Robust H_∞ Fuzzy Output Feedback Control for Uncertain Discrete-time Nonlinear Systems

Li-Kui Wang and Xiao-Dong Liu

Abstract

In this paper, a generalized non-quadratic Lyapunov function and non-parallel distributed compensation (non-PDC) law are proposed for robust H_∞ fuzzy control of discrete-time uncertain nonlinear systems. The focus is on designing a robust output feedback observer such that the observer error system is robustly asymptotically stable and has a guaranteed H_∞ performance. A new basis-dependent idea is introduced to solve the fuzzy observer-based control problem, which is different from the quadratic framework that entails fixed matrices for the entire membership function, or the linearly basis-dependent framework that uses linear convex combinations of s matrices. This idea is realized by carefully selecting the structure of the matrices involved in the products with system matrices. Linear matrix inequality (LMI) conditions are obtained for the existence of observer and based on these the results are casted into a convex optimization problem, which can be readily solved via standard numerical software. The effectiveness and the superiority of the proposed approach are demonstrated by two examples borrowed from the literature.

Keywords: Takagi-Sugeno's fuzzy model, Non-quadratic lyapunov function, H_∞ performance, Linear matrix inequality, State observer.

1. Introduction

As an alternative method to conventional control approach for complex control systems, fuzzy logic control has received much attention in the past decades. It has been shown that fuzzy logic control is one of the most useful techniques for utilizing the qualitative knowledge of a system to design controllers. Among various model-based fuzzy control approaches, the method based on the Takagi-Sugeno (T-S) fuzzy model has become popular today for its applications on the stability and performance of many practical nonlinear systems (see [1-4] and the references therein). Up to now, many important issues have been studied for T-S fuzzy control systems, such as, stability analysis [9-14], H_∞ performance [15], time-delay [21] and switched systems [22]. One of the approaches to deal with these issues is the Lyapunov theory and the quadratic Lyapunov function is the main technique [5-9]. To overcome the conservatism arisen from the use of a single Lyapunov matrix, more effective Lyapunov methods have been presented. See, for example, piecewise quadratic Lyapunov functions [10, 11, 14] the weighting-dependent Lyapunov functions [12, 13], Polya's theorem [26] etc. It is noted that all of the aforementioned research have been focused on PDC law, on the other hand, using the non-PDC law design methods along with non-quadratic Lyapunov functions, some conditions were proposed in [16]. Recently, based on an extended non-quadratic Lyapunov function with more variables, some conditions, which were less conservative than those in [16], were provided in [17, 18].

Since T-S fuzzy models are used to describe complex nonlinear systems, there may be uncertainties resulting from the modeling procedure. When parameter uncertainties appear, the problems of robust stability analysis and robust stabilization for fuzzy systems have been studied in [19, 20]. In addition, considering the approximate errors the observer is necessary [6, 20]. By applying the non-PDC law design methods, the robust H_2 fuzzy observer-based control problem was studied in [23]. Different from previous two-stage procedure in [24], the method proposed in [23] guarantees that the observer gain matrices and the control gain matrices can be obtained directly by solving a set of LMIs.

In this paper, we study the robust H_∞ fuzzy observer-based control problem for discrete-time uncertain T-S fuzzy systems. Differently from previous results one of the contributions of this paper is that we develop a generalized non-quadratic Lyapunov function

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which contains some exiting results as special case, for example, as $q=0$, it is the quadratic Lyapunov function, as $q=1$, it is the non-quadratic Lyapunov function of [16] and for $q=2$, it is the Lyapunov function of [17]. In addition, a more generalized non-PDC law and new partitioned slack variables are designed to reduce the conservatism.

In this paper, A^T denotes the transpose of A. A star $(*)$ in a symmetric matrix denotes the transposed element in the symmetric position. The symbol I_n stands for the identity matrix in $R^{n\times n}$. $l_2[\mathbf{0},\infty)$ refers to the space of square summable infinite vector sequences. $\|\bullet\|_2$ stands for the l_2 norm.

2. Problem Statement and Preliminaries

The discrete-time T-S fuzzy system under investigation is described as follows:

$$x(k+1)=\sum_{i=1}^{r}h_i(\theta(k))\{(A_i+\Delta A_i(k))x(k) + (B_i+\Delta B_i(k))u(k)+N_i\omega(k)\} \tag{1}$$

$$z(k)=\sum_{i=1}^{r}h_i(\theta(k))\{C_{1i}x(k)+D_iu(k)+M_i\omega(k)\} \tag{2}$$

$$y(k)=\sum_{i=1}^{r}h_i(\theta(k))C_{2i}x(k) \tag{3}$$

$x(k)$ is the state, $u(k)\in R^m$ is the control input, $z(k)\in R^p$ is the controlled output, $y(k)\in R^p$ is the output, $\omega(k)\in R^s$ is an exogenous disturbance input. Matrices $A_i\in R^{n\times n}$, $B_i\in R^{n\times m}$, $N_i\in R^{n\times s}$, $C_{1i}\in R^{p\times n}$, $C_{2i}\in R^{q\times n}$, $D_i\in R^{p\times m}$, $M_i\in R^{p\times s}$ are system matrices. $i=1,2,\cdots,r$ and r is the number of IF-THEN rules. $\Delta A_i(k)$ and $\Delta B_i(k)$ represent the time-varying uncertainties which have the following structure:

$$[\Delta A_i(k)\ \ \Delta B_i(k)]=HF(k)[E_{1i}\ \ E_{2i}],\ i=1,2,\cdots r \tag{4}$$

where $F(k)\in R^{\alpha\times\beta}$ is an unknown matrix function with Lebesgue measurable elements and satisfying

$$F^T(k)F(k)\le I_\beta \tag{5}$$

$H\in R^{n\times\alpha}$, $E_{1i}\in R^{\beta\times n}$ and $E_{2i}\in R^{\beta\times m}$ are known constant matrices with appropriate dimensions that specify how the uncertain parameters in $F(k)$ enter the nominal matrices A_i and B_i. $h_i(\theta(k))$ is the basis functions which satisfies

$$h_i(\theta(k))\ge 0,\sum_{i=1}^{r}h_i(\theta(k))=1$$

The following fuzzy observer is employed to deal with the state estimation of the system (1)-(3):

$$\hat{x}(k+1)=\sum_{i=1}^{r}h_i(\theta(k))\{(A_i+B_iu(k))x(k)\} + L(k)\{y(k)-\hat{y}(k)\},\ \hat{x}(0)=0 \tag{6}$$

$$\hat{y}(k)=\sum_{i=1}^{r}h_i(\theta(k))C_{2i}x(k) \tag{7}$$

$L(k)$ is to be designed and the estimation error is denoted as $e(k)=x(k)-\hat{x}(k)$.

The aim of this paper is to construct the control law, $u(k)$, $L(k)$ such that the closed-loop system globally asymptotically stable for any nonzero $\omega(k)\in l_2[0,\infty)$ and all admissible uncertainties $\Delta A_i(k)$ and $\Delta B_i(k)$. In addition, the H_∞ performance γ ($\|z(k)\|_2<\gamma\|\omega(k)\|$) is guaranteed as small as possible.

For simplicity, the basis functions $h_i(\theta(k))$ in (1) are represented as h_i and $h_i(\theta(k+1))=h_{+i}$.

Let

$$H^{\{q,r\}}=\{h_{i_1}h_{i_2}\cdots h_{i_q}\,|\,i_1\le i_2\cdots i_q,\ i_1,i_2\cdots i_q\in\{1,2,\cdots,r\}\} \tag{8}$$

$$H_+^{\{q,r\}}=\{h_{+i_1}h_{+i_2}\cdots h_{+i_q}\,|\,i_1\le i_2\cdots i_q,\ i_1,i_2\cdots i_q\in\{1,2,\cdots,r\}\} \tag{9}$$

and the corresponding subscript is

$$\wp^{\{q,r\}}=\{i_1i_2\cdots i_q\,|\,i_1\le i_2\cdots i_q,i_1,i_2,\cdots i_q\in\{1,2,\cdots,r\}\}$$

There are $\dfrac{(q+r-1)!}{q!(r-1)!}$ elements in $H\{q,r\}$. For example, as *r=3, q=3*

$$H^{\{3,3\}}=\{h_1^3,h_1^2h_2,h_1^2h_3,h_1h_2^2,h_1h_2h_3,h_1h_3^2,\ h_2^3,h_2^2h_3,h_2h_3^2,h_3^3\}$$

$$\wp^{\{3,3\}}=\{111,112,113,122,123,133,222,223,233,333\}$$

Definition 1: The HPB-MF is defined as follows

$$f\left(H^{\{q,r\}},S_{i_1i_2\cdots i_q}\right)=\sum_{h_{i_1}h_{i_2}\cdots h_{i_q}\in H^{\{q,r\}}}h_{i_1}h_{i_2}\cdots h_{i_q}S_{i_1i_2\cdots i_q} \tag{10}$$

$$f\left(H_+^{\{q,r\}},S_{i_1i_2\cdots i_q}\right)=\sum_{h_{+i_1}h_{+i_2}\cdots h_{+i_q}\in H^{\{q,r\}}}h_{+i_1}h_{+i_2}\cdots h_{+i_q}S_{i_1i_2\cdots i_q} \tag{11}$$

$S_{i_1i_2\cdots i_q}$ are matrix variables and q is the degree of HPB-MF.

For example as *r=3, q=3* one has

$$f\left(H^{\{3,3\}},S_{i_1i_2i_3}\right)=h_1^3S_{111}+h_1^2h_2S_{112}+h_1^2h_3S_{113} + h_1h_2^2S_{122}+h_1h_2h_3S_{123}+h_2^3S_{222} + h_2^2h_3S_{223}+h_3^3S_{333}$$

and for the special cases $q=1$, one gets

$$f\left(H^{\{1,r\}},S_{i_1}\right)=S(H)=h_1S_1+h_2S_2+\cdots+h_rS_r \tag{12}$$

Definition 2: For each $h_{i_1} h_{i_2} \cdots h_{i_q} \in H^{\{q,r\}}$

$$h_{i_1} h_{i_2} \cdots h_{i_q} = h_1^{d_1} h_2^{d_2} \cdots h_r^{d_r}, d_1 + d_2 + \cdots d_r = q \tag{13}$$

The function $g(i_1 i_2 \cdots i_q)$ dependent on the value of $i_1, i_2, \cdots i_q$ is defined as follows

$$g(i_1 i_2 \cdots i_q) = \frac{q!}{d_1! d_2! \cdots d_r} \tag{14}$$

For example, as $r=3$, $q=3$, $i_1 = i_2 = 1$, $i_3 = 2$, one has $h_1 h_1 h_2 = h_1^2 h_2, d_1 = 2, d_2 = 1, d_3 = 0$, then

$$g(112) = \frac{3!}{2!1!0!} = 3$$

Definition 3: The $f\left(H^{\{q,r\}}, S_{i_1 i_2 \cdots i_q}\right)$ is said to be positive definite (semi-definite) in $H^{\{q,r\}}$ if there exist some matrices $S_{i_1 i_2 \cdots i_q}$ such that $f\left(H^{\{q,r\}}, S_{i_1 i_2 \cdots i_q}\right) > 0$ ($f\left(H^{\{q,r\}}, S_{i_1 i_2 \cdots i_q}\right) \ge 0$) for all $h_{i_1} h_{i_2} \cdots h_{i_q} \in H^{\{q,r\}}$.

The following well known lemma is useful in this paper.

Lemma1 [25]: Let A, H, E and F be real matrices of appropriate dimensions with $F^T F \le I$. For any matrix $P > 0$ and scalar $\varepsilon > 0$ such that $P - \varepsilon H H^T > 0$, then we have

$$(A + HEF)^T P^{-1} (A + HEF) \le A^T \left(P - \varepsilon H H^T\right) A + \varepsilon^{-1} E^T E$$

3. Main Results

Theorem 1: Consider the system (1)-(3). For a given scalar $\gamma > 0$, if there exist matrices, $Y_{i_1 i_2 \cdots i_{q-1}} \in R^{m \times n}$, $G^1_{i_1 i_2 \cdots i_{q-1}} \in R^{n \times n}$, $G^2 \in R^{n \times n}$, $Q^2_{i_1 i_2 \cdots i_{q-1}} \in R^{n \times n}$, $W_{i_2 \cdots i_q} \in R^{n \times q}$, $\varepsilon > 0$ and positive matrices $P^1_{i_1 i_2 \cdots i_q} \in R^{n \times n}$, $Q^3_{i_1 i_2 \cdots i_{q-1}} \in R^{n \times n}$, satisfying $\Psi^1_{j_1 j_2 \cdots j_q} < 0, \Psi^2_{j_1 j_2 \cdots j_q} < 0$, where

$$\Psi^1_{j_1 j_2 \cdots j_q} = \begin{bmatrix} G^1_{i_1 i_2 \cdots i_{q-1}} + \left(G^1_{i_1 i_2 \cdots i_{q-1}}\right)^T - P^1_{i_1 i_2 \cdots i_q} & * & * & * & * & * & * & * & * \\ J_1 & J_2 & * & * & * & * & * & * & * \\ 0 & 0 & J_3 & * & * & * & * & * & * \\ A_{i_q} G^1_{i_1 i_2 \cdots i_{q-1}} + B_{i_q} Y_{i_1 i_2 \cdots i_{q-1}} & \mathrm{A} & \mathrm{N} & J_4 & * & * & * & * & * \\ 0 & J_{10} & \mathrm{N} & \dfrac{Q^2_{j_1 j_2 \cdots j_q}}{g(j_1 j_2 \cdots j_q)} & J_5 & * & * & * & * \\ 0 & 0 & 0 & \varepsilon \mathrm{H}^T & J_{11} & J_6 & * & * & * \\ E_{1i_q} G^1_{i_1 i_2 \cdots i_{q-1}} + E_{2i_q} Y_{i_1 i_2 \cdots i_{q-1}} & \mathrm{E}_1 & 0 & 0 & 0 & 0 & J_7 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_8 & * \\ C_{1i_q} G^1_{i_1 i_2 \cdots i_{q-1}} + D_{i_q} Y_{i_1 i_2 \cdots i_{q-1}} & \Gamma_1 & \mathrm{M} & 0 & 0 & 0 & 0 & 0 & J_9 \end{bmatrix}$$

$$i_1 i_2 \cdots i_q \in \wp^{\{q,r\}}, i_1 = i_2 = \cdots = i_q, j_1 j_2 \cdots j_q \in \wp^{\{q,r\}}$$

$$\Psi^2_{j_1 j_2 \cdots j_q} = \begin{bmatrix} \left\{ \begin{matrix} G^1_{i_1 i_2 \cdots i_{q-1}} + \left(G^1_{i_1 i_2 \cdots i_{q-1}}\right)^T \\ + G^1_{i_2 i_3 \cdots i_q} + \left(G^1_{i_2 i_3 \cdots i_q}\right)^T - P^1_{i_1 i_2 \cdots i_q} \end{matrix} \right\} & * & * & * & * & * & * & * & * \\ J_1 & J_2 & * & * & * & * & * & * & * \\ 0 & 0 & J_3 & * & * & * & * & * & * \\ \left\{ \begin{matrix} A_{i_q} G^1_{i_1 i_2 \cdots i_{q-1}} + B_{i_q} Y_{i_1 i_2 \cdots i_{q-1}} \\ + A_{i_1} G^1_{i_2 i_3 \cdots i_q} + B_{i_1} Y_{i_2 i_3 \cdots i_q} \end{matrix} \right\} & \mathrm{A} & \mathrm{N} & J_4 & * & * & * & * & * \\ 0 & J_{10} & \mathrm{N} & \dfrac{g(i_1 i_2 \cdots i_q) Q^2_{j_1 j_2 \cdots j_q}}{g(j_1 j_2 \cdots j_q)} & J_5 & * & * & * & * \\ 0 & 0 & 0 & g(i_1 i_2 \cdots i_q) \varepsilon \mathrm{H}^T & J_{11} & J_6 & * & * & * \\ \left\{ \begin{matrix} E_{1i_q} G^1_{i_1 i_2 \cdots i_{q-1}} + E_{2i_q} Y_{i_1 i_2 \cdots i_{q-1}} \\ + E_{1i_1} G^1_{i_2 i_3 \cdots i_q} + E_{2i_q} Y_{i_2 i_3 \cdots i_q} \end{matrix} \right\} & \mathrm{E}_1 & 0 & 0 & 0 & 0 & J_7 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_8 & * \\ \left\{ \begin{matrix} C_{1i_q} G^1_{i_1 i_2 \cdots i_{q-1}} + D_{i_q} Y_{i_1 i_2 \cdots i_{q-1}} \\ C_{1i_1} G^1_{i_2 i_3 \cdots i_q} + D_{i_q} Y_{i_2 i_3 \cdots i_q} \end{matrix} \right\} & \Gamma_1 & \mathrm{M} & 0 & 0 & 0 & 0 & 0 & J_9 \end{bmatrix}$$

$$i_1 i_2 \cdots i_q \in \wp^{\{q,r\}} \setminus \{i_1 = i_2 = \cdots = i_q\}, j_1 j_2 \cdots j_q \in \wp^{\{q,r\}}$$

$$J_1 = g(i_1 i_2 \cdots i_q) I_n - Q^2_{i_1 i_2 \cdots i_q}, J_{10} = \left(G^2\right)^T \mathrm{A} - W_{i_1 i_2 \cdots i_{q-1}} C_{2i_q}$$

$$J_3 = g(i_1 i_2 \cdots i_q) \gamma^2 I_s, J_4 = \frac{g(i_1 i_2 \cdots i_q)}{g(j_1 j_2 \cdots j_q)} P^1_{j_1 j_2 \cdots j_q}, J_9 = g(i_1 i_2 \cdots i_q) I_p$$

$$J_6 = g(i_1 i_2 \cdots i_q) \varepsilon I_{\alpha+s}, J_7 = g(i_1 i_2 \cdots i_q) \varepsilon I_\beta, J_8 = g(i_1 i_2 \cdots i_q) \varepsilon I_s$$

$$J_5 = \frac{g(i_1 i_2 \cdots i_q)}{g(j_1 j_2 \cdots j_q)} Q^3_{j_1 j_2 \cdots j_q}, J_2 = g(i_1 i_2 \cdots i_q)\left(G^2 + \left(G^2\right)^T\right) - Q^2_{i_1 i_2 \cdots i_q}$$

$$J_{11} = g(i_1 i_2 \cdots i_q) \varepsilon \mathrm{H}^T G^2, J_{12} = \left(G^2\right)^T \mathrm{A} - W_{i_1 i_2 \cdots i_{q-1}} C_{2i_q} - W_{i_2 i_3 \cdots i_q} C_{2i_1}$$

$$\mathrm{A} = \frac{(q-1)!}{d_1 d_2 \cdots d_r}\left(A_{i_1} + \cdots A_{i_q}\right), \mathrm{N} = \frac{(q-1)!}{d_1 d_2 \cdots d_r}\left(N_{i_1} + \cdots N_{i_q}\right)$$

$$\Gamma_1 = \frac{(q-1)!}{d_1 d_2 \cdots d_r}\left(C_{1i_1} + \cdots C_{1i_q}\right), \mathrm{M} = \frac{(q-1)!}{d_1 d_2 \cdots d_r}\left(M_{i_1} + \cdots M_{i_q}\right)$$

$$\mathrm{E}_1 = \frac{(q-1)!}{d_1 d_2 \cdots d_r}\left(E_{1i_1} + E_{1i_2} + \cdots E_{1i_q}\right), \mathrm{H} = \begin{bmatrix} H & 0 \end{bmatrix}$$

then there exists a fuzzy observer-based controller of the following form

$$u(k) = f\left(H^{\{q-1,r\}}, Y_{i_1 i_2 \cdots i_{q-1}}\right) f^{-1}\left(H^{\{q-1,r\}}, G_{i_1 i_2 \cdots i_{q-1}}\right) \hat{x}(k) \tag{15}$$

with the observer gains

$$L(k) = f\left(H^{\{q-1,r\}}, L_{i_1 i_2 \cdots i_{q-1}}\right) \tag{16}$$

where

$$L_{i_1 i_2 \cdots i_{q-1}} = \left(G^2\right)^{-T} W_{i_1 i_2 \cdots i_{q-1}}$$

such that the closed-loop fuzzy system (1)-(3) is globally asymptotically stable with γ disturbance attenuation.

Proof: Applying the generalized control law (15) and $L(k)$ (16), it follows from (12) that the closed-loop system can be described by

$$\xi(k+1)=\left(A_c+H_cF(k)E_c\right)\xi(k)+N_c\omega(k) \tag{17}$$

$$z(k)=C_c\xi(k)+M(H)\omega(k) \tag{18}$$

$$y(k)=C_2(H)x(k) \tag{19}$$

where

$$A_c=\begin{bmatrix} A(H)+B(H)U & -B(H)U \\ 0 & A(H)-f\left(H^{\{q-1,r\}},L_{i_1i_2\cdots i_{q-1}}\right)C_2(H)\end{bmatrix}$$

$$N_c=\begin{bmatrix} N(H) \\ N(H)\end{bmatrix},\ C_c=\begin{bmatrix} C_1(H)+D(H)U & -D(H)U\end{bmatrix}$$

$$H_c=\begin{bmatrix} H \\ H\end{bmatrix},\ E_c=\begin{bmatrix} E_1(H)+E_2(H)U & -E_2(H)U\end{bmatrix}$$

$$\xi(k)=\begin{bmatrix} x(k) \\ e(k)\end{bmatrix}, U=f\left(H^{\{q-1,r\}},Y_{i_1i_2\cdots i_{q-1}}\right)f^{-1}\left(H^{\{q-1,r\}},G^1_{i_1i_2\cdots i_{q-1}}\right)$$

Choosing the parameter-dependent Lyapunov candidate function as

$$V(\xi(k))=\xi(k)^T f^{-1}\left(H^{\{q,r\}},\mathrm{P}_{i_1i_2\cdots i_q}\right)\xi(k)$$

one has

$$\Delta V(\xi(k))=V(\xi(k+1))-V(\xi(k))=\hat{\xi}(k)^T\Omega_1\hat{\xi}(k)$$

where

$$\Omega_1=\left(\tilde{A}_c+\tilde{H}_c\tilde{F}(k)\tilde{E}_c\right)^T f^{-1}\left(H_+^{\{q,r\}},\mathrm{P}_{j_1j_2\cdots j_q}\right)\left(\tilde{A}_c+\tilde{H}_c\tilde{F}(k)\tilde{E}_c\right)-\Xi_1$$

$$\Xi_1=\begin{bmatrix} f^{-1}\left(H^{\{q,r\}},\mathrm{P}_{i_1i_2\cdots i_q}\right) & 0 \\ 0 & 0\end{bmatrix}, \hat{\xi}(k)=\begin{bmatrix}\xi(k)\\ \omega(k)\end{bmatrix}, \tilde{A}_c=\begin{bmatrix} A_c & N_c\end{bmatrix}$$

$$\tilde{H}_c=\begin{bmatrix} H_c & 0\end{bmatrix}, \tilde{F}(k)=\begin{bmatrix} F(k) & 0 \\ 0 & I_s\end{bmatrix}, \tilde{E}_c=\begin{bmatrix} E_c & 0 \\ 0 & 0\end{bmatrix} \tag{20}$$

Using Lemma 1 one has

$$z(k)^Tz(k)-\gamma^2\omega(k)^T\omega(k)+\Delta V(\xi(k))=\hat{\xi}(k)^T\Omega_2\hat{\xi}(k)$$

where

$$\Omega_2=\tilde{A}_c^T\left(f\left(H_+^{\{q,r\}},\mathrm{P}_{j_1j_2\cdots j_q}\right)-\varepsilon\tilde{H}_c\tilde{H}_c^T\right)\tilde{A}_c+\varepsilon^{-1}\tilde{E}_c^T\tilde{E}_c-\Xi_2 \tag{21}$$

$$\Xi_2=\begin{bmatrix} f^{-1}\left(H^{\{q,r\}},\mathrm{P}_{i_1i_2\cdots i_q}\right)-C_c^TC_c & -C_c^TM(H) \\ M(H)^TC_c & \gamma^2I_s-M(H)^TM(H)\end{bmatrix}$$

$$f\left(H_+^{\{q,r\}},\mathrm{P}_{j_1j_2\cdots j_q}\right)-\varepsilon\tilde{H}_c\tilde{H}_c^T>0,\varepsilon>0 \tag{22}$$

applying Schur's complement along with Lemma 1 and using the fact that $G^TP^{-1}G\geq G^T+G-P$ which holds for all square matrices G and P of compatible dimensions, one has $\Omega_2<0$ if the following inequality holds

$$\begin{bmatrix} \mathrm{X} & 0 & * & * & * & * & * \\ 0 & \gamma^2I_s & * & * & * & * & * \\ A_cf\left(H^{\{q-1,r\}},\mathrm{I}_{i_1i_2\cdots i_{q-1}}\right) & N_c & f\left(H_+^{\{q,r\}},\mathrm{P}_{j_1j_2\cdots j_q}\right) & * & * & * & * \\ 0 & 0 & \varepsilon\tilde{H}_c^T & \varepsilon I_{\alpha+s} & * & * & * \\ E_cf\left(H^{\{q-1,r\}},\mathrm{I}_{i_1i_2\cdots i_{q-1}}\right) & 0 & 0 & 0 & \varepsilon I_\beta & * & * \\ 0 & 0 & 0 & 0 & 0 & \varepsilon I_s & * \\ C_cf\left(H^{\{q-1,r\}},\mathrm{I}_{i_1i_2\cdots i_{q-1}}\right) & M(H) & 0 & 0 & 0 & 0 & I_p \end{bmatrix}<0 \tag{23}$$

where

$$\mathrm{X}=f^T\left(H^{\{q-1,r\}},\mathrm{I}_{i_1i_2\cdots i_{q-1}}\right)+f\left(H^{\{q-1,r\}},\mathrm{I}_{i_1i_2\cdots i_{q-1}}\right)-f\left(H^{\{q,r\}},\mathrm{P}_{i_1i_2\cdots i_q}\right)$$

We can partition $f\left(H^{\{q-1,r\}},\mathrm{I}_{i_1i_2\cdots i_{q-1}}\right), f\left(H^{\{q,r\}},\mathrm{P}_{i_1i_2\cdots i_q}\right)$ as follows

$$f\left(H^{\{q,r\}},\mathrm{P}_{i_1i_2\cdots i_q}\right)=\begin{bmatrix} f\left(H^{\{q,r\}},P^1_{i_1i_2\cdots i_q}\right) & \\ f\left(H^{\{q,r\}},P^2_{i_1i_2\cdots i_q}\right) & f\left(H^{\{q,r\}},P^3_{i_1i_2\cdots i_q}\right)\end{bmatrix} \tag{24}$$

$$f\left(H^{\{q-1,r\}},\mathrm{I}_{i_1i_2\cdots i_{q-1}}\right)=\begin{bmatrix} f\left(H^{\{q-1,r\}},G^1_{i_1i_2\cdots i_{q-1}}\right) & \left(G^2\right)^{-1} \\ 0 & \left(G^2\right)^{-1}\end{bmatrix} \tag{25}$$

Pre- and postmultiplying (23) by

$$diag\left(I_n,\left(G^2\right)^T,I_s,I_n,\left(G^2\right)^T,I_{\alpha+s},I_\beta,I_s,I_p\right)$$

and its transpose respectively and let

$$Q^2_{i_1i_2\cdots i_q}=\left(G^2\right)^TP^2_{i_1i_2\cdots i_q},W_{i_1i_2\cdots i_{q-1}}=\left(G^2\right)^TL_{i_1i_2\cdots i_{q-1}}$$

$$Q^3_{i_1i_2\cdots i_q}=\left(G^2\right)^TP^3_{i_1i_2\cdots i_q}G^2$$

one has $\Psi<0$, where

$$\Psi=\begin{bmatrix} \mathrm{X}_1 & * & * & * & * & * & * & * & * \\ \mathrm{X}_2 & \mathrm{X}_3 & * & * & * & * & * & * & * \\ 0 & 0 & \gamma^2I_s & * & * & * & * & * & * \\ A_c^1 & A_c^2 & N(H) & f\left(H_+^{\{q,r\}},P^1_{j_1j_2\cdots j_q}\right) & * & * & * & * & * \\ 0 & A_c^3 & N(H) & f\left(H_+^{\{q,r\}},Q^2_{j_1j_2\cdots j_q}\right) & f\left(H_+^{\{q,r\}},Q^3_{j_1j_2\cdots j_q}\right) & * & * & * & * \\ 0 & 0 & 0 & \varepsilon\mathrm{H}^T & \varepsilon\mathrm{H}^TG^2 & \varepsilon I_{\alpha+s} & * & * & * \\ E_c^1 & E_c^2 & 0 & 0 & 0 & 0 & \varepsilon I_\beta & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon I_s & * \\ C_c^1 & C_c^2 & M(H) & 0 & 0 & 0 & 0 & 0 & I_p \end{bmatrix} \tag{26}$$

$$\mathrm{X}_1=f\left(H^{\{q-1,r\}},G^1_{i_1i_2\cdots i_{q-1}}\right)+f\left(H^{\{q-1,r\}},G^1_{i_1i_2\cdots i_{q-1}}\right)^T-f\left(H^{\{q,r\}},P_{i_1i_2\cdots i_q}\right)$$

$$\mathrm{X}_2=I_n-f\left(H^{\{q,r\}},Q^2_{i_1i_2\cdots i_q}\right), \mathrm{X}_3=G^2+\left(G^2\right)^T-f\left(H^{\{q,r\}},Q^3_{i_1i_2\cdots i_q}\right)$$

$$A_c^1=A(H)f\left(H^{\{q-1,r\}},G^1_{i_1i_2\cdots i_{q-1}}\right)+B(H)f\left(H^{\{q-1,r\}},Y_{i_1i_2\cdots i_{q-1}}\right)$$

$$A_c^2=A(H), A_c^3=\left(G^2\right)^TA(H)-f\left(H^{\{q-1,r\}},W_{i_1i_2\cdots i_{q-1}}\right)C_2(H)$$

$$E_c^1=E_1(H)f\left(H^{\{q-1,r\}},G^1_{i_1i_2\cdots i_{q-1}}\right)+E_2(H)f\left(H^{\{q-1,r\}},Y_{i_1i_2\cdots i_{q-1}}\right)$$

$$C_c^1=C_1(H)f\left(H^{\{q-1,r\}},G^1_{i_1i_2\cdots i_{q-1}}\right)+D(H)f\left(H^{\{q-1,r\}},Y_{i_1i_2\cdots i_{q-1}}\right)$$

$$E_c^2=E_1(H), C_c^2=C_1(H)$$

Note

$$\Psi = \sum_{j_1 j_2 \cdots j_q \in \wp^{\{q,r\}}} h_{+j_1} h_{+j_2} \cdots h_{+j_q} g\left(j_1 j_2 \cdots j_q\right) \Psi_{j_1 j_2 \cdots j_q}$$

where

$$\Psi_{j_1 j_2 \cdots j_q} = \begin{bmatrix} X_1 & * & * & * & * & * & * & * & * \\ X_2 & X_3 & * & * & * & * & * & * & * \\ 0 & 0 & \gamma^2 I_s & * & * & * & * & * & * \\ A_c^1 & A_c^2 & N(H) & \dfrac{P^1_{j_1 j_2 \cdots j_q}}{g\left(j_1 j_2 \cdots j_q\right)} & * & * & * & * & * \\ 0 & A_c^3 & N(H) & \dfrac{Q^2_{j_1 j_2 \cdots j_q}}{g\left(j_1 j_2 \cdots j_q\right)} & \dfrac{Q^3_{j_1 j_2 \cdots j_q}}{g\left(j_1 j_2 \cdots j_q\right)} & * & * & * & * \\ 0 & 0 & 0 & \varepsilon \mathrm{H}^T & \varepsilon \mathrm{H}^T G^2 & \varepsilon I_{\alpha+s} & * & * & * \\ E_c^1 & E_c^2 & 0 & 0 & 0 & 0 & \varepsilon I_\beta & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon I_s & * \\ C_c^1 & C_c^2 & M(H) & 0 & 0 & 0 & 0 & 0 & I_p \end{bmatrix} \quad (27)$$

since $g\left(j_1 j_2 \cdots j_q\right) > 0$, one has $\Psi < 0$ if $\Psi_{j_1 j_2 \cdots j_q} < 0$. Exploiting the possible combinations in $H^{\{q,r\}}$ one gets

$$\Psi_{j_1 j_2 \cdots j_q} = \sum_{\substack{i_1 i_2 \cdots i_q \in \wp^{\{q,r\}} \\ i_1 = i_2 = \cdots = i_q}} h_{i_1} h_{i_2} \cdots h_{i_q} \Psi^1_{j_1 j_2 \cdots j_q} + \sum_{\substack{i_1 i_2 \cdots i_q \in \wp^{\{q,r\}} \\ \setminus \{i_1 = i_2 = \cdots = i_q\}}} h_{i_1} h_{i_2} \cdots h_{i_q} \Psi^2_{j_1 j_2 \cdots j_q}$$

with

$$\begin{aligned} \mathrm{A} &= g\left(i_2 i_3 \cdots i_q\right) A_{i_1} + g\left(i_1 i_3 \cdots i_q\right) A_{i_2} + \cdots g\left(i_1 i_2 \cdots i_{q-1}\right) A_{i_q} \\ &= \frac{(q-1)!}{d_1! d_2! \cdots d_r}\left(A_{i_1} + A_{i_2} + \cdots A_{i_q}\right) \end{aligned}$$

and N, M, E_1, Γ_1 are similar to A. Note if $i_1 = i_2 = \cdots = i_q$, one gets $g\left(i_1 i_2 \cdots i_q\right) = 1$ and $\mathrm{A} = A_{i_1}$, $\mathrm{N} = N_{i_1}$, $\mathrm{M} = M_{i_1}$, $\mathrm{E}_1 = E_{1 i_1}$, $\Gamma_1 = C_{1 i_1}$. For example, if $r = 3, q = 3, i_1 = i_2 = i_3 = 1$, we have

$$\begin{aligned} \mathrm{A} &= g\left(i_2 i_3 \cdots i_q\right) A_{i_1} + g\left(i_1 i_3 \cdots i_q\right) A_{i_2} + \cdots g\left(i_1 i_2 \cdots i_{q-1}\right) A_{i_q} \\ &= \frac{2!}{3!0!0!}\left(A_1 + A_1 + A_1\right) = A_1 \end{aligned}$$

It is easy to verify that if the LMIs in Theorem 1 hold, then $\Omega_2 < 0$.

On the other hand, let

$$J_N = \sum_{k=0}^{N-1}\left(z(k)^T z(k) - \gamma^2 \omega(k)^T \omega(k)\right) \quad (28)$$

For any nonzero $\omega(k) \in l_2[0\ \infty)$ and zero initial condition $x(0) = 0$, one gets

$$J_N = \sum_{k=0}^{N-1}\left(z(k)^T z(k) - \gamma^2 \omega(k)^T \omega(k) + \Delta V(x(k))\right) - V(x(N)) \quad (29)$$

and hence, $\Omega < 0$ implies $J_N < 0$ which guarantees that the closed-loop system (1), (2) is asymptotically stable with γ disturbance attenuation for all admissible uncertainties.

The following Theorem 2 shows that lower disturbance attenuation can be obtained by increasing q.

Theorem 2: For some given value of q, the corresponding theorem obtained from Theorem 1 is expressed as LMIs and denoted as $T_\infty^{(q-1,q-1,q)}$. For any integer $\hat{q}$ satisfying $\hat{q} > q$, $T_\infty^{(\hat{q}-1,\hat{q}-1,\hat{q})}$ is feasible if $T_\infty^{(q-1,q-1,q)}$ is feasible.

Proof : Suppose the LMIs in $T_\infty^{(q-1,q-1,q)}$ are feasible, let

$$f\left(H^{\{q,r\}}, \mathrm{I}_{i_1 i_2 \cdots i_q}\right) = \left(\sum_{i=1}^r h_i\right) f\left(H^{\{q-1,r\}}, \mathrm{I}_{i_1 i_2 \cdots i_{q-1}}\right) \quad (30)$$

$$f\left(H^{\{q,r\}}, Y_{i_1 i_2 \cdots i_q}\right) = \left(\sum_{i=1}^r h_i\right) f\left(H^{\{q-1,r\}}, Y_{i_1 i_2 \cdots i_{q-1}}\right) \quad (31)$$

$$f\left(H^{\{q+1,r\}}, \mathrm{P}_{i_1 i_2 \cdots i_{q+1}}\right) = \left(\sum_{i=1}^r h_i\right) f\left(H^{\{q,r\}}, \mathrm{P}_{i_1 i_2 \cdots i_q}\right) \quad (32)$$

$$f\left(H^{\{q,r\}}, L_{i_1 i_2 \cdots i_q}\right) = \left(\sum_{i=1}^r h_i\right) f\left(H^{\{q-1,r\}}, L_{i_1 i_2 \cdots i_{q-1}}\right) \quad (33)$$

the LMIs in $T_\infty^{(q,q,q+1)}$ can be obtained by linear combination of those in $T_\infty^{(q-1,q-1,q)}$. That is, for any solutions satisfying $T_\infty^{(q-1,q-1,q)}$ will be bound to satisfy $T_\infty^{(q,q,q+1)}$. With recursion, For any integer $\hat{q}$ satisfying $\hat{q} > q$, $T_\infty^{(\hat{q}-1,\hat{q}-1,\hat{q})}$ is also feasible.

Remark 1: The major differences between the proposed theorems $T_\infty^{(q-1,q-1,q)}$ and the theorem of [23] are as follows. Wu [23] develops a method to deal with the robust H_2 output feedback control problem, while in this paper we extend the method to solve the robust H_∞ output feedback control problem. The slack variables introduced in [23] are linear dependent on the basis function, but in this paper they can be quadratic, cubic or even higher degree on the basis function. Specifically, only the (1, 2), (2, 1) and (2, 2) blocks of the matrix variables I are required to be fixed, while the Lyapunov function matrices and the (1, 1) blocks of the slack matrix variables are relaxed to be polynomially dependent on the basis function. As the degree increases more variables are generated which enable us have the potential to obtain less conservative results. In this paper, we develop a more general Lyapunov function which contains some exiting results as special case, for example, as $q=0$, it is the quadratic Lyapunov function, as $q=1$, it is the non-quadratic Lyapunov function of [16] and for $q=2$, it is the Lyapunov function of [17]. Note, the following non-quadratic Lyapunov function

$$V(x)=x(k)^T\left(\sum_{i=1}^{r}h_i\mathrm{I}_i\right)^{-T}\left(\sum_{i=1}^{r}h_i\mathrm{P}_i\right)\left(\sum_{i=1}^{r}h_i\mathrm{I}_i\right)^{-1}x(k)$$

and non-PDC law

$$u(k)=\left(\sum_{i=1}^{r}h_iY_i\right)\left(\sum_{i=1}^{r}h_iG_i^1\right)^{-1}\hat{x}(k)$$

are utilized in [23] to cope with robust H_2 control problem. The extension of the method in [23] to deal with the robust H_∞ output control problem is straightforward. Using the Lyapunov function and control law given in [23] and following the same line in Theorem 1 one gets the following Theorem 3

Theorem 3: Consider the fuzzy system (1)-(3). For a given scalar $\varepsilon>0$, if there exist matrices $Y_i\in R^{m\times n}$, $G_i^1\in R^{n\times n}$, $G^2\in R^{n\times n}$, $Q_i^2\in R^{n\times n}$, $W_i\in R^{n\times q}$ and positive matrices $P_i^1\in R^{n\times n}$, $Q_i^3\in R^{n\times n}$ satisfying the following LMIs

$$\Phi_{lii}>0, l,i\in\{1,2,\cdots r\} \tag{34}$$

$$\frac{1}{r-1}\Phi_{lii}+\frac{1}{2}\left(\Phi_{lij}+\Phi_{lji}\right)>0, l,i,j\in\{1,2,\cdots r\}, i\neq j \tag{35}$$

$$\Phi_{lij}=\begin{bmatrix}
P_i^1 & * & * & * & * & * & * & * & *\\
Q_i^2 & Q_i^3 & * & * & * & * & * & * & *\\
0 & 0 & \gamma^2 I_s & * & * & * & * & * & *\\
A_iG_j^1+B_iY_j & A_i & N_i & J_1 & * & * & * & * & *\\
0 & \left(G^2\right)^T A_i-W_iC_{2j} & N_i & I_n-Q_l^2 & G^2+\left(G^2\right)^T-Q_l^3 & * & * & * & *\\
0 & 0 & 0 & \varepsilon \mathrm{H}^T & \varepsilon \mathrm{H}^T G^2 & \varepsilon I_{\alpha+s} & * & * & *\\
E_{1i}G_j^1+E_{2i}Y_j & E_{1i} & 0 & 0 & 0 & 0 & \varepsilon I_\beta & * & *\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon I_s & *\\
C_{1i}G_j^1+D_iY_j & C_{1i} & M_i & 0 & 0 & 0 & 0 & 0 & I_p
\end{bmatrix}>0$$

$J_1=G_l^1+\left(G_l^1\right)^T-P_l^1$, then there exists a fuzzy observer-based controller of the following form

$$u(k)=f\left(H^{\{1,r\}},Y_{i_1}\right)f^{-1}\left(H^{\{1,r\}},G_{i_1}\right)\hat{x}(k)$$

with the observer gain matrices $L_i=\left(G^2\right)^{-T}W_i$ such that the closed-loop fuzzy system (1)-(3) is globally asymptotically stable with γ disturbance.

4. Simulation example

In this section, we compare our results with other method using example borrowed from the literatures. All the experiments have been performed in a Celeron (R) 2.8 GHz, 512 MB RAM, using LMI solver and m-file of MATLAB 7.0.

Example 1: In this example, the system under consideration is a nonlinear system modified from example 1 in [15]

$$\begin{aligned}x_1(k+1)=&\,x_1(k)+\Delta a_1(k)x_1(k)-x_1(k)x_2(k)+0.01\omega_2(k)\\&-0.03\omega_1(k)+\left(5+x_1(k)\right)u(k)\end{aligned} \tag{36}$$

$$x_2(k+1)=-x_1(k)-0.5x_2(k)+2x_1(k)u(k)+0.01\omega_2(k) \tag{37}$$

$$\begin{aligned}z(k)=&-0.1x_1(k)-0.5x_2(k)+0.5u(k)+0.01\omega_1(k)\\&+0.01\omega_2(k)\end{aligned} \tag{38}$$

$$y(k)=0.1x_1(k) \tag{39}$$

where $\Delta a_1(k)$ is the uncertain parameters satisfying $\Delta a_1(k)\in[-0.1, 0.1]$.

Under the assumption that, the nonlinear system is exactly represented by the T-S fuzzy system given by

Rule 1: If $x_1(k)$ is about h_1 then

$$x(k+1)=(A_1+\Delta A_1(k))x(k)+(B_1+\Delta B_1(k))u(k)+N_1\omega(k)$$
$$z(k)=C_{11}x(k)+D_1u(k)+M_1\omega(k)$$
$$y(k)=C_{21}x(k)$$

Rule 2: If $x_1(k)$ is about h_2 then

$$x(k+1)=(A_2+\Delta A_2(k))x(k)+(B_2+\Delta B_2(k))u(k)+N_2\omega(k)$$
$$z(k+1)=C_{12}x(k)+D_2u(k)+M_2\omega(k)$$
$$y(k)=C_{22}x(k)$$

$$h_1=\frac{x_1(k)+\beta}{2\beta},\ h_2=1-h_1,\ x_1(k)\in[-\beta\ \beta]$$

$$A_1=\begin{bmatrix}1 & -\beta\\ -1 & -0.5\end{bmatrix}, A_2=\begin{bmatrix}1 & \beta\\ -1 & -0.5\end{bmatrix}, D_1=D_2=0.5,$$

$$B_1=\begin{bmatrix}5+\beta\\ 2\beta\end{bmatrix}, B_2=\begin{bmatrix}5-\beta\\ -2\beta\end{bmatrix}, M_1=M_2=\begin{bmatrix}0.01\\ 0.01\end{bmatrix}^T,$$

$$N_1=N_2=\begin{bmatrix}-0.03 & 0.01\\ 0.00 & 0.01\end{bmatrix}, C_{11}=C_{12}=\begin{bmatrix}-0.1 & -0.05\end{bmatrix}$$

$$C_{21}=C_{22}=\begin{bmatrix}0.1 & 0\end{bmatrix}$$

and $\Delta A_1(k), \Delta A_2(k), \Delta B_1(k), \Delta B_2(k)$ can be represented in the form of (4) with

$$H=\begin{bmatrix}1\\ 0\end{bmatrix}, E_{11}=E_{12}=\begin{bmatrix}0.1 & 0\end{bmatrix}, E_{21}=E_{22}=0$$

Since γ is related to the level of disturbance attenuation, the aim here is to compute the minimum value of γ for some given β, ε and any admissible uncertainties $\Delta a_1(k)\in[-0.1, 0.1]$. The minimum value of γ is obtained by means of the following convex optimization problem:

$$\gamma_{\min}=\min\gamma$$

$$\text{s.t. } T_\infty^{(q-1,q-1,q)} \text{ with different value of } q$$

Note the LMI solver utilized here is MINCX(LMIS, C, OPTIONS) where the OPTIONS is set to [1e-5 100 0 0 0].

As $\varepsilon = 0.01$, applying $T_\infty^{(q-1,q-1,q)}$ with different value of q, the results are shown in Table 1. From Table 1 we can see the method in [19] is less conservative than $T_\infty^{(0,0,1)}$ but more conservative than $T_\infty^{(q-1,q-1,q)}$ with $q > 2$. The method in [23] is less conservative than $T_\infty^{(0,0,1)}$, $T_\infty^{(1,1,2)}$, $T_\infty^{(2,2,3)}$ but as q increases $T_\infty^{(q-1,q-1,q)}$ guarantees a larger feasible area and achieves a smaller $\gamma_{\min}$ than the method in [23]. For example, the method in [23] is infeasible for $\beta \ge 1.7$ while $T_\infty^{(3,3,4)}$ is feasible for $\beta = 1.7$. In addition, applying $T_\infty^{(3,3,4)}$ with $\beta = 1.6986$ one gets $\gamma_{\min} = 0.3916$, which is smaller than the one obtained by the method in [23] ($\gamma_{\min} = 3.3592$). Note, as $\beta = 0.8550$, applying $T_\infty^{(2,2,3)}$, $T_\infty^{(3,3,4)}$ one gets the same result $\gamma_{\min} = 0.0349$. This shows the bound of the H_∞ performance under this condition is tight and need not continue the search for lower performance $\gamma_{\min}$ by increasing q.

Table 1. The value of $\gamma_{\min}$ obtained by different methods.

β	$\gamma_{\min}$ ([23])	$\gamma_{\min}$ ([19])	$\gamma_{\min}$ ($T_\infty^{(0,0,1)}$)	$\gamma_{\min}$ ($T_\infty^{(1,1,2)}$)	$\gamma_{\min}$ ($T_\infty^{(2,2,3)}$)	$\gamma_{\min}$ ($T_\infty^{(3,3,4)}$)
0.8550	0.0349	0.0350	11.5254	0.0350	0.0349	0.0349
1.4945	0.0417	2.8319	no	0.4841	0.0427	0.0415
1.5000	0.0422	no	no	1.0715	0.0433	0.0419
1.6670	0.1513	no	no	no	2.9435	0.1372
1.6986	3.3592	no	no	no	no	0.3916
1.7000	no	no	no	no	no	0.4431

Applying $T_\infty^{(2,2,3)}$ with $\beta = 1.67$, $\varepsilon = 0.001$ one gets $\gamma_{\min} = 1.5677$ and the following results

$$G_{11}^1 = \begin{bmatrix} 0.0112 & 0.0443 \\ -0.0089 & 0.0688 \end{bmatrix}, G_{12}^1 = \begin{bmatrix} 0.0284 & -0.0349 \\ -0.0144 & 0.1102 \end{bmatrix}$$

$$G_{12}^2 = \begin{bmatrix} 0.0034 & 0.0053 \\ -0.0041 & 0.0363 \end{bmatrix}, Y_{11} = \begin{bmatrix} -0.0044 & 0.0088 \end{bmatrix}$$

$$Y_{12} = \begin{bmatrix} -0.0055 & 0.0004 \end{bmatrix}, Y_{22} = \begin{bmatrix} 0.0012 & -0.0175 \end{bmatrix}$$

$$L_{11} = \begin{bmatrix} 10.0298 & -10.0001 \end{bmatrix}^T, L_{12} = \begin{bmatrix} 20.0949 & -20.0000 \end{bmatrix}^T$$

$$L_{22} = \begin{bmatrix} 10.0561 & -10.0002 \end{bmatrix}^T$$

then, we have the following observer-based controller

$$u(k) = f\left(H^{\{q-1,r\}}, Y_{i_1 i_2 \cdots i_{q-1}}\right) f^{-1}\left(H^{\{q-1,r\}}, G^1_{i_1 i_2 \cdots i_{q-1}}\right)\hat{x}(k)$$
$$= \left(h_1^2 Y_{11} + h_1 h_2 Y_{12} + h_2^2 Y_{22}\right)\left(h_1^2 G_{11}^1 + h_1 h_2 G_{12}^1 + h_2^2 G_{22}^1\right)\hat{x}(k)$$

with the observer gains

$$L(k) = h_1^2 L_{11} + h_1 h_2 L_{12} + h_2^2 L_{22}$$

Utilizing the above results and let

$$\omega(k) = \begin{bmatrix} (\text{rand}(\bullet) - 0.3)/(1 + 0.01k) \\ (\text{rand}(\bullet) - 0.3)/(1 + 0.01k) \end{bmatrix}, \Delta a_1(k) = 0.1\sin(k)$$

the simulations are shown in Figure 1-3. Figure 1 and Figure 2 show the states response with the initial state $x(0) = \begin{bmatrix} 1.67 & 1 \end{bmatrix}, \hat{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ respectively. The solid lines show the actual states trajectories $x_1(k), x_2(k)$ and the dotted lines show those of the estimated states $\hat{x}_1(k), \hat{x}_2(k)$ of the fuzzy observer. Figure 3 presents the control action of $u(k)$. From these simulations, it can be seen the designed fuzzy observer-based controller ensures the asymptotic stability of the closed-system (36)-(39) and guarantees a prescribed H_∞ performance level under the uncertain parameter $\Delta a_1(k) = 0.1\sin(k)$.

Example 2: Consider an inverted pendulum controlled by a DC motor via a gear train shown in Figure 4, whose fuzzy modeling was done in [27]. With a sampling time T = 0.02, we have the discrete-time model as follows:

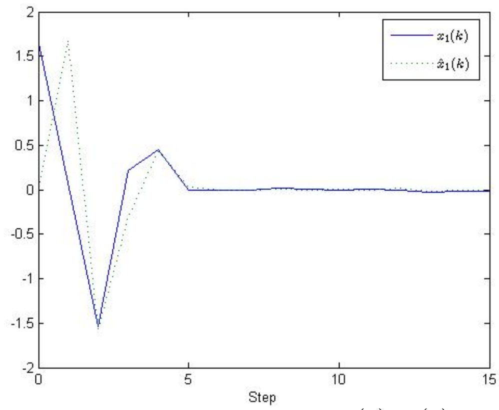

Figure 1. State responses $x_1(k), \hat{x}_1(k)$.

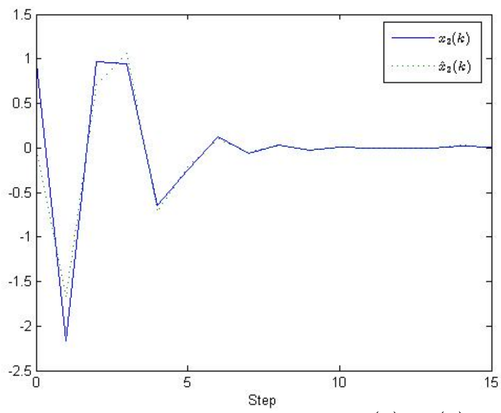

Figure 2. State responses $x_2(k), \hat{x}_2(k)$.

$$x(k+1) = \sum_{i=1}^{2} h_i(\theta(k))(A_i x(k) + u(k))$$

$$y(k) = \sum_{i=1}^{2} h_i(\theta(k)) C_{2i} x(k)$$

with

$$A_1 = \begin{bmatrix} 1.0002 & 0.02 & 0.02 \\ 0.196 & 1.0001 & 0.0181 \\ -0.0184 & -0.1813 & 0.8170 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0.02 & 0.0002 \\ 0 & 0.9981 & 0.0181 \\ 0 & -0.1811 & 0.8170 \end{bmatrix}$$

For simulation, we add some disturbance terms, uncertainties and a controlled output.

$$x(k+1) = \sum_{i=1}^{2} h_i(\theta(k))\{(A_i + \Delta A_i(k))x(k)$$
$$+ (B_i + \Delta B_i(k))u(k) + N_i\omega(k)\}$$
$$z(k) = \sum_{i=1}^{2} h_i(\theta(k))\{C_{1i}x(k) + D_i u(k) + M_i\omega(k)\}$$

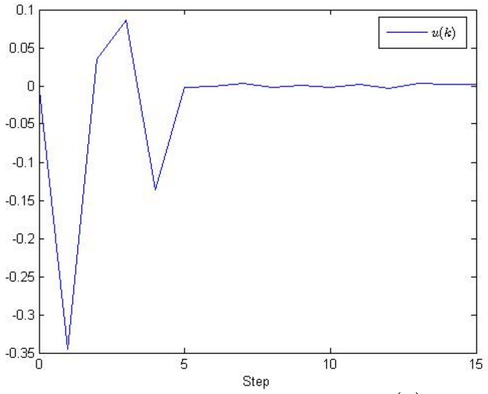

Figure 3. The control action $u(k)$.

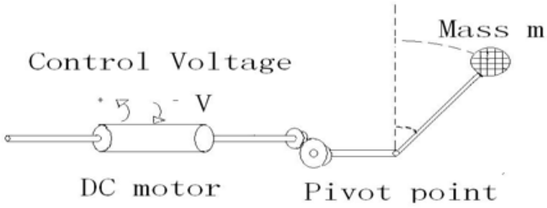

Figure 4. An inverted pendulum controlled by a DC motor.

with

$$N_1 = [-0.054 \quad -0.094 \quad 0]^T, N_2 = -N_1$$
$$C_{11} = C_{12} = [0.054 \quad 0.005 \quad 0.1]^T$$
$$D_1 = D_2 = 0.5, M_1 = M_2 = 0.1$$

and $\Delta A_1(k), \Delta A_2(k), \Delta B_1(k), \Delta B_2(k)$ can be represented in the form of (4) with

$H = [0.145 \quad 0 \quad 0], E_{11} = E_{12} = [0.1 \quad 0 \quad 0], E_{21} = E_{22} = 0$.

For this practical example, as $\varepsilon = 0.01$, the method in [19] is infeasible, while using $T_\infty^{(3,3,4)}$ one gets $\gamma_{\min} = 25.8622$ which is lower than the one obtained by the method in [23] ($\gamma_{\min} = 26.9113$). This example further shows the effectiveness of our method over others.

5. Conclusions

In this paper, we have studied the robust H_∞ fuzzy observer-based control problem for discrete-time uncertain T-S fuzzy systems. In order to get less conservative result, a structured polynomial basis function-dependent approach is proposed and some sufficient conditions for the existence of a generalized non-PDC law have been obtained. These conditions guarantee a larger stability region and lower disturbance attenuation than other method. Two examples browed from the literature show the effectiveness of the proposed approach.

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