

Vulnerable Option Pricing under Heterogeneity and Its Applications in Taiwan Warrant Market

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Abstract

The European option pricing formula proposed by Black and Scholes in 1973 is a new milestone in financial field. However, many basic assumptions used to establish this pricing formula are unrealistic in real trading market. Though many researchers after Black and Scholes have proposed numerous solutions to resolve the drawback of Black-Scholes' formula, few have mentioned the topics of heterogeneous beliefs of investors and the counterparty risk at the same time. This study proposes a new method to resolve these problems and then investigates the robustness of the formula by using the warrant in Taiwan stock market. The results clearly demonstrate how the proposed model leads to a straightforward extension of the existing asset pricing models in a fairly easy way.

Keywords: *Fuzzy Option, Heterogeneous Belief, Default, Credit Risk.*

1. Introduction

When investors trade in financial markets, they assume that the trading counterparty will always meet their obligations. However, in the real world, this assumption is not always the case. Market investors are thus always vulnerable to default risk, which is not negligible when they trading in an over-the-counter (OTC) market. The price of the OTC option must thus be discounted to reflect a credit risk premium. Consequently, accounting for this premium is important when pricing OTC options, and the credit risk has to be considered in the model when there is a chance that default may occur.

Many researchers have proposed formulae to evaluate options with default risk. The financial term for options with default risk is "vulnerable option". Johnson and Stulz [10] first proposed the concept of vulnerable options. In the model of Johnson and Stulz, options are the

only liability of the firm and default occurs when their value exceeds the value of the assets of the trading counterparty. Assuming options to be the only liability of the firm is also a drawback of the model of Johnson and Stulz. Articles proposing solutions to this problem include in [8, 10, 13]. Hull and White proposed a model for pricing vulnerable options to allow the trading counterparty to assume other liabilities with equal priority. Consequently, in the event of default, only a proportion of the original claims are paid to the option holders. Klein [11] proposed an improved method of pricing vulnerable Black-Scholes options based on assumptions which are widely appropriate. Klein also established an analytic pricing formula which allows not only for correlation between the asset underlying an option and those of the counterparty, but also for the option writer to have other liabilities. Liao and Huang [14] refined the model of Klein and provided a closed-form valuation formula for vulnerable options subject to interest rate and credit risks. Although all the above models are good at evaluating the credit risk of the trading counterparty, they suffer a common drawback that is important but seldom mentioned. That is, the underlying assets of the vulnerable option, such as firm value or collateral, are generally illiquid or even nontraded, and consequently it is inappropriate to treat them as liquid assets when pricing vulnerable options. Hence, this investigation aims to consider this issue and use the fuzzy measure to determine the liquidity risk when pricing options which may simultaneously face both credit risk and liquidity risk.

Few models exploit the idea of using fuzzy measure concepts to fuzzify credit risk models, though that of [3, 5] is an exception. The credit risks models of [4] are the most innovative developed to date and to our knowledge were the first one to use fuzzy logic to represent liquidity risk. Han and Zheng [9] followed Cherubini and Della Lunga and further used fuzzy logic to explain the heterogeneity of investor trading behaviors. Classic economic theory assumes that all market participants have "identical rationality", and that investor information, preference, expectations, and investment strategies are homogenous, and that all agents exhibit similar investment behavior. Since most option pricing models follow

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Manuscript received 28 Dec. 2008; revised 20 Feb. 2010; accepted 27 Aug. 2010.

the assumption of the “identical rationality”, option prices are derived naturally from the unique probability distribution of the uncertain future payoff. However, the assumption that all investors have “identical rationality” is unrealistic in real world situations, and in any case market participants with identical rationality and receiving identical market information will still adopt different trading strategies. Indeed, if all market participants followed a clear and homogeneous investment strategy, the rationale for trading would vanish. Since expectations and perceptions regarding future market conditions vary among individual investors, it is reasonable to assume that their trading behaviors are imprecise. This study incorporates the strengths of the models of [5, 13, 20] to construct a powerful option pricing formula suitable for application under conditions involving credit risk, liquidity risk and market participants exhibiting imprecise investment behavior. Other liabilities besides options are permitted in the capital structure of the writer and the proposed model considers a correlation between the assets of the writer and the underlying assets, as well as allowing for the assets of the writer to be illiquid or non-tradable, and for investors to exhibit nonidentical rationality. Rather than employing the traditional probability measure to design the model, this study adopts a new measure, the λ -additive fuzzy measures, to establish the vulnerable option pricing model. Liquidity risk is introduced by representing the upper and lower bounds of the price of the contingent claim, which are calculated as the upper and lower Choquet integrals with respect to a subadditive function.

The fuzzy set theory was first proposed by [21] and subsequently has developed very rapidly. According to the initial definition, a fuzzy set describes the set of an event without clear boundaries. A fuzzy measure is a generalized version of a classical measure. The generalization is obtained by replacing the additivity axiom of the classical measure with weaker axioms of monotonicity and continuity. Classical measure theory is closely related to probability theory, while a generalized theory based on two types of nonadditive measure was initially proposed by [7] and subsequently refined by [19]. Additionally, the notion of fuzzy random variables was initially proposed by [12] and subsequently refined by Puri and Ralescu [17]. Recently, Tsaur and Wang [18] used triangular form of the membership function to devise a fuzzy regression model with a regression interval close to the sum of the radius values of collected interval data. This study utilizes a specific fuzzy measure, the λ -additive fuzzy measure, as proposed by [3, 5].

The remainder of this paper is organized as follows.

Section 2 briefly describes fuzzy set and option pricing theory. Section 3 then proposes the Black-Scholes option model under a fuzzy environment and extends the fuzzy option to include fuzzy vulnerable options. Next, Section 4 includes numerical results and an application for pricing Taiwan’s warrants with credit risk while Section 5 presents conclusions.

2. Fuzzy Set and Option Pricing Theory

A. Fuzzy Set Theory

Fuzzy set theory was first presented by Zadeh in the mid-1960s, and has since been applied in a wide range of scientific areas. The fuzziness of a symbol lies in the lack of well-defined boundaries of the set of objects to which it applies, a condition that is frequently encountered in real life. Additionally, a fuzzy subset obviously has no well-defined boundaries. This means that fuzzy sets can be used to formalize the inaccuracies that exist in human decision making, and as a representation of vague, uncertain, or imprecise knowledge. Specifically, in an economic context, Ponsard [16] proposed that “the use of fuzzy subsets theory leads to results that could not be obtained by classical methods”. Zadeh [21] developed an algebra for what he called fuzzy sets. This algebra was created to handle imprecise elements in decision-making processes, and is the formal body of theory that allows the treatment of practically all decisions in an uncertain environment. Later, Bellman and Zadeh [1] stated that “informally, a fuzzy set is a class of objects in which there is no sharp boundary between those objects that belong to the class and those that do not”.

To date, financial applications of fuzzy sets have included consumption behavior analysis (e.g. consumer preference matching; consumption decomposition; spending trend prediction; subjective price evaluation; prediction of satisfaction level), the credit card business (e.g. credit fraud detection; automated fraud explanation; credit spending analysis), market forecasts (e.g. stock price forecast; consumer price forecast; industry/regional growth prediction), market demand analysis, market fluctuation forecast, market simulations (e.g. pricing strategy study) and case studies (e.g. scenario simulations) [13]. Consequently, this study is confident in its use of fuzzy sets with regard to a vulnerable option pricing model.

In this paper, probability is considered as the uncertainty that something will occur or not, and the concept of fuzziness is utilized because it can express this uncertainty. The imprecision we encounter when judging or estimating future cash flows is not stochastic in nature, and the use of probability theory gives us a misleading

level of precision. The proposed model incorporates subjective judgments and statistical uncertainties, and thus may give investors a better understanding of the problems they face when making investment decisions. Before the proposed model is constructed, the following section presents a brief introduction of some terms related to fuzzy measurements.

Let Ω represent a nonempty set, and let Σ denote σ -algebra on Ω , while $\{\Omega, \Sigma\}$ is a measurable space. Some basic concepts are outlined here to introduce fuzzy set theory. The fuzzy measure is defined as follows.

Definition 1: Fuzzy measure

Given a measurable space $\{\Omega, \Sigma\}$, a finite monotone set function μ is a fuzzy measure if and only if:

- (1). $\mu(\emptyset) = 0$.
- (2). if $A_i, A_j \in \Sigma$ and $A_i \subset A_j$ then $\mu(A_i) \leq \mu(A_j)$
- (3). (a) if $\{A_n\} \in \Sigma, A_1 \subset A_2 \subset \dots$ and $\bigcup_{n=1}^{\infty} A_n \in \Sigma$ then

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right).$$

- (b) if $\{A_n\} \in \Sigma, A_1 \supset A_2 \supset \dots$ and $\bigcap_{n=1}^{\infty} A_n \in \Sigma$ then

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right).$$

In addition to the above property, if $\mu(\Omega) = 1$, we call it a regular fuzzy measure. This kind of nonidentical rationality can be seen in the subjective probabilities assigned by investors to their own average weighted price, and this subjective probability can be transformed from an identical mathematical expression based on fuzzy measures. This study thus applied a specific fuzzy measure known as the λ -additive fuzzy measure, following [3, 5].

Definition 2: λ -additive fuzzy measure

A λ -additive fuzzy measure is defined on $\{\Omega, \Sigma\}$, denoted by μ , which possesses the property that for arbitrary $\lambda \in (-1, \infty), A_i, A_j \in \Sigma, A_i \cap A_j = \emptyset$,

$$\mu(A_i \cup A_j) = \mu(A_i) + \mu(A_j) + \lambda \mu(A_i) \mu(A_j)$$

Furthermore, when $A_n \in \Sigma (n = 1, 2, 3, \dots)$ are disjointed with each other, the following is obtained

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \begin{cases} \frac{1}{\lambda} \left\{ \prod_{n=1}^{\infty} [1 + \lambda \mu(A_n)] - 1 \right\}, & \lambda \neq 0 \\ \sum_{n=1}^{\infty} \mu(A_n), & \lambda = 0 \end{cases}$$

When $\lambda \neq 0$, a correspondence exists between the λ -additive fuzzy measures and probability measures. If ν is an λ -additive fuzzy measure on Σ , then $P = \ln_{1+\lambda}(1 + \lambda \nu)$ is a probability measure on Σ , and if P is a probability measure, then

$$\nu = \frac{1}{\lambda} \left[(1 + \lambda)^P - 1 \right] \text{ is a } \lambda\text{-additive fuzzy measure [20].}$$

Consequently, by selecting the parameter λ , each investor assigns their own subjective measure when pricing assets.

Definition 3: Duality of the λ -additive fuzzy measure

To recover a duality result between the sub-additive and super-additive measures of the λ -additive fuzzy measure, let $\lambda^* = -\frac{\lambda}{1+\lambda}$. It can be verified that

$$\mu^\lambda(A) + \mu^{\lambda^*}(A^C) = 1,$$

where A^C denotes the complement to any subset of A . This means that for any sub-additive measure, it is possible to construct a super-additive measure, and thus the sum of the two measures over two complement sets equals 1. In other words, if μ^λ is a λ -additive measure on Σ , then μ^{λ^*} is a λ^* -additive measure on Σ .

To calculate the fuzzy vulnerable option value it is necessary to learn how to perform integration in fuzzy measure theory. Several principal fuzzy integrals exist, along with other integrals modified from these. This article mainly discusses the Choquet integral, because it is the most natural fuzzy integral, and the fuzzy vulnerable option prices are thus derived from the Choquet expectation.

Definition 4: Choquet integral [6]

The fuzzy integrals used in this study follow the concept of integration proposed by [4], and are defined as follows. Letting $f(X(\omega))$ denote the measurable real function on Ω , then for $f(X) \geq 0$, the Choquet integral based on fuzzy measure μ is:

$$(c) \int f(X) d\mu = \int \mu(\omega \in \Omega : f(X) \geq x) dx,$$

where the right-hand side is the Riemann integral and $(c) \int$ means ‘‘Choquet integral’’.

B. Option Pricing Theory

During the early 1970s, Black and Scholes [2] and, independently, Metron [15] used the geometric Brownian motion to construct a theory for pricing stock options. The Black-Scholes formula has become indispensable in modern financial market practice.

First, this study briefly reviews the Black-Scholes model. Asset prices S_t are assumed to follow geometric Brownian motion and are represented by the equation:

$$dS_t = \mu_s S_t dt + \sigma_s S_t dz_t^p, \tag{1}$$

where the process z_t^p is a standard Brownian motion,

μ_s denotes drift and σ_s represents the underlying stock volatility.

Generally, a European call option gives the holder the right, but not the obligation, to purchase a stock at a specified time T for a specified price K . Let S_t denote the price of the underlying asset at expiration time T . Then the payoff, C_T , of a European style call option at time T is given by

$$C_T = \text{Max}(S_T - K, 0). \tag{2}$$

This means that the call option is exercised if $S_T > K$ and otherwise is abandoned. The above call option is sometimes called a plain vanilla or standard option. Let $r > 0$ represent the instantaneous interest rate, namely interest factor, on a bond. Furthermore, let a bond price process $\{R_t\}_{t \geq 0}$ satisfy the ordinary differential equation:

$$dR_t = rR_t dt, \tag{3}$$

with $R_0 = 1$.

A probability measure Q is called an equivalent martingale measure to probability measure P for the discounted price process $\tilde{S}_t = e^{-rt} S_t$ if

$$\tilde{S}_t = E^Q[\tilde{S}_t | F_s] = E^Q[\tilde{S}_t], \tag{4}$$

for each $0 \leq s \leq t \leq T$ and $Q \sim P$, where F_t denotes the history of the process up to time t . Under Q , $dz^Q = dz^P - \frac{(r-\mu_s)}{\sigma_s} dt$ is a standard Brownian motion and it holds that

$$dS_t = rS_t dt + \sigma_s S_t dz^Q. \tag{5}$$

According to the Fundamental Theorem of Asset Pricing, an arbitrage-free price C_t of an option at time t is given by the conditional expectation of the discounted payoff under an equivalent martingale measure Q ,

$$C_t = e^{-r(T-t)} E_t^Q [\text{Max}(S_T - K, 0)]. \tag{6}$$

Some calculations yield the Black-Scholes formula as,

$$C_t = S_t N(d_1) - Ke^{-r(T-t)} N(d_2), \tag{7}$$

$$d_2 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma_s^2\right)(T-t)}{\sigma_s \sqrt{T-t}} = d_1 + \sigma_s \sqrt{T-t}, \tag{8}$$

where $N(\cdot)$ denotes the cumulative distribution function of a standard normal variable.

However, when selling or buying stocks on a financial market, there sometimes exists a difference between actual prices and the theoretical value derived from the Black-Scholes method, making predicting actual future prices challenging. The difficulty derives not only from randomness of financial stochastic systems but also un-

certainty, which cannot be represented using probability theory alone. When markets are unstable and rapidly changing, losses/errors between the model and reality frequently become bigger. Explaining these losses/errors only using probabilistic methods is difficult because of the existence of uncertainties that differ from probabilistic ones where something exists or not with probability.

3. The Model

This section follows [11] in building up a fuzzified version of the vulnerable option pricing model. This model resolves the problems of [10,11], who considered illiquid or not tradable assets, such as collateral assets, as liquid or tradable. In addition, this study also adopts the idea of [4] and the pricing method of [9] to develop a method for pricing vulnerable options by using the fuzzy measure theory to represent liquidity risk. This study focuses on the situation where the probability measure used to price contingent claims is not precisely known. This theory enables the different values of long and short positions to be explained.

A. Klein's [11] model

Let P denote the natural measure and let S_t and V_t , respectively, represent the values of the assets underlying and those of the counterparty which follow geometric Brownian motion:

$$dS = \mu_s S dt + \sigma_s S dz^P, \tag{9}$$

$$dV = \mu_v V dt + \sigma_v V db^P, \tag{10}$$

where μ_s and μ_v denote the instantaneous expected rates of return, and volatility σ_s and σ_v represent the instantaneous standard deviations of return on the assets underlying the option and those of the counterparty, respectively; furthermore ρ denotes the correlation between S_t and V_t .

Vulnerable options are options on which the writer may default, and fuzzy vulnerable options are those subject to credit risk under a fuzzy environment. The main contribution of this study is focusing on fuzzy Black-Scholes options with correlated credit risk. To our knowledge, the proposed model is the first valuation model of vulnerable options in a fuzzy environment which is able to account for liquidity risk and credit risk for the option in an incomplete market.

Under the assumption of identical rationality, vulnerable call and put prices are as follows:

$$vC_t = e^{-r(T-t)} E_t^Q [A_T],$$

$$A_t = \text{Max}(S_t - K, 0) \left\{ \left[1 | V_t \geq D^* \right] + \left[\frac{(1-\alpha)V_t}{D} | V_t < D^* \right] \right\}, \tag{11}$$

and

$$vp_t = e^{-r(T-t)} E_t^Q [B_T],$$

$$B_T = \text{Max}(K - S_T, 0) \left\{ \left[1 | V_T \geq D^* \right] + \left[\frac{(1-\alpha)V_T}{D} | V_T < D^* \right] \right\}, \quad (12)$$

where S_t denotes the value of the assets underlying the option; V_T represents the value of the assets of the counterparty; D is the amount of claims; D^* may be less than D due to the possibility of a counterparty continuing operations even while V_T is less than D ; α is the deadweight costs associated with bankruptcy expressed as a percentage of the value of the assets of the counterparty; r denotes the risk-free interest rate; $T-t$ represents the time to maturity; and Q is the classical risk-neutral probability measure.

Theorem 1: (Klein [11])

The pricing formula of vulnerable call options is as follows:

$$vc_t = S_t N_2(a_1, b_1, \rho) - e^{-r(T-t)} K N_2(a_2, b_2, \rho) + \frac{(1-\alpha)V_t}{D} \left\{ S_t e^{(r+\rho\sigma_s\sigma_v)(T-t)} N_2(c_1, d_1, -\rho) - K N_2(c_2, d_2, -\rho) \right\}, \quad (13)$$

where $N_2(\cdot)$ denotes the cumulative bivariate normal distribution function, and the parameters are as follows:

$$a_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r + \frac{1}{2}\sigma_s^2)(T-t)}{\sigma_s \sqrt{T-t}} = a_2 + \sigma_s \sqrt{T-t},$$

$$b_1 = \frac{\ln\left(\frac{V_t}{D}\right) + (r - \frac{1}{2}\sigma_v^2 + \rho\sigma_s\sigma_v)(T-t)}{\sigma_v \sqrt{T-t}} = b_2 + \rho\sigma_s \sqrt{T-t},$$

$$c_1 = a_2 + (\sigma_s + \rho\sigma_v) \sqrt{T-t},$$

$$d_1 = -\frac{\ln\left(\frac{V_t}{D}\right) + (r + \frac{1}{2}\sigma_v^2 + \rho\sigma_s\sigma_v)(T-t)}{\sigma_v \sqrt{T-t}},$$

$$c_2 = a_2 + \rho\sigma_v \sqrt{T-t}, \quad d_2 = -\left[b_2 + \sigma_v \sqrt{T-t} \right].$$

The pricing formula of vulnerable put options can be similarly derived. In Klein’s model, it is assumed that V is a tradable security, although it is not likely to be traded directly. This study uses fuzzy measure theory to incorporate the liquidity risk into V .

B. Fuzzy Vulnerable Option

In the economy dominated by nonidentical rationalities examined in this study, despite sharing the same information, investors make forecasts and evaluate identical assets independently, often making very different assessments from one another. Investors understand the whole market from their own perspective and then build their own unique ones, known as “individual markets”. In these, either no arbitrage opportunities exist and equilibrium is maintained, or assets are purchased at low prices and sold at high ones. In the individual market

equilibrium state, each investor prices all the assets in the marketplace, including underlying assets and their derivatives.

Theorem 2: Vulnerable options with each investor’s own subjective measure λ

Under the assumption of nonidentical rationality, the vulnerable call options, vc , is

$$vc_t^\lambda = e^{-r(T-t)} CE_t^{Q^\lambda} [A_T] \begin{cases} = E_t^Q [A_T], & \lambda = 0 \\ \neq E_t^Q [A_T], & \lambda \neq 0 \end{cases}, \quad (14)$$

where $E_t^Q[\cdot]$ and $CE_t^{Q^\lambda}[\cdot]$ denote the expectations of a risk-neutral investor under the martingale measure and Choquet expectation based on the subjective measures of that investor, respectively. $\lambda=0$ means that investors are risk neutral, and they can obtain the same results as in [11] in the second subsection with Choquet integration when $\lambda=0$.

This approach allows for other liabilities and a proportional recovery of nominal claims in default, but has the advantage of explicitly relating the payout ratio to the value of the assets of the counterparty. This approach also allows correlation between the assets of the counterparty and the asset underlying the option. These assumptions are appropriate in many business situations and should improve the quality of the pricing of vulnerable options.

The relationship of the λ -additive fuzzy measure is

$$\mu_\lambda(Q) = \begin{cases} \lambda \left[(1+\lambda)^Q - 1 \right], & \lambda \neq 0 \\ Q, & \lambda = 0 \end{cases}, \quad \lambda' = -\frac{\lambda}{\lambda+1}. \quad (15)$$

Theorem 3: λ -interval bounds of a fuzzy vulnerable call option

According to Definition 3, it is convenient to induce membership of a fuzzy subset, which is the fuzzy vulnerable option. In the case of a nonadditive measure, particularly a sub-additive λ , the lower bound of the expected value is computed using the lower Choquet integral with respect to this measure, and the upper bound is obtained by computing the corresponding upper Choquet integral. Consequently, the λ -interval bounds of the fuzzy vulnerable call option price are as follows:

$$\left[vc_t^\lambda, vc_t^{\lambda'} \right] = \left[e^{-r(T-t)} CE_t^{Q^\lambda} (A_T), e^{-r(T-t)} CE_t^{Q^{\lambda'}} (A_T) \right] \quad (16)$$

where vc_t^λ is the vulnerable call option with measure λ .

When a market participant returns from their individual market to the real market, they can compare their own price vc^λ with the market price realized as vc , which is the reference point. If $vc^\lambda = vc$, the participant will take no action, if $vc^\lambda > vc$, they will go long on asset, and if $vc^\lambda < vc$, they will short the asset. vc^λ is the reserve price of the participant. The final market

price ex post can stay between the two reserve prices for the long and short positions. Thus the final market price ex post is expected collectively as an interval. In markets with sufficient investors, λ may vary continuously, making the final market price ex post naturally form a fuzzy set.

$\lambda \neq 0$ means investors are not risk-neutral. On the other hand, λ represents investor risk attitude, and the heterogeneous expectations essentially derive from this. This study uses Choquet integration to obtain the fuzzy vulnerable options formula in integration form, and then derives their approximate analytic solutions. Although the approximation can speed up the calculation, such solutions are only effective when λ is approximately 0. If necessary, investors can still obtain the prices of fuzzy vulnerable calls and puts using the integrated formula rather than the approximate solutions when λ is not approximately 0.

$$\begin{aligned}
 vc_t^{\lambda \neq 0} &= e^{-r(T-t)} \int_K \frac{1}{\lambda} [(1+\lambda)^{N_2(b'_1, b_2, \rho)} - 1] dx \\
 &+ \frac{(1-\alpha)V_t}{D} \int_K \frac{1}{\lambda} [(1+\lambda)^{N_2(d'_1, d_2, \rho)} - 1] dx \\
 &\doteq S_t \{ \mu[N_2(a_1, b_1, \rho)](1+\lambda)^{N_2(a_2, b_2, \rho) - N_2(a_1, b_1, \rho)} \\
 &+ e^{(r+\sigma_s \sigma_v)(T-t)} \frac{(1-\alpha)V_t}{D} \mu[N_2(c_1, d_1, -\rho)](1+\lambda)^{N_2(c_2, d_2, -\rho) - N_2(c_1, d_1, -\rho)} \} \\
 &- Ke^{-r(T-t)} \{ \mu[N_2(a_1, b_1, \rho)] + \frac{(1-\alpha)V_t e^{r(T-t)}}{D} \mu[N_2(c_2, d_2, -\rho)] \}. \quad (17)
 \end{aligned}$$

This formula is able to describe investor risk attitudes and subjective beliefs, making the fuzzy measure superior to other option pricing approaches. The difference between the chosen measure and other methods is that the chosen parameter, λ , can also be mapped to one membership function degree and the bounds of option prices to which the investor can refer when making investment decisions. The selection of λ is not arbitrary, but rather should be selected according to investor personality and prevailing market atmosphere.

For the λ -interval bounds of fuzzy vulnerable options, we provide another viewpoint: Overconfident investors believe more strongly in their own assessments of an asset's value than those of other investors. In this way, overconfidence leads to heterogeneous beliefs or differences in opinions. Finally, based on the integrated form of the fuzzy vulnerable options formula, this study applies partial differentiation to it with respect to several parameters, such as stock price, volatility, risk-free rate, and time to maturity, to obtain the Greeks of those variables.

4. Numerical Examples and Applications

A. Numerical Examples

The effects of λ on the fuzzy Black-Scholes call price and fuzzy vulnerable call price are shown in Tables 1. Table 1 lists the numerical cases of fuzzy vulnerable calls. The curves of Figure 1 to 2 show the relationship between option prices and λ in the tables.

Table 1. Value of fuzzy vulnerable call options.

Parameter Settings	Fuzzy vulnerable call options						
	$\alpha = 0$			$\alpha = 1$			
	$\lambda = 0$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 0$	$\lambda = 0.5$	$\lambda = 1$	
Base Case	Lower bound	4.46	3.74	3.28	3.45	2.91	2.56
	Upper bound	4.46	5.25	5.84	3.45	4.04	4.49
S	40	0.77	0.63 0.92	0.54 1.05	0.67	0.55 0.81	0.48 0.91
	60	11.49	9.83 13.27	8.73 14.60	8.06	6.95 9.25	6.21 10.13
	5	3.31	2.84 3.81	2.53 4.18	0.18	0.15 0.22	0.13 0.26
V	15	4.57	3.93 5.27	3.50 5.78	4.57	3.93 5.27	3.50 5.78
	40	10.86	9.32 12.50	8.31 12.72	7.48	6.46 8.59	5.79 9.34
	60	1.35	1.11 1.62	0.96 1.83	1.16	0.95 1.39	0.82 1.56
σ_s	0.1	2.09	1.77 2.45	1.56 2.72	1.57	1.34 1.83	1.18 2.02
	0.4	6.83	5.71 8.06	4.99 9.00	5.38	4.52 6.33	3.96 7.04
	0.1	4.54	3.82 5.32	3.36 5.91	3.74	3.17 4.37	2.80 4.83
σ_v	0.4	4.37	3.66 5.15	3.20 5.91	3.30	2.77 3.87	2.43 4.30
	0.48	3.71	3.11 4.36	2.72 4.86	2.86	2.41 3.35	2.12 3.71
	0.88	5.11	4.29 6.02	3.76 6.71	3.98	3.35 4.66	2.95 5.17
ρ	-0.5	3.93	3.28 4.65	2.86 5.19	1.09	0.90 1.30	0.78 1.47
	0	4.24	3.53 5.02	3.07 5.62	2.27	1.89 2.68	1.65 3.00

Parameters: $S = 50$, $V = 8$, $D = D^* = 8$, $K = 50$, $\sigma_s = \sigma_v = 0.25$, $r = 0.028$, $T - t = 0.68$, $\rho = 0.5$.

The numerical results indicate that when λ approaches 0, the upper and lower bounds of the λ -interval both closely approximate the classical Black-Scholes prices in all cases. Consequently, the λ -interval converges to the Black-Scholes price when λ tends to zero. λ can be considered an indication of risk attitude, and zero- λ represents neutral risk attitude type. When the λ is zero, the option prices are the same as [2] and the vulnerable option prices are the same as in [11].

In Table 1, for the base case, $\lambda=1$ represents that subjective investor expectations of the market indicates significant variation in investor expectations of the market. In this situation the price bounds calculated using the proposed model are (3.28, 5.84). On the contrary, the model of Klein [11], which is based on the assumption that all market participants have identical rationality is employed, derives an option value of 4.46, and the dif-

ference between this value and the price bound derived from the proposed model is about (26.39%, 31.07%). This means that using the option value calculated from [11] as the trading criterion would produce a price error of approximately 26%. Consequently, it is practical to propose the fuzzy vulnerable option pricing formula to derive the upper and lower bounds of the option value since in reality none of the market participants are identical.

The credit risk causes the call prices of the base cases of fuzzy vulnerable call to be all lower than the classical fuzzy Black-Scholes call prices listed in Table 1. This result is instinctive, since risk-averse buyers are only willing to pay less for the calls because of the possibility of the writer defaulting at maturity.

Financial and fuzzy theory holds that the upper bound of the λ -interval is the buyer's (long) position and the lower bound is the writer's (short) position. The fuzzy model permits traders to take three different positions, namely buyer, writer, and broker. Additionally, there are three traders, namely hedgers, speculators, brokers. Hedgers generally take long positions for hedging efficiency, meaning they usually occupy the upper bound of the λ -interval. Speculators may take either short or long positions for trading profit, and thus occupy both the upper and lower bounds of the λ -interval. Brokers take both short and long positions and their net positions tend to zero. Brokers receive the liquidity risk premium, which is the difference between the upper and lower bounds of the λ -interval.

In model setting, hedgers and brokers are generally risk-averse, while speculators are risk-seeking, although all may be risk-neutral. Interestingly, the risk preference level which traders tend to be more risk-averse or risk-seeking increases with λ . For example, hedgers taking long positions are risk-averse, because they pay a risk premium to hedge risk. To simplify, we can only see the positive λ , which can represent a λ -interval together with its dual parameter λ^* .

Figure 1(a) shows this situation, with the upper bound increases with λ , meaning hedgers are willing to pay a higher risk premium. Additionally, speculators holding short positions are risk-loving, because they sell their calls lower than the Black-Scholes price to assume risk. The lower bound reduces when λ increases, which means speculators' risk-seeking level increases λ . However, speculators holding long positions are still risk-seeking, because they buy their calls higher than the Black-Scholes price to assume risk. The λ -interval can be viewed as the ask-bid spread or credit spread, and can be a measure of liquidity risk. The λ -interval increases with broker risk-aversion, consistent with the financial concept that the bid-ask spread increases with liquidity risk. In the model, liquidity risk is subjective and set by

the broker, and thus individual option prices or option price intervals vary according to rationality.

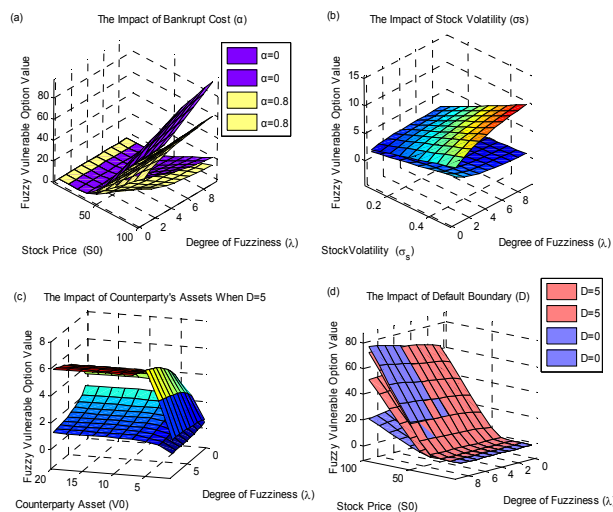


Figure 1. The relationship between λ , stock price, credit risk for fuzzy vulnerable call.

Figure 1(b) shows that the volatility of the basis asset S increases with the option value. A more volatile basis asset of an option is associated with a higher possibility of obtaining a higher option value. For example, as the volatility of the basis asset S increases from 0.25 to 0.4, the value of the vulnerable option increases from (3.74, 5.25) to (5.71, 8.06). Figure 1(c) shows that when the value of the assets of the counterparty is sufficiently large, even if the counterparty defaults, the option holders can still recover their losses from the assets of the counterparty and the default risk does not change the value of the option. Figure 1(d) shows options subject to credit risk are more risky than risk-free options. That is, the bid-ask spread of the fuzzy vulnerable call exceeds that of the fuzzy Black-Scholes call; and the effect of the default risk (D) reduces the values of the upper and lower bounds of the fuzzy vulnerable call option. This means that as there is the chance of default, the investors become more conservative. When default boundary approaches zero ($D \rightarrow 0$), fuzzy vulnerable call degenerates to fuzzy Black-Scholes call and Figure 1(d) also shows that the upper bound of the fuzzy vulnerable call price is higher than that of the fuzzy call price when λ is sufficiently large. At first glance, this seems unreasonable, but in fact it occurs because the credit risk increases the required liquidity risk premium and thus enlarges the λ -interval.

Figure 2 shows the hedge ratios of fuzzy vulnerable call, and reveals similar patterns to Figure 1. The base cases of the fuzzy vulnerable call hedge ratios are less than those for the fuzzy call, and the upper surfaces of the fuzzy vulnerable call hedge ratios exceed those of the fuzzy call. This phenomenon results from the credit risk

property of the vulnerable call.

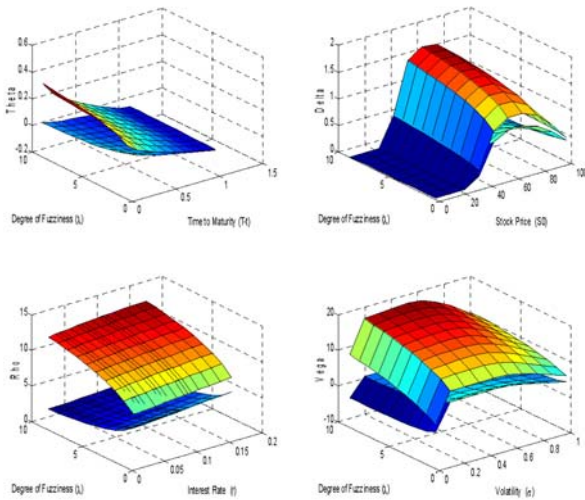


Figure 2. The hedge ratios of fuzzy vulnerable call.

B. Pricing Taiwan’s Warrants with Credit Risk

We can apply the fuzzy vulnerable option model proposed in this article to price the Taiwan’s warrants with credit risk and we show how we can use this approach to yield bid-ask quotes in a derivative market, in our case: A market for warrants.

First, we briefly introduce the representative company- Macronix, which distributed its shares through the initial public offering (IPO) in Taiwan equity market in March 1995 (Ticker: 2337). Macronix is the largest and the most advanced supplier of ROM products worldwide. As one of the top world-class suppliers of nonvolatile memory semiconductors, Macronix currently produces a wide range of ROM and NOR flash products and solutions with various densities in embedded, consumer, and enterprise applications. The main reason for us to choose Macronix as the sample firm is it had suffered from the financial crises in 2008 but survived with a significant capital reduction.

Second, in this dataset, we may immediately compute the upper and lower bounds, i.e. bid and ask quotes, of the options $[vc_i^{\lambda}, vc_i^{\lambda^*}]$, where the lower and upper Choquet integrals involved are evaluated precisely as described in (16) above. In Table 2 we provide the results obtained by ‘fuzzifying’ the vulnerable Black-Scholes model and practice the fuzzy measure with using the data taken from quotes of the Macronix stock warrants reported by Taiwan Economic Journal (TEJ) Database. The data are referred to a snapshot taken at the closed prices of the market on 1 March 2010. We collected the implied volatilities data reported on the TEJ Database for a set of American-style warrants with different strike prices. It is important to note that for an American-style warrant on nondividend payout, the

American-style warrant price is the same as the European style. Data on the risk-free rate and the time to expiration were also collected from the TEJ Database. Also the fractional recovery rate of firm value at default is assumed to be 85% and prescribed by the empirical investigation on the bankruptcy costs. In the table we present the Black-Scholes prices, along with upper and lower bounds which are recovered using the Choquet integrals. We also report the difference between upper and lower bounds, representing the bid-ask spread predicted. The value of λ was calibrated to 4, in order to obtain plausible values of the bid-ask spread, i.e. around 2 NT dollars.

Finally, in Table 2 the estimated the bid-ask spreads predicted by the model decrease with the higher levels of the strike price; meanwhile, the implied volatilities display a tendency to increase, as found in the option pricing theory in most cases. When K increases from 17.60 to 20, the price wedge decreases and shifts leftward towards zero (from (0.69, 2.30) to (0.25, 1.02)). When implied volatility (σ_{im}) increases from 0.3071 to 0.3541, the price wedge grows (from (0.25, 1.02) to (0.52, 2.03)). Thus, the implied volatility of underlying stock prices increases option price wedge and uncertainty. Despite the market prices of warrants are included by the bounds of the fuzzy vulnerable Black-Scholes, the model still overestimates the market prices of warrants by about 30-40% due to the ignorance of credit risk.

Table 2. Fuzzy vulnerable Black-Scholes model.

Warrant Code	Market Price	B-S Model	Fuzzy Bounds	Bid-Ask Spreads	Strike	σ_{im}	Expiry Date
03113	0.27	0.55	0.25	0.76	20.00	30.71	07/26
			1.02				
03139	1.04	1.39	0.69	1.61	17.60	30.71	07/27
			2.30				
03640	0.98	0.56	0.25	0.76	19.50	30.71	09/02
			1.02				
06983	0.55	1.14	0.52	1.50	19.50	35.41	06/21
			2.03				

Source: TEJ database. Date: 1 March 2010.

The value of the underlying asset was equal to 17.50, the risk-free rate was 2.5% on a yearly basis, and time to expiration was 15 days. The value λ of used is 4.

This example clearly demonstrates how the fuzzy measure approach leads to a straightforward extension of the existing asset pricing models in a fairly easy way. The important argument worthy to point out is that by resorting to the fuzzy measure we introduce the second dimension to the asset valuation problem, which has to deal with the limits of the set of the probability measures. This second dimension results in the determination of the bounds for the prices of assets, as opposed to the precise ‘crisp’ values, and these values may be considered as bid ask spreads.

5. Conclusion

When pricing or hedging derivatives in financial markets, especially OTC markets, important and common issues are that the real market is not “perfect” and “complete” as assumed in classical economic theory or the ideal trading assumptions of [2]. Default risk and liquidity risk always exist in OTC markets. Numerous researchers have discussed credit or liquidity risk, but few have considered both risks. This study discusses both these risks when pricing options in the OTC market, and uses fuzzy logic to construct an option pricing model to evaluate the price of an option when default risk and liquidity risk are taken into consideration. Besides the proposed evaluation model, important concepts, such as “nonidentical rationality” and “over-confidence” of market participants, which differ from the assumptions of classical economic theory, are also proposed to explain liquidity risk when evaluating financial derivatives in over-the-counter markets. The concept of “nonidentical rationality” assumes that risk preferences vary among market investors. This study also applied a λ -additive fuzzy measure and Choquet integration to include this fuzziness in the option pricing model. This investigation introduced liquidity risk by representing the upper and lower bounds of the price of the contingent claim, calculated as the upper and lower Choquet integral with respect to a subadditive function.

Finally, we apply the fuzzy vulnerable option model proposed in this article to price the Taiwan’s warrants with credit risk and we show how we can use this approach to yield bid-ask quotes in a derivative market, in our case: A market for warrants.

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