{j=1}^{n} (x_j)^{k_j}}$ , $k_j \in Z^+$ , by using $\underbrace{[F_1 \otimes \cdots \otimes F_1]}_{k_1} \otimes \cdots \otimes \underbrace{[F_n \otimes \cdots \otimes F_n]}_{k_n}$ . Its smaller approximation accuracy is mainly determined by the maximum ones of sub-approximators.

## 6. Conclusions

Compared with other existing approaches, the main contributions of this paper is that (1) if the fuzzy logic approximators for sub-functions are obtained, then the new fuzzy logic approximators can be constructed by using the old membership functions generated by the mathematic operations of original ones. This avoids the second partition on the input space; (2) the approximation accuracy of the new fuzzy logic approximator is determined by the bigger one of approximation accuracies of original sub-approximators. In a word, if you have obtained some sub-approximators by using any way, then the whole approximator can be obtained by using the methods in this paper.

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