

On Convex Fuzzy Processes and Their Generalizations

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Abstract

In this paper, we give the correlation and difference between M-convex fuzzy process and SLW-convex fuzzy process, and present some basic properties of these convex fuzzy processes, which improve some known results about the connection between those convex fuzzy processes and their graphs. In addition, we introduce a new notion of s-convex fuzzy process as a generalization of M-convex fuzzy process, discuss the relationship between QYS-s-convex fuzzy processes and CRO-s-convex fuzzy processes, and give some important connections between these s-convex fuzzy processes and their graphs. Two examples are given to illustrate the validity of the main results.

Keywords: Convex fuzzy sets, Convex fuzzy processes, Fuzzy analysis, S-convex fuzzy processes.

1. Introduction

As a suitable mathematical model to handle vagueness and uncertainty, fuzzy set theory is emerging as a powerful theory and has attracted the attention of many researchers and practitioners who contributed to its development and applications [1-9]. Convexity plays a most useful role in the theory and applications of fuzzy sets. Therefore, the research on convexity and generalized convexity is one of the most important aspects of fuzzy set theory [10-13].

In 2000, Matloka [14] introduced the notion of convex fuzzy process. Soon after that, Syan, Low and Wu [15] gave another definition and showed that the new-type of convex fuzzy process shares many properties with the

class of convex fuzzy processes in [14]. To avoid confusion, we will call the former M-convex fuzzy process and the latter SLW-convex fuzzy process in this paper. In the sequel we will clarify the relationship between M-convex fuzzy process and SLW-convex fuzzy process, and give some necessary and sufficient conditions for fuzzy mappings to be M-convex fuzzy processes and SLW-convex fuzzy processes, respectively.

In 2004, Chalco-Cano, Rojas-Medar and Osuna-Gomez [16] introduced the concept of s-convex fuzzy process which is a generalization of SLW-convex fuzzy process. In this paper, we will present a new type of s-convex fuzzy process which can be thought of as a generalization of M-convex fuzzy process. For the two concepts of s-convex fuzzy process, we will call the new one QYS-s-convex fuzzy process but the old one CRO-s-convex fuzzy process in order to avoid confusion. We will investigate the relationship between them, and give some necessary and sufficient conditions for fuzzy mappings to be the two types of s-convex fuzzy processes, respectively.

2. Preliminaries

Let R^n denote the n -dimensional Euclidean space, and let $F(R^n)$ denote the set of all nonempty fuzzy sets in R^n .

A fuzzy set $\mu: R^n \rightarrow [0,1]$ is called convex [11], if

$$\mu(\lambda y_1 + (1 - \lambda)y_2) \geq \min\{\mu(y_1), \mu(y_2)\}$$

for all $y_1, y_2 \in R^n$ and $\lambda \in (0,1)$.

A fuzzy set $\mu: R^n \rightarrow [0,1]$ is called a fuzzy cone [10] if $\mu(\lambda y) = \mu(y)$ for all $y \in R^n$ and $\lambda > 0$. A convex fuzzy cone is a fuzzy cone, which is also a convex fuzzy set.

For $\mu, \nu \in F(R^n)$ and $\lambda > 0$, owing to Zadeh's extension principle [17], scalar multiplication and addition are defined for any $y \in R^n$ by $\lambda\mu(y) = \mu(y/\lambda)$ and

$$(\mu + \nu)(y) = \sup_{y_1, y_2: y_1 + y_2 = y} \min\{\mu(y_1), \nu(y_2)\}.$$

The fuzzy set μ is said to be included in ν , denoted by $\mu \subseteq \nu$, if $\mu(y) \leq \nu(y)$ for all $y \in R^n$.

The graph of a fuzzy mapping $A: R^m \rightarrow F(R^n)$, de-

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noted by G_A , is a fuzzy set in $R^m \times R^n$ such that for any $(x, y) \in R^m \times R^n$, $G_A(x, y) = A(x)(y)$ [14].

A mapping A from R^m to $F(R^n)$ is called M-convex fuzzy process [14] if it satisfies the conditions

- (a) $A(x_1 + x_2) \supseteq A(x_1) + A(x_2), \forall x_1, x_2 \in R^m$,
- (b) $A(\lambda x) = \lambda A(x), \forall \lambda > 0, x \in R^m$.

A mapping A from R^m to $F(R^n)$ is called SLW-convex fuzzy process [15] if it satisfies the condition

$$(c) \quad \sup_{y_1, y_2: \lambda y_1 + (1-\lambda)y_2 = y} \min\{A(x_1)(y_1), A(x_2)(y_2)\} \geq A(\lambda x_1 + (1-\lambda)x_2)(y)$$

for all $x_1, x_2 \in R^m$, $\lambda \in (0,1)$ and $y \in R^n$.

One of the principal results of [15] is

Theorem 1 [15]: A mapping $A : R^m \rightarrow F(R^n)$ is an SLW-convex fuzzy process if and only if it satisfies the condition

$$(c') \quad A(\lambda x_1 + (1-\lambda)x_2) \supseteq \lambda A(x_1) + (1-\lambda)A(x_2)$$

for all $x_1, x_2 \in R^m$, $\lambda \in (0,1)$.

The CRO-s-convex fuzzy process was introduced by Chalco-Cano, Rojas-Medar and Osuna-Gomez [16]. Let $s \in (0,1]$, a fuzzy mapping $F : R^m \rightarrow F(R^n)$ is called CRO-s-convex fuzzy process if for all $\alpha \in (0,1)$ and $x_1, x_2 \in R^m$, it satisfies the condition

$$(d) \quad \alpha^s F(x_1) + (1-\alpha)^s F(x_2) \subseteq F(\alpha x_1 + (1-\alpha)x_2).$$

Motivated both by the work [16] and by the importance of the concept of M-convex fuzzy process [14], we present the concept of QYS-s-convex fuzzy process here.

Definition 1: Let $s \in (0,1]$, a fuzzy mapping $F : R^m \rightarrow F(R^n)$ is called QYS-s-convex fuzzy process if it satisfies the following conditions

- (e) $F(x_1 + x_2) \supseteq F(x_1) + F(x_2), \forall x_1, x_2 \in R^m$,
- (f) $F(\alpha x) = \alpha^s F(x), \forall \alpha > 0, x \in R^m$.

3. Fuzzy Convex Processes

We first discuss the relationship between M-fuzzy convex processes and SLW-convex fuzzy processes. In [15] the authors have given the following result.

Theorem 2 [15]: Let $A : R^m \rightarrow F(R^n)$ be an M-convex fuzzy process. Then A is an SLW-convex fuzzy process.

For the converse, we give the conclusion as follows.

Theorem 3: Let $A : R^m \rightarrow F(R^n)$ be an SLW-convex fuzzy process satisfying (b). Then A is an M-convex fuzzy process.

Proof: We know from Theorem 1 that the mapping A

satisfies (c'). Now let $x_1, x_2 \in R^m$, and $\lambda = 1/2$. From (b) and (c'), we obtain

$$\begin{aligned} \frac{1}{2} A(x_1 + x_2) &= A\left(\frac{1}{2}(x_1 + x_2)\right) \supseteq \\ \frac{1}{2} A(x_1) + \frac{1}{2} A(x_2) &= \frac{1}{2}(A(x_1) + A(x_2)), \end{aligned}$$

which implies $A(x_1 + x_2) \supseteq A(x_1) + A(x_2)$. Thus, A is an M-convex fuzzy process.

Corollary 1: A mapping $A : R^m \rightarrow F(R^n)$ is an M-convex fuzzy process if and only if it satisfies (b) and (c), or (b) and (c').

Proof: It follows quickly from Theorem 1-3.

Next, we present some important connections between M-convex fuzzy processes and their graphs.

Theorem 4 [14]: The graph of an M-convex fuzzy process $A : R^m \rightarrow F(R^n)$ is a convex fuzzy cone in $R^m \times R^n$.

In fact, the converse of this theorem is also true.

Theorem 5: If the graph of a fuzzy mapping $A : R^m \rightarrow F(R^n)$ is a convex fuzzy cone in $R^m \times R^n$, then A is an M-convex fuzzy process.

Proof: Taking into account the definitions of the graph and fuzzy cone we observe that for any $\lambda > 0$ and

$$(x, y) \in R^m \times R^n,$$

$$\begin{aligned} A(\lambda x)(y) &= G_A(\lambda x, y) = G_A(\lambda(x, y/\lambda)) \\ &= G_A(x, y/\lambda) = A(x)(y/\lambda) = \lambda A(x)(y). \end{aligned}$$

Thus the mapping A satisfies (b).

Now, we will prove that the mapping A satisfies (c') too. Let $x_1, x_2 \in R^m$, $\lambda \in (0,1)$. For any given point $y_0 \in R^n$, from the definitions of addition and scalar multiplication, we obtain

$$\begin{aligned} &(\lambda(A(x_1) + (1-\lambda)A(x_2)))(y_0) \\ &= \sup_{y_1, y_2: \lambda y_1 + (1-\lambda)y_2 = y_0} \min\{\lambda A(x_1)(y_1), (1-\lambda)A(x_2)(y_2)\} \\ &= \sup_{y_1, y_2: \lambda y_1 + (1-\lambda)y_2 = y_0} \min\{A(x_1)\left(\frac{y_1}{\lambda}\right), A(x_2)\left(\frac{y_2}{1-\lambda}\right)\} \\ &= \sup_{y_1, y_2: \lambda y_1 + (1-\lambda)y_2 = y_0} \min\{A(x_1)(y_1), A(x_2)(y_2)\} \quad (1) \end{aligned}$$

Since the graph G_A is a convex fuzzy set, for any $(x_1, y_1), (x_2, y_2) \in R^m \times R^n$ and $\lambda \in (0,1)$ such that $\lambda y_1 + (1-\lambda)y_2 = y_0$, we have

$$\begin{aligned} &G_A(\lambda x_1 + (1-\lambda)x_2, y_0) \\ &= G_A(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \\ &= G_A(\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2)) \end{aligned}$$

$$\geq \min\{G_A(x_1, y_1), G_A(x_2, y_2)\}.$$

In other words, we get

$$\begin{aligned} & A(\lambda x_1 + (1-\lambda)x_2)(y_0) \\ &= A(\lambda x_1 + (1-\lambda)x_2)(\lambda y_1 + (1-\lambda)y_2) \\ &\geq \min\{A(x_1)(y_1), A(x_2)(y_2)\}. \end{aligned} \tag{2}$$

Notice that

$$\begin{aligned} & A(\lambda x_1 + (1-\lambda)x_2)(y_0) \\ &= \sup_{y_1, y_2: \lambda y_1 + (1-\lambda)y_2 = y_0} A(\lambda x_1 + (1-\lambda)x_2)(\lambda y_1 + (1-\lambda)y_2). \end{aligned}$$

From (2), we have that for any $y_1, y_2 \in R^n$ satisfying $\lambda y_1 + (1-\lambda)y_2 = y_0$,

$$\begin{aligned} & \sup_{y_1, y_2: \lambda y_1 + (1-\lambda)y_2 = y_0} A(\lambda x_1 + (1-\lambda)x_2)(\lambda y_1 + (1-\lambda)y_2) \\ &\geq \min\{A(x_1)(y_1), A(x_2)(y_2)\}. \end{aligned}$$

Now by taking the supremum on the right of the above inequality and observing (1), we have

$$\begin{aligned} & A(\lambda x_1 + (1-\lambda)x_2)(y_0) \\ &= \sup_{y_1, y_2: \lambda y_1 + (1-\lambda)y_2 = y_0} A(\lambda x_1 + (1-\lambda)x_2)(\lambda y_1 + (1-\lambda)y_2) \\ &\geq \sup_{y_1, y_2: \lambda y_1 + (1-\lambda)y_2 = y_0} \min\{A(x_1)(y_1), A(x_2)(y_2)\} \\ &= (\lambda A(x_1) + (1-\lambda)A(x_2))(y_0). \end{aligned}$$

Since y_0 is an arbitrary given point in R^n , we complete the whole proof here.

Corollary 2: A fuzzy mapping A is an M-convex fuzzy process if and only if its graph is a convex fuzzy cone.

Proof: It follows quickly from Theorem 4 and 5.

Similarly, we can get the connection between SLW-convex fuzzy processes and their graphs.

Theorem 6 [15]: The graph of an SLW-convex fuzzy process A is a convex fuzzy set in $R^m \times R^n$.

Theorem 7: If the graph of a fuzzy mapping A is a convex fuzzy set in $R^m \times R^n$, then A is an SLW-convex fuzzy process.

Proof: It is similar to the proof of Theorem 5.

Corollary 3: A fuzzy mapping A is an SLW-convex fuzzy process if and only if its graph is a convex fuzzy set.

Proof: It follows quickly from Theorem 6 and 7.

Example 1: Define the fuzzy mapping $A : (0, \infty) \rightarrow F(R)$ as follows

$$A(x)(y) = \begin{cases} 1, & \text{if } y \geq x, \\ y/x, & \text{if } 0 \leq y < x, \\ 0, & \text{if } y < 0. \end{cases}$$

Now we show that the graph

$$G_A(x, y) = \begin{cases} 1, & \text{if } y \geq x, \\ y/x, & \text{if } 0 \leq y < x, \\ 0, & \text{if } y < 0, \end{cases}$$

is a fuzzy convex cone in $(0, \infty) \times R$. In fact, for any $(x, y) \in (0, \infty) \times R$ and $\lambda > 0$, we have

$$\begin{aligned} & G_A(\lambda(x, y)) = G_A(\lambda x, \lambda y) \\ &= G_A(x, y) = \begin{cases} 1, & \text{if } y \geq x, \\ y/x, & \text{if } 0 \leq y < x, \\ 0, & \text{if } y < 0, \end{cases} \end{aligned}$$

which implies G_A is a fuzzy cone. Let $\lambda > 0$ and $(x_1, y_1), (x_2, y_2) \in (0, \infty) \times R$. In order to prove G_A is a convex fuzzy set, we now distinguish the following three cases.

Case 1: $\lambda y_1 + (1-\lambda)y_2 \geq \lambda x_1 + (1-\lambda)x_2$.

It is clear that

$$\begin{aligned} & G_A(\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2)) \\ &= G_A(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \\ &= 1 \geq \min\{G_A(x_1, y_1), G_A(x_2, y_2)\}. \end{aligned}$$

Case 2: $\lambda y_1 + (1-\lambda)y_2 \leq 0$.

In this case we have that at least one of the two points y_1 and y_2 is not positive. Without loss of generality, suppose $y_1 \leq 0$. Then

$$\begin{aligned} & G_A(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \\ &= 0 = \min\{G_A(x_1, y_1), G_A(x_2, y_2)\}. \end{aligned}$$

Case 3: $0 < \lambda y_1 + (1-\lambda)y_2 < \lambda x_1 + (1-\lambda)x_2$.

If $y_1 < x_1$ and $y_2 \geq x_2$, then we have that

$$\begin{aligned} & G_A(\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2)) \\ &= (\lambda y_1 + (1-\lambda)y_2) / (\lambda x_1 + (1-\lambda)x_2) \\ &\geq (\lambda y_1 + (1-\lambda)y_2) / (\lambda x_1 + (1-\lambda)y_2) \\ &\geq y_1 / x_1 = \min\{G_A(x_1, y_1), G_A(x_2, y_2)\}. \end{aligned}$$

If $y_1 < x_1$ and $y_2 < x_2$, then, without loss of generality, we suppose $y_1/x_1 \leq y_2/x_2$. Therefore, we have

$$\begin{aligned} & (\lambda y_1 + (1-\lambda)y_2) / (\lambda x_1 + (1-\lambda)x_2) - y_1/x_1 \\ &= (1-\lambda)(x_1 y_2 - y_1 x_2) / x_1(\lambda x_1 + (1-\lambda)x_2) \geq 0, \end{aligned}$$

which implies

$$\begin{aligned} & G_A(\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2)) \\ &\geq \min\{G_A(x_1, y_1), G_A(x_2, y_2)\}. \end{aligned}$$

From what have been discussed above, we now can see that G_A is a fuzzy convex cone in $(0, \infty) \times R$. Thus according to Theorem 5, A is an M-convex fuzzy process.

4. S-convex Fuzzy Processes

Now, we discuss the relationship between QYS-s-convex fuzzy processes and CRO-s-convex fuzzy processes. Theorem 3.2 in [16] is as follows.

Theorem 8 [16]: Let $s \in (0,1]$. Let $F : R^m \rightarrow F(R^n)$ be a fuzzy mapping satisfying (e) and (f). Then F is a CRO-s-convex fuzzy process.

Consequently, by the definition in Preliminaries, we have that if F is a QYS-s-convex fuzzy process, and then it is a CRO-s-convex fuzzy process. For the converse, we give the conclusion as follows.

Theorem 9: Let $s \in (0,1]$. Let $F : R^m \rightarrow F(R^n)$ be a CRO-s-convex fuzzy process satisfying (f). Then F is a QYS-s-convex fuzzy process.

Proof: Now let $x_1, x_2 \in R^m$ and $\alpha = 1/2$. From (d) and (f), we obtain

$$\begin{aligned} \frac{1}{2^s} F(x_1 + x_2) &= F\left(\frac{1}{2}(x_1 + x_2)\right) \supseteq \\ \frac{1}{2^s} F(x_1) + \frac{1}{2^s} F(x_2) &= \frac{1}{2^s} (F(x_1) + F(x_2)), \end{aligned}$$

which implies $F(x_1 + x_2) \supseteq F(x_1) + F(x_2)$. Therefore, F is a QYS-s-convex fuzzy process.

Corollary 4: Let $s \in (0,1]$. A fuzzy mapping F is a QYS-s-convex fuzzy process if and only if it satisfies (d) and (f).

Proof: It follows quickly from Theorem 8 and 9.

Next, we present some important connections between these two types of s-convex fuzzy processes and their graphs.

Theorem 10: Let $s \in (0,1]$. A fuzzy mapping F is a CRO-s-convex fuzzy process if and only if its graph G_F satisfies the condition

$$\begin{aligned} G_F(\alpha x_1 + (1-\alpha)x_2, \alpha^s y_1 + (1-\alpha)^s y_2) \\ \geq \min\{G_F(x_1, y_1), G_F(x_2, y_2)\} \end{aligned} \tag{3}$$

for all $(x_1, y_1), (x_2, y_2) \in R^m \times R^n$ and $\alpha \in (0,1)$.

Proof: Suppose $F : R^m \rightarrow F(R^n)$ is a CRO-s-convex fuzzy process.

Let $\alpha \in (0,1)$ and $(x_1, y_1), (x_2, y_2) \in R^m \times R^n$. Then there holds

$$\begin{aligned} G_F(\alpha x_1 + (1-\alpha)x_2, \alpha^s y_1 + (1-\alpha)^s y_2) \\ = F(\alpha x_1 + (1-\alpha)x_2)(\alpha^s y_1 + (1-\alpha)^s y_2) \\ \geq (\alpha^s F(x_1) + (1-\alpha)^s F(x_2))(\alpha^s y_1 + (1-\alpha)^s y_2) \\ \geq \min\{\alpha^s F(x_1)(\alpha^s y_1), (1-\alpha)^s F(x_2)((1-\alpha)^s y_2)\} \\ = \min\{F(x_1)(y_1), F(x_2)(y_2)\} \end{aligned}$$

$$= \min\{G_F(x_1, y_1), G_F(x_2, y_2)\}.$$

For the converse, let $x_1, x_2 \in R^m$ and $\alpha \in (0,1)$. From the definitions of addition and scalar multiplication, for any given point $y \in R^n$ we obtain

$$\begin{aligned} (\alpha^s F(x_1) + (1-\alpha)^s F(x_2))(y) \\ = \sup_{y_1, y_2: y_1 + y_2 = y} \min\{\alpha^s F(x_1)(y_1), (1-\alpha)^s F(x_2)(y_2)\} \\ = \sup_{y_1, y_2: y_1 + y_2 = y} \min\{F(x_1)\left(\frac{y_1}{\alpha^s}\right), F(x_2)\left(\frac{y_2}{(1-\alpha)^s}\right)\} \\ = \sup_{y_1, y_2: \alpha^s y_1 + (1-\alpha)^s y_2 = y} \min\{F(x_1)(y_1), F(x_2)(y_2)\}. \end{aligned} \tag{4}$$

Since the graph G_F satisfies (3), for any $\alpha \in (0,1)$ and $(x_1, y_1), (x_2, y_2) \in R^m \times R^n$ such that $\alpha^s y_1 + (1-\alpha)^s y_2 = y$, we have

$$\begin{aligned} G_F(\alpha x_1 + (1-\alpha)x_2, \alpha^s y_1 + (1-\alpha)^s y_2) \\ \geq \min\{G_F(x_1, y_1), G_F(x_2, y_2)\}, \end{aligned}$$

that is,

$$\begin{aligned} F(\alpha x_1 + (1-\alpha)x_2)(\alpha^s y_1 + (1-\alpha)^s y_2) \\ \geq \min\{F(x_1)(y_1), F(x_2)(y_2)\}. \end{aligned} \tag{5}$$

Now taking into account (4) and (5), we obtain

$$\begin{aligned} F(\alpha x_1 + (1-\alpha)x_2)(y) \\ = \sup_{y_1, y_2: \alpha^s y_1 + (1-\alpha)^s y_2 = y} F(\alpha x_1 + (1-\alpha)x_2)(\alpha^s y_1 + (1-\alpha)^s y_2) \\ \geq \sup_{y_1, y_2: \alpha^s y_1 + (1-\alpha)^s y_2 = y} \min\{F(x_1)(y_1), F(x_2)(y_2)\} \\ = (\alpha^s F(x_1) + (1-\alpha)^s F(x_2))(y), \end{aligned}$$

which completes the whole proof.

Theorem 11: Let $s \in (0,1]$. A fuzzy mapping F is a QYS-s-convex fuzzy process if and only if its graph G_F satisfies (3) and the following condition

$$G_F(\alpha x, \alpha^s y) = G_F(x, y) \tag{6}$$

for all $(x, y) \in R^m \times R^n$ and $\alpha \in (0,1)$.

Proof: We first prove that Condition (6) is equivalent to (f). Suppose that Condition (f) holds. Then for any

$(x, y) \in R^m \times R^n$ and $\alpha \in (0,1)$, we have

$$\begin{aligned} G_F(\alpha x, \alpha^s y) &= F(\alpha x)(\alpha^s y) \\ &= \alpha^s F(x)(\alpha^s y) = F(x)(y) = G_F(x, y). \end{aligned}$$

Conversely, for any $x \in R^m, y \in R^n$ and $\alpha \in (0,1)$, we have

$$\begin{aligned} \frac{1}{\alpha^s} F(\alpha x)(y) &= F(\alpha x)(\alpha^s y) \\ &= G_F(\alpha x, \alpha^s y) = G_F(x, y) = F(x)(y) \end{aligned}$$

which implies $F(\alpha x)(y) = \alpha^s F(x)(y)$. Since Condition (3) is equivalent to (d) which follows from Theorem 10, we get the desired result by Corollary 4.

Example 2: This example is partly quoted from [18]. Let us consider the fuzzy mapping $F : (0, \infty) \rightarrow F(R)$ that associates to each $x \in (0, \infty)$ the points of the real line “much bigger than \sqrt{x} ”. Now, we define the fuzzy mapping $F_1 : (0, \infty) \rightarrow F(R)$ as follows

$$F_1(x)(y) = \begin{cases} y/\sqrt{x} - 1, & \text{if } \sqrt{x} \leq y \leq 2\sqrt{x}, \\ 1, & \text{if } y \geq 2\sqrt{x}, \\ 0, & \text{if } y \leq \sqrt{x}. \end{cases}$$

For F_1 and $x = 4$, we have that the points of the real line “much bigger than $\sqrt{4} = 2$ ” is the fuzzy sets

$$F_1(4)(y) = \begin{cases} y/2 - 1, & \text{if } 2 \leq y \leq 4, \\ 1, & \text{if } y \geq 4, \\ 0, & \text{if } y \leq 2. \end{cases}$$

This means that the points after 4 are “much bigger than 2”, while the points in the interval (2, 4) are partially “much bigger than 2”, i.e., they have a degree of membership to the fuzzy set $F_1(4)$. Therefore, F_1 models the fuzzy mapping F . In [18], the authors have shown that F_1 is a CRO-1/2-convex fuzzy process. Now we show that F_1 is also a QYS-1/2-convex fuzzy process. According to Theorem 9, it is enough to merely prove that F_1 satisfies (f). Let $\alpha > 0$, $x \in (0, \infty)$ and $y \in R$.

From the definition of F_1 , we get

$$\begin{aligned} F_1(\alpha x)(y) &= \begin{cases} y/\sqrt{\alpha x} - 1, & \text{if } \sqrt{\alpha x} \leq y \leq 2\sqrt{\alpha x}, \\ 1, & \text{if } y \geq 2\sqrt{\alpha x}, \\ 0, & \text{if } y \leq \sqrt{\alpha x}, \end{cases} \\ &= \begin{cases} \frac{y}{\sqrt{\alpha}} \frac{1}{\sqrt{x}} - 1, & \text{if } \sqrt{x} \leq \frac{y}{\sqrt{\alpha}} \leq 2\sqrt{x}, \\ 1, & \text{if } \frac{y}{\sqrt{\alpha}} \geq 2\sqrt{x}, \\ 0, & \text{if } \frac{y}{\sqrt{\alpha}} \leq \sqrt{x}, \end{cases} \\ &= F_1(x)\left(\frac{y}{\sqrt{\alpha}}\right) \\ &= \alpha^{\frac{1}{2}} F_1(x)(y). \end{aligned}$$

Therefore, F_1 is a QYS-1/2-convex fuzzy process.

5. Conclusions

In this present investigation, we have got the correlation and difference between M-convex fuzzy process and SLW-convex fuzzy process, and have given the complete connections between these convex fuzzy processes and their graphs. For their generalizations, we have also shown the relationship between QYS-s-convex fuzzy processes and CRO-s-convex fuzzy processes, and have obtained the important connections between these s-convex fuzzy processes and their graphs. Two examples are given to illustrate the validity of the main results.

Several possible applications of our results may be suggested. We briefly mention some of them. An analytical approach to fuzzy systems certainly requires the fuzzy mappings as a prerequisite. Recalling the application of convex process was first studied in economics, convex fuzzy processes may prove useful in such a field. Fuzzy mappings are a convenient way to define ill-defined correspondences between variables which characterize a system. The possibility of simple criterions for convex fuzzy processes by their graphs, on such representation makes them even more attractive. This paper continues the authors’ research in fuzzy convex analysis and fuzzy complex analysis [12, 19]. We have introduced some new and more general definitions in the area of fuzzy convexity [12]. In the next phase of investigations, we will combine the results obtained in this paper and the ones in [12], and try to solve the open problem in fuzzy analysis that we proposed in [19]. In addition, the mathematics of the introduced concept certainly deserves further investigation. It would be interesting to deeply analyze issues related to continuity of QYS-s-convex fuzzy process like the work for CRO-s-convex fuzzy process in [20], and it would be also interesting to study inequalities for QYS-s-convex fuzzy processes like the work for CRO-s-convex fuzzy process in [18]. In our further research, we intend to address these issues.

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