Study of fuzzy algebraic hypersystems from a general viewpoint

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Abstract

Using the idea of quasi-coincidence of a fuzzy point with a fuzzy set, the concept of an (α,β)-fuzzy subalgebraic hypersystem in an algebraic hypersystem, which is a generalization of a fuzzy subalgebraic system, is introduced, and related properties are investigated. In particular, the study of $(\in,\in\vee q)$-fuzzy subalgebraic hypersystems of an algebraic hypersystem are dealt with. Finally, we consider the concept of implication-based fuzzy subalgebraic hypersystems.

Keywords: Algebraic hypersystem, (α,β) - fuzzy subalgebraic hypersystem, $(\in,\in\vee q)$-fuzzy subalgebraic hypersystem, fuzzy logic, implica-tion operator.

1. Introduction

After the introduction of fuzzy sets by Zadeh [18], reconsideration of the concept of classical mathematics began. On the other hand, because of the importance of group theory in mathematics, as well as its many areas of application, the notion of fuzzy subgroups was defined by Rosenfeld [14] and its structure was investigated. Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and so on.

The study of algebraic hyperstructures (or hypersystems) is a well established branch of classical algebraic theory. Hyperstructure theory was born in1934 when Marty [13] defined hypergroups, began to analyse their properties and applied them to groups, rational functions and algebraic functions. A comprehensive review of the theory of hyperstructures appears in [4, 5, 15]. The relations between fuzzy sets and algebraic hyperstructures (structures) have been already considered by Corsini, Davvaz, Leoreanu, Ameri, Zahedi and others, for instance, see [1, 5, 6-12, 16, 19].

In this paper, we introduce the concept of an (α,β) - fuzzy subalgebraic hypersystem, which is a generalization of a fuzzy subalgebraic system, in an algebraic hypersystem using the idea of quasi-coincidence of a fuzzy point with a fuzzy set, and investigate related properties. Moreover, we consider the characterization and some of the fundamental properties of such subalgebraic hypersystems. In particular, we deal with the study of $(\in,\in\vee q)$ - fuzzy subalgebraic hypersystems of an algebraic hypersystem. Finally, we consider the concept of implication based fuzzy subalgebraic hypersystems.

2. Preliminaries

Let H be a non-empty set and f a mapping $U(F;t)=\{x\in H | F(x)\geq t\}$, where $\mathcal{P}^*(H)$ is the set of all the non-empty subsets of H. Then f is called a *binary hyperoperation* on H. In general, a mapping $f: H^n \to \mathcal{P}^*(H)$ is called an *n-ary hyperoperation* and n is called *the order of hyperoperation*.

A non-empty set and one or more n-ary hyperoperations on the set will be called an *algebraic hypersystem*. We shall denote an algebraic hypersystem by $\langle H,\Gamma\rangle$, where H is a non-empty set and $\Gamma=\{f_1,f_2,\cdots\}$ is a set of hyperoperations on H.

The algebraic hypersystem $(H,\{f_i\}_{i\geq 1})$ induces a universal algebra $(\mathcal{P}^*(H),\{f_i\}_{i\geq 1})$ with the operations $F_i(A_1,...,A_n)=\cup\{f_i(a_1,...,a_n) | a_k\in A_k, 1\leq i\leq n\}$ for $A_1,...,A_n\in\mathcal{P}^*(H)$.

If Γ is a singleton $\Gamma=\{f\}$ and f is a 2-ary hyperoperation, the algebraic hypersystem is called *hypergroupoid*, the hyperoperation is denoted by $\circ$ and the image of the pair (x,y) is denoted by $x\circ y$. Hypergroups, polygroups and hyperrings are algebraic hypersystems

Let $\langle H,\Gamma\rangle$ be an algebraic hypersystem. A subset S of H is said to be closed under the n-ary hyperoperation f if $(x_1,...,x_n)\in S^n$ implies $f(x_1,...,x_n)\in S$. S is called a subalgebraic hypersystem of H, if S is closed under any hyperoperation in Γ.

Definition 2.1: Let $\langle H,\Gamma\rangle$ be an algebraic hypersystem and F a fuzzy subset of H. Then F is called a *fuzzy*

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subalgebraic hypersystem of H, if for any n-ary hyperopera

$$(HF1)\inf_{y\in f(x_1,\dots,x_n)} F(y) \ge \min\{F(x_1),F(x_2),\dots,F(x_n)\}.$$

Example 2.2:
(i) Consider $H=\{e,a,b\}$ and define $\circ$ on H with the following table 1:

Table 1.

$\circ$	e	a	b
e	$\{e\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{a,b\}$	$\{e,a\}$
b	$\{b\}$	$\{e,a\}$	$\{e\}$

Define a fuzzy set $F:H\to[0,1]$ by $F(a)\le F(b)\le F(e)$. Then F is afuzzy subalgebraic hypersystem of H.
(ii) Let $(H=\{1,-1,i,-i\},\circ)$, where $\circ$ defined on H with the following table 2:

Table 2.

$\circ$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Note that every algebraic system is an algebraic hypersystem. Define $F:H\to[0,1]$ by $F(1)=1$, $F(-1)=0.5$ and $F(i)=F(-i)=0$. Clearly F is a fuzzy subalgebraic system of H.
(iii) Let $(H=\{x,y,z,t\},\{f_1,f_2\})$ be an algebraic hypersystem shown in table 3.

Table 3.

f_1	x	y	z	t
x	$\{e\}$	$\{y\}$	$\{z\}$	$\{t\}$
y	$\{y\}$	$\{x,y\}$	$\{z\}$	$\{t\}$
z	$\{z\}$	$\{z\}$	$\{x,y,t\}$	$\{z,t\}$
t	$\{t\}$	$\{t\}$	$\{z,t\}$	$\{x,y,t\}$

f_2	x	y	z	t
x	x	y	z	t
y	x	y	z	t
z	x	y	z	t
t	x	y	z	t

We define a fuzzy set $F:H\to[0,1]$ by $F(x)=0.7$, $F(y)=0.5$ and $F(z)=F(t)=0.3$. Routine calculations give that F is a fuzzy subalgebraic hypersystem of H

Let F be a fuzzy set. For every $t\in[0,1]$, the set $U(F;t)=\{x\in H \mid F(x)\ge t\}$ is called *the level subset of* F.

Theorem 2.3 [1]: Let H be an algebraic hypersystem and F a fuzzy set of H. Then F is a fuzzy subalgebraic hypersystem of H if and only if for any $t\in[0,1]$, $U(F;t)(\ne\Phi)$ is a subalgebraic hypersystem of H.

A fuzzy set F of an algebraic hypersystem H of the form

$$F(y)=\begin{cases} t(\ne 0) & \text{if} \quad y=x \\ 0 & \text{if} \quad y\ne x \end{cases}$$

is said to be *fuzzy point with support x and value t and is denoted by* $U(x;t)$.

A fuzzy point $U(x;t)$ is said to *belong to* (resp. be *quasi-coincident with*) a fuzzy set F, written as $U(x;t)\in F$ (resp. $U(x;t)qF$) if $F(x)\ge t$ (resp. $F(x)+t>1$).

If $U(x;t)\in F$ or (resp. and) $U(x;t)qF$, then we write $U(x;t)\in\vee q$ (resp. $\in\wedge q$) F. The symbol $\overline{\in\vee q}$ means $\in\vee q$ does not hold. Using the notion of "belongingness ($\in$)" and "quasi-coincidence (q)" of fuzzy points with fuzzy subsets, the concept of (α,β)-fuzzy subsemigroup where α and β are any two of $\{\in,q,\in\vee q,\in\wedge q\}$ with $\alpha\ne\in\wedge q$ is introduced in [3]. It is noteworthy that the most viable generalization of Rosenfeld's fuzzy is the notion of $(\in,\in\vee q)$-fuzzy subgroup. The detailed study with $(\in,\in\vee q)$-fuzzy subgroup has been considered in [2].

3. (α,β)-fuzzy subalgebraic hypersystems

In what follows, let (H,Γ) be an algebraic hypersystem, $f\in\Gamma$ any n-ary hyperoperation on H, and α and β will denote any one of $\in,q,\in\vee q$ or $\in\wedge q$ unless otherwise specified. Based on [2, 3], we can extend the concept of (α,β)-fuzzy subgroups to the concept of (α,β)-fuzzy subalgebraic hypersystems in an algebraic hypersystems.

Definition 3.1: A fuzzy set F of H is called an (α,β)*-fuzzy subalgebraic hy-persystem* of H with $\alpha\ne\in\wedge q$, if it satisfies for all $t_i\in(0,1]$ and $x_i\in H\,(i=1,2,\dots,n)$, $(HF2)\,U(x_i;t_i)\alpha F$ implies $U(y;\min\{t_1,t_2,\dots,t_n\})\beta F$, for all $y\in f(x_1,x_2,\dots,x_n)$.

Let F be a fuzzy set of H such that $F\le 0.5$ for all $x\in H$. Let $x\in H$ and $t\in(0,1]$ be such that $U(x;t)\in\wedge qF$, then $F(x)\geq t$ and $F(x)+1>1$. It follows that $1<F(x)+t\leq F(x)+F(x)=2F(x)$, which implies $F(x)>0.5$. This means that $\{U(x;t)|U(x;t)\in\wedge qF\}=\Phi$. Therefore the case $\alpha=\in\wedge q$ in Definition 3.1 will be omitted.

Proposition 3.2: Every $(\in\vee q,\in\vee q)$-fuzzy subalgebraic hypersystem of H is an $(\in,\in\vee q)$-fuzzy subalgebraic hypersystem of H.

Proof. Let F be an $(\in\vee q,\in\vee q)$-fuzzy subalgebraic hypersystem of H. Let $x_i\in H$ and $t\in(0.1](i=1,2,...,n)$ be such that $U(x;t)\in F$, then $U(x;t)\in\vee qF$. Since F is an $(\in\vee q,\in\vee q)$-fuzzy subalgebraic hypersystem, it follows that $U(y;\min\{t_1,t_2,\cdots,t_n\})\in\vee qF$, for all $y\in f(x_1,x_2,\cdots,x_n)$. Consequently, F is an $(\in,\in\vee q)$-fuzzy subalgebraic hypersystem of H.

Proposition 3.3: Every $(\in,\in)$-fuzzy subalgebraic hypersystem of H is an $(\in,\in\vee q)$-fuzzy subalgebraic hypersystem of H.

Proof. Straightforward.

The converse of Propositions 3.2 and 3.3 are not true. Consider Klein's 4-group $H=\{e,x,y,z\}$. Let $F:H\rightarrow[0,1]$ be defined by $F(e)=0.6, F(x)=0.7$ and $F(z)=F(z)=0.4$ Then F is an $(\in,\in\vee q)$-fuzzy subalgebraic hypersystem of H. We note that F is not an (α,β)-fuzzy algebraic hypersystem of H where $(\alpha,\beta)\neq(\in,\in),(q,\in\vee q)$, $(\in\vee q,\in\vee q)$.

Lemma 3.4: If A is a subalgebraic hypersystem of H, then the characteristic function χ_A of A is an $(\in,\in)$-fuzzy subalgebraic hypersystem of H.

Proof: Let $x_i\in H$ and $t\in(0.1](i=1,2,...,n)$ be such that $U(x_i;t_i)\in\chi_A$, then $\chi_A(x_i)\geq t_i>0$, which implies that $\chi_A(x_i)=1$, for all $i=1,2,...,n$. Thus $x_i\in A$, for all $i=1,2,...,n$, and so, for all $y\in f(x_1,x_2,\cdots,x_n)$, we have $y\in A$. It follows that $\chi_A(y)=1\geq\min\{t_1,t_2,\cdots,t_n\}$, so that $U(y;\min\{t_1,t_2,\cdots,t_n\})\in\chi_A$. Therefore χ_A is an $(\in,\in)$-fuzzy subalgebraic hypersystem of *H*.

Theorem 3.5: For any subset A of H, χ_A is an $(\in,\in\vee q)$-fuzzy subalgebraic hypersystem of H if and only if A is a subalggebraic hypersystem of H.

Proof. Let χ_A be an $(\in,\in\vee q)$-fuzzy subalgebraic hypersystem of H. Let $x_i\in A, i=1,2,...,n,$ then $U(x_i;1)\in\chi_A$, which implies, $U(y;1)=U(y;\min\{1,1,\cdots,1\})\in\vee q\chi_A$, for all $y\in f(x_1,x_2,\cdots,x_n)$. Hence $\chi_A(y)>0$ for all $y\in f(x_1,x_2,\cdots,x_n)$, and so $y\in A$. Therefore A is a subalgebraic hypersystem of H.

Conversely, if A is a subalgebraic hypersystem of H, then χ_A is an $(\in,\in)$-fuzzy subalgebraic hypersystem of H by Lemma 3.4, and therefore χ_A is an $(\in,\in\vee q)$-fuzzy subalgebraic hypersystem of H by Proposition 3.3.

Now, we give the main result on a general (α,β)-fuzzy subalgebraic hypersystem of algebraic hypersystems.

Theorem 3.6: Let F be a non-zero (α,β)-fuzzy subalgebraic hypersystem of H. Then the set $F_0=\{x\in H|F(x)>0\}$ is a subalgebraic hypersystem of H.

Proof: Let $(x_i)\in F_0$, for all $i=1,2,...,n$, then $F(x_i)>0, i=1,2,...,n$. Assume that $F(y)=0$, for all $y\in f(x_1,x_2,\cdots,x_n)$. If $\alpha\in\{\in,\in\vee q\}$, then $U(x_i;F(x_i))\alpha F$, for all $i=1,2,...,n$, but $U(y;\min\{F(x_1),F(x_1),\cdots,F(x_1)\})\in\overline{\beta}F$ for every $\beta\in\{\in,q,\in\vee q,\in\wedge q\}$, a contradiction. Note that $U(x_i;1)qF(i=1,2,...,n)$, but for all $y\in f(x_1,x_2,\cdots,x_n)$; $U(y;\min\{1,1,\cdots,1\})=U(y;1)\overline{\beta}F$ for every $\beta\in\{\in,q,\in\vee q,\in\wedge q\}$, this is a contradiction. Hence, for all $y\in f(x_1,x_2,\cdots,x_n), F(x)>0$, that is, $y\in F_0$, and so $f(x_1,x_2,\cdots,x_n)\subseteq F_0$. This completes the proof.

A fuzzy set F of an algebraic hypersystem H is said to be *proper* if $\operatorname{Im}F$ has at least two elements. Two fuzzy sets are said to be *equivalent* if they have same family of level subsets. Otherwise, they are said to be *non-equivalent*.

Theorem 3.7: Let H have proper subalgebraic hypersystem. A proper $(\in,\in)$-fuzzy subalgebraic hypersystem F of H such that card $\operatorname{Im}F\geq 3$, can be expressed as the union of two proper non-equivalent $(\in,\in)$-fuzzy subalgebraic hypersystem of H.

Proof. Let F be a proper $(\in,\in)$-fuzzy subalgebraic hypersystem of H with $\operatorname{Im}F=\{t_0,t_1,\cdots,t_n\}$, where $t_0>t_1>\cdots>t_n$ and $n\geq 2$. Then $U(F;t_0)\subseteq U(F;t_1)\subseteq\cdots\subseteq U(F;t_n)=H$. $U(F; t0)$ is the chain of $\in$-level subalgebraic hypersystems of F.

Define fuzzy sets A and B in H by

$$A(x)=\begin{cases} r_1 & if \quad x\in U(F;t_1)\\ t_2 & if \quad x\in U(F;t_2)\backslash U(F;t_1)\\ \vdots & \\ t_n & if \quad x\in U(F;t_n)\backslash U(F;t_{n-1})\end{cases}$$

and

$$B(x)=\begin{cases} t_0 & if \quad x\in U(F;t_0) \\ t_1 & if \quad x\in U(F;t_1)\backslash U(F;t_0) \\ r_2 & if \quad x\in U(F;t_3)\backslash U(F;t_1) \\ t_4 & if \quad x\in U(F;t_4)\backslash U(F;t_3) \\ \vdots & \\ t_n & if \quad x\in U(F;t_n)\backslash U(F;t_{n-1}) \end{cases}$$

respectively, where $t_2 < r_1 < t_1$ and $t_4 < r_2 < t_2$. Then A and B are $(\in,\in)$-fuzzy subalgebraic hypersystems of H with $U(F;t_1)\subseteq U(F;t_2)\subseteq\cdots\subseteq U(F;t_n)=H$ and $U(F;t_0)\subseteq U(F;t_1)\subseteq\cdots\subseteq U(F;t_n)=H$ are respectively chains of $\in$ -level subalgebraic hypersystems, and $A,B\le F$. Thus A and B are non-equivalent, and obviously $A\bigcup B=F$. This completes the proof.

4. $(\in,\in\vee q)$-fuzzy subalgebraic hypersystems

In this section, we mainly discuss some fundamental aspects of $(\in,\in\vee q)$-fuzzy subalgebraic hypersystems of an algebraic hypersystem H.

Definition 4.1: A fuzzy set F of H is said to be an $(\in,\in\vee q)$-*fuzzy subalgebraic hypersystem* of H if for all $t_i\in(0,1]$ and $x_i\in H\,(i=1,2,...,n)$, $(HF3)\ U(x_i;t_i)\in F$ implies $U(y;\min\{t_1,t_2,...,t_n\})\in\vee qF$, for all $y\in f(x_1,x_2,\cdots,x_n)$.

Note that if F is a fuzzy subalgebraic hypersystem of H according to Definition 2.1, then F is an $(\in,\in\vee q)$-fuzzy subalgebraic hypersystem of H according to Definition 4.1. But the converse is not true shown by the following example:

Example 4.2:

(i) Let $H=\{e,a,b\}$ be an algebraic hypersystem defined by the following table 4

Table 4.

$\circ$	e	a	b
e	$\{e\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{a,b\}$	$\{e,a\}$
b	$\{b\}$	$\{e,a\}$	$\{e\}$

Define $F:H\to[0,1]$ by $F(e)=0.8, F(a)=0.7$ and $F(b)=0.6$. Then it is easy to see that F is an $(\in,\in\vee q)$-fuzzy subalgebraic hypersystem of H, but is not a fuzzy subalgebraic hypersystem of H.

(ii) Consider Example 2.2 (ii), and define $0.5\le F(i)=F(-i)\le F(-1)=F(1)$ Then F not only is an $(\in,\in\vee q)$-fuzzy subalgebraic system of H, but also is a fuzzy subalgebraic system of H.

(iii) Consider Example 2.2 (iii), then F is an $(\in,\in\vee q)$-fuzzy subalgebraic system of H.

Theorem 4.3: A fuzzy set F of H is an $(\in,\in\vee q)$-fuzzy subalgebraic hypersystem of H if and only if for all $x_i\in H\,(i=1,2,...,n)$,

$$(HF4)\ \min\{F(x_1),F(x_2),\cdots,F(x_n),0.5\}\le \inf_{y\in f(x_1,x_2,\cdots,x_n)} F(y).$$

Proof: Assume that F is an $(\in,\in\vee q)$-fuzzy subalgebraic hypersystem of H. Let $x_i\in H\,(i=1,2,...,n)$, we consider the following cases:

(i) $\min\{F(x_1),F(x_2),\cdots,F(x_n)\}<0.5$,

(ii) $\min\{F(x_1),F(x_2),\cdots,F(x_n)\}\ge 0.5$.

Case (i): Assume that there exists $y\in f(x_1,x_2,\cdots,x_n)$ such that $F(y)<\min\{F(x_1),F(x_2),\cdots,F(x_n),0.5\}$, which implies $F(y)<\min\{F(x_1),F(x_2),\cdots,F(x_n)\}$.

Choose t such that $F(y)<t<\min\{F(x_1),F(x_2),\cdots,F(x_n)\}$. Then $U(x_i;t_i)\in F$, but $U(y;t)\overline{\in\vee q}F$, which contradicts $(HF3)$.

Case (ii): Assume that $F(y)<0.5$ for some $y\in f(x_1,x_2,\cdots,x_n)$. Then $U(x_i;0.5)\in F, i=1,2,...,n$. But $U(y;0.5)\overline{\in\vee q}F$, a contradiction. Hence $(HF4)$ holds.

Conversely, let $U(x_i;t_i)\in F,(i=1,2,...,n)$, then $F(x_i)\ge t_i\,(i=1,2,...,n)$.

For any $y\in f(x_1,x_2,\cdots,x_n)$, we have

$$F(y)\ge\min\{F(x_1),F(x_2),\cdots,F(x_n),0.5\}$$
$$\ge\min\{t_1,t_2,\cdots,t_n,0.5\}.$$

If $\min\{t_1,t_2,\cdots,t_n\}>0.5$, then $F(y)\ge 0.5$, which implies,

$$F(y)+\min\{t_1,t_2,\cdots,t_n\}>1$$

If $\min\{t_1,t_2,\cdots,t_n\}\le 0.5$, then $F(y)\ge\min\{t_1,t_2,\cdots,t_n\}$. Therefore $U(y;\min\{t_1,t_2,\cdots,t_n\})\in\vee qF$ for all $y\in f(x_1,x_2,\cdots,x_n)$. This completes the proof.

Theorem 4.4: Let F be an $(\in,\in\vee q)$-fuzzy subalgebraic hypersystem of H .Then for all $0<t\le 0.5, U(F;t)$ is an empty set or a subalgebraic hypersystem of H. Conversely, if F is a fuzzy set of H such that $U(F;t)(\ne\Phi)$ is a subalgebraic hypersystem of H for all $0<t\le 0.5$, then F is an $(\in,\in\vee q)$-fuzzy subalgebraic hypersystem of H.

Proof: Let F be an $(\in,\in\vee q)$-fuzzy subalgebraic hypersystem of H and $0<t\le 0.5$. Let $x_i\in U(F;t)$, $i=1,2,...,n$, then $F(x_i)\ge t$. Now

$$\inf_{y\in f(x_1,x_2,\cdots,x_n)} F(y)\ge\min\{F(x_1),F(x_2),\cdots,F(x_n),0.5\}\ge\min\{t,0.5\}=t.$$

Therefore for every $y\in f(x_1,x_2,\cdots,x_n)$, we have $F(y)\ge t$, and so $y\in U(F;t)$, which implies,

$f(x_1, x_2, \cdots, x_n) \subseteq U(F;t)$. Therefore $U(F;t)$ is a subalgebraic hypersystem of H.

Conversely, let F be a fuzzy set of H such that $U(F;t)(\neq \Phi)$ is a subalgebraic hypersystem of H for all $0 < t \leq 0.5$. For every $x_i \in H(i=1,2,...,n)$, we can write $F(x_i) \geq \min\{F(x_1), F(x_2), \cdots, F(x_n), 0.5\} = t_0, i=1,2,...,n$, then $x_i \in U(F;t_0)$, and so $f(x_1, x_2, \cdots, x_n) \supseteq U(F;t_0)$. Therefore for every $y \in f(x_1, x_2, \cdots, x_n)$, we have $F(y) \geq t_0$, which implies

$$\inf_{y \in f(x_1, x_2, \cdots, x_n)} F(y) \geq t_0.$$

Therefore, F is an $(\in, \in \vee q)$-fuzzy subalgebraic hypersystem of H.

Naturally, a corresponding result should be considered when $U(F;t)$ is a subalgebraic hypersystem of H for all $t \in (0.5, 1]$.

Theorem 4.5: Let F be a fuzzy set of H. Then $U(F;t)(\neq \Phi)$ is a subalgebraic hypersystem of H for all $t \in (0.5, 1]$ if and only if

$$(HF5) \min\{F(x_1), F(x_2), \cdots, F(x_n)\} \leq \inf_{y \in f(x_1, x_2, \cdots, x_n)} (\{F(y), 0.5\}),$$

for all $x_i \in H(i=1,2,...,n)$.

Proof: Assume that $U(F;t)$ is a subalgebraic hypersystem of H. If there exist $x_i, y \in H(i=1,2,...,n)$ with $y \in f(x_1, x_2, \cdots, x_n)$ such that $\max\{F(y), 0.5\} < \min\{F(x_1), F(x_2), \cdots, F(x_n)\} = t$ then $t \in (0.5, 1], F(y) < t, x_i \in U(F;t), i=1,2,...,n$. Since $x_i \in U(F;t)$ and $U(F;t)$ is a subalgebraic hypersystem, so $y \in f(x_1, x_2, \cdots, x_n) \subseteq U(F;t)$ and $F(y) \geq t$ for all $y \in f(x_1, x_2, \cdots, x_n)$, which is a contradiction with $F(y) < t$. Therefore $\min\{F(x_1), F(x_2), \cdots, F(x_n)\} \leq \max\{F(y), 0.5\}$, for all $x_i, y \in H(i=1,2,...,n)$ with $y \in f(x_1, x_2, \cdots, x_n)$, which implies,

$$\min\{F(x_1), F(x_2), \cdots, F(x_n)\} \leq \inf_{y \in f(x_1, x_2, \cdots, x_n)} (\max\{F(y), 0.5\}),$$

for all $x_i \in H$. Hence $(HF5)$ holds.

Conversely, assume that $t \in (0.5, 1]$ and $x_i \in U(F;t)(i=1,2,...,n)$. Then

$$0.5 < t < \min\{F(x_1), F(x_2), \cdots, F(x_n)\} \leq \inf_{y \in f(x_1, x_2, \cdots, x_n)} (\max\{F(y), 0.5\})$$

It follows that for every $y \in f(x_1, x_2, \cdots, x_n)$, $0.5 < t \leq \max\{F(y), 0.5\}$, and so $t \leq F(y)$, which implies $y \in U(F;t)$. Hence $f(x_1, x_2, \cdots, x_n) \subseteq U(F;t)$, that is, $U(F;t)$ is a subalgebraic hypersystem of H.

Let F be a fuzzy set of an algebraic hypersystem H and $J = \{\alpha | \alpha \in (0,1]$ and $U(F;\alpha)$ is an empty set or a subalgebraic hypersystem of $H\}$. In particular, if $J = (0,1]$, then F is an ordinary fuzzy subalgebraic hypersystem of H (Theorem 2.3); if $J = (0, 0.5)$, F is an $(\in, \in \vee q)$-fuzzy subalgebraic hypersystem of H (Theorem 4.4).

In [17], Yuan et al. gave the definition of a fuzzy subgroup with thresholds which is a generalization of Rosenfeld's fuzzy subgroup, and Bhkat and Das's fuzzy subgroup. Based on [17], we can extend the concept of a fuzzy subgroup with thresholds to the concept of fuzzy sublagebraic hypersystems with thresholds in the following way:

Definition 4.6: Let $s, t \in [0,1]$ and $s < t$, then a fuzzy set F of H is called a *fuzzy subalgebraic hypersystem with thresholds* (s,t) of H if it satisfies:

$$(HF6) \min\{F(x_1), F(x_2), \cdots, F(x_n), t\} \leq \inf_{y \in f(x_1, x_2, \cdots, x_n)} (\max\{F(y), s\}),$$

for all $x_i \in H, i=1,2,...,n$.

Remark: If F is a fuzzy subalgebraic hypersystem with thresholds of H, then we can conclude that F is an ordinary fuzzy subalgebraic hypersystem when $s=0, t=1$; and F is an $(\in, \in \vee q)$-fuzzy subalgebraic hypersystem when $s=0, t=0.5$.

Now, we characterize fuzzy subalgebraic hypersystems with thresholds by their level subalgebraic hypersystems.

Theorem 4.7: A fuzzy set F of H is a fuzzy subalgebraic hypersystem with thresholds (s,t) of H if and only if $U(F;\alpha)(\neq \Phi)$ is a subalgebraic hypersystem of H for all $\alpha \in (s,t]$.

Proof: Let F be a fuzzy subalgebraic hypersystem with thresholds (s,t) of H and $\alpha \in (s,t]$. Let $x_i \in U(F;\alpha)$, then $F(x_i) \geq \alpha, i=1,2,...,n$. Now

$$\inf_{y \in f(x_1, x_2, \cdots, x_n)} (\max\{F(y), s\}) \geq \min\{F(x_1), F(x_2), \cdots, F(x_n), t\}$$
$$\geq \min\{\alpha, t\} \geq \alpha > s.$$

So for every $\alpha \in f(x_1, x_2, \cdots, x_n)$, we have $\max\{F(\alpha), s\} > \alpha > s$, which implies $F(y) > \alpha$, and so $y \in U(F;\alpha)$. Hence $f(x_1, x_2, \cdots, x_n) \subseteq U(F;\alpha)$. Therefore $U(F;\alpha)$ is a subalgebraic hypersystem of H for all $\alpha \in (s,t]$.

Conversely, let F be a fuzzy set of H such that $U(F;\alpha)(\neq \Phi)$ is a subalgebraic hypersystem of H for all $\alpha \in (s,t]$. If there exist $x_i(i=1,2,...,n), y \in H$ with $y \in f(x_1, x_2, \cdots, x_n)$ such that $\max\{F(\alpha), s\} < \min\{F(x_1), F(x_2), \cdots, F(x_n), t\} = \alpha$, then $\alpha \in (s,t], F(y) < \alpha, x_i \in U(F;\alpha)$,

$i=1,2,...,n$. Since $U(F;\alpha)$ is a subalgebraic hypersystem of H and $x_i \in U(F;\alpha)$ for all $y \in f(x_1,x_2,\cdots,x_n)$, so $f(x_1,x_2,\cdots,x_n) \subseteq U(F;\alpha)$. Hence $F(y) \geq \alpha$, for all $y \in f(x_1,x_2,\cdots,x_n)$. This is a contradiction with $F(y) < \alpha$. Therefore $\min\{F(x_1),F(x_2),\cdots,F(x_n),t\} \leq \max\{F(y),s\}$, for all $x_i, y \in H$ with $y \in f(x_1,x_2,\cdots,x_n)$. This proves that F is a fuzzy subalgebraic hypersystem with thresholds (s,t) of H.

5. Implication-Based fuzzy subalgebraic hypersystems

Fuzzy logic is an extension of set theoretic variables (or terms of the linguistic variable truth). Some operators, like $\wedge,\vee,\neg,\rightarrow$ in fuzzy logic are also defined by using truth tables, the extension principle can be applied to derive definitions of the operators.

In the fuzzy logic, truth value of fuzzy proposition P is denoted by $[P]$. In the following, we display the fuzzy logical and corresponding set-theoretical notions:

(5.1) $[x \in A] = A(x)$;

(5.2) $[x \notin A] = 1 - A(x)$;

(5.3) $[P \wedge Q] = \min\{[P],[Q]\}$;

(5.4) $[P \vee Q] = \max\{[P],[Q]\}$;

(5.5) $[P \rightarrow Q] = \min\{1, 1-[P]+[Q]\}$;

(5.6) $[\forall x P(x)] = \inf\{P(x)\}$;

(5.7) $\models P$ if and only if $[P]=1$ for all valuations.

The truth valuation rules given in (5.4) are those in the Lukasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We only show a selection of them in the next table 5:

Table 5.

Name	Definition of Implication Operators
Early Zadeh	$I_m(\alpha,\beta) = \max\{1-\alpha, \min\{\alpha,\beta\}\}$
Lukasiewicz	$I_a(\alpha,\beta) = \min\{1, 1-\alpha+\beta\}$
Standard Star(Godel)	$I_g(\alpha,\beta) = \begin{cases} 1 & if \quad \alpha \leq \beta \\ \beta & if \quad \alpha > \beta \end{cases}$
Contraposition of Godel	$I_{cg}(\alpha,\beta) = \begin{cases} 1 & if \quad \alpha \leq \beta \\ 1-\alpha & if \quad \alpha > \beta \end{cases}$
Gaines-Rescher	$I_{gr}(\alpha,\beta) = \begin{cases} 1 & if \quad \alpha \leq \beta \\ 0 & if \quad \alpha > \beta \end{cases}$
Kleene-Dienes	$I_b(\alpha,\beta) = \max\{1-\alpha, \beta\}$

Definition 5.1: A fuzzy set F of H is called a *fuzzifying subalgebraic hypersystem* of H if it satisfies, for all $x_i \in H, i=1,2,...,n$, $\models [\wedge[x_i \in F] \rightarrow [\forall y \in f(x_1,x_2,\cdots,x_n), y \in F]]$.

Clearly, Definition 5.1 is equivalent to Definition 2.1. Therefore fuzzifying subalgebraic hypersystem is an ordinary fuzzy subalgebraic hypersystem.

In [16], the concept of t-tautology is introduced, i.e., $\models_t P$ if and only if $[P] \geq t$ for all valuations.

Now, we can extend the concept of implication-based fuzzy subalgebraic hypersystems in the following way:

Definition 5.2: Let F be a fuzzy set of H and $t \in (0,1]$ is a fixed number. Then F is called a *t-implication-based fuzzy subalgebraic hypersystem* of H, if for all $x_i \in H, i=1,2,...,n, \models [\wedge[x_i \in F] \rightarrow [\forall y \in f(x_1,x_2,\cdots,x_n), y \in F]]$.

Now, let I be an implication operator, then we have

Corollary 5.3: A fuzzy set F of H is a t-implication-based fuzzy subalgebraic hypersystem of H if and only if for all $x_i \in H, i=1,2,...,n$,

$$I\left(\wedge F(x_i), \inf_{y \in f(x_1,x_2,\cdots,x_n)} F(y)\right) \geq t$$

Let F be a fuzzy set of H, then we have the following results:

Theorem 5.4: (i) Let $I = I_{gr}$, the F is an 0.5-implication-based fuzzy subalgebraic hypersystem of H if and only if F is a fuzzy subalgebraic hypersystem with thresholds $(r=0, s=1)$ of H;

(ii) Let $I = I_g$, then F is an 0.5-implication-based fuzzy subalgebraic hypersystem of H if and only if F is a fuzzy subalgebraic hypersystem with thresholds $(r=0, s=0.5)$ of H;

(iii) Let $I = I_{cg}$, then F is an 0.5-implication-based fuzzy subalgebraic hypersystem of H if and only if F is a fuzzy subalgebraic hypersystem with thresholds $(r=0.5, s=1)$ of H.

6. Conclusions

The aim of this paper is to introduce and study a new sort of fuzzy subalgebraic hypersystems of algebraic hypersystems and to investigate related properties. Also, we consider the definition of implication operators in the Lukusiewicz system of continuous-valued logic for fuzzy subalgebraic hypersystems.

In our future work, we will focus on considering other types with relations among them, and some applications in information sciences and general systems.

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