Fuzzy Decomposition Method by Mapping Analysis

Huan-Wen Tzeng

Abstract

The max-min composition of fuzzy relation equation takes the extreme value as the solution; such a characteristic induces the solution of decomposition to be a possible region. The solution region has just one maximum and probably more than one minimum. The maximum solution has the 'possibility' characteristic, and the minimum has the 'necessity' trait. For the needs of greater efficiency and more effectiveness in regard to performance, this paper proposes a new analytical method for the use of decomposition. The algorithm of logical mapping analysis is used for the new method. Mapping status, such as Unique Mapping, Overlapped Mapping, Redundant mapping, and Together Mapping, is introduced to construct the algorithm of logical mapping analysis. Then, by the principle of just one single freedom for every maximum and minimum solution, the status of overlapped and redundant mappings will be eliminated and the together mapping status will be separated as an individual presentation. This proposed method not only discovers all of the real solutions, but is also easily programmed for computer analysis.

Keywords: Decomposition, fuzzy relation equation, mapping analysis, freedom of solution.

1. Introduction

In the case of fuzzy relation equation composition,

$$\underset{\sim}{X} \circ \underset{\sim}{R} = \underset{\sim}{B} \tag{1}$$

where $\underset{\sim}{X}$, $\underset{\sim}{R}$, and $\underset{\sim}{B}$ are fuzzy matrices, and "$\circ$" is the max-min composition. When matrices $\underset{\sim}{X}$ and $\underset{\sim}{R}$ are given and matrix $\underset{\sim}{B}$ is to be determined, the problem is slight, the solution exists and it is unique [1].

In (1), when the matrix $\underset{\sim}{X}$ is to be determined, the problem of decomposition becomes significant. And the solution of decomposition will be a region; the solution's region is shown as Figure 1 [1,2].

The single and absolute character of maximum solution means it has the width-first possibility characteristic [3]. The solution involves every possible element to achieve the solution vector $\underset{\sim}{X}$.

The minimum solution may not be singular [1]. It also means that the minimum solution has the depth-first necessity characteristic [3]. It only takes the necessary element to consist solution $\underset{\sim}{X}$. Also, it needs to eliminate every redundant element.

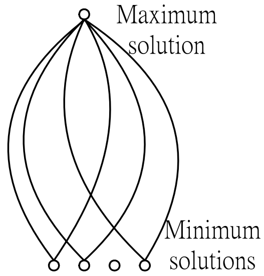

Figure 1. The solution region of decomposition.

Fuzzy decomposition has been an interesting study topic for many years. Table one shows the results of this paper in comparison with some traditional methods used for deriving the maximum and minimum solutions; however, most of them (method A-E) have the same characteristic of large computing quantity and complexity in regard to programming for computer analysis [1,2,4-6]. Two of them could not solve the correct solution in some examples. Since most traditional methods are very complex to program, it is hard to attest to their performance with all of the examples in this paper. So, the mark "U" is used to represent the uncertainty of their performance status.

Method F is proposed in this paper, with the intention of finding a new algorithm for solving the fuzzy relation equations with greater efficiency and more effectiveness.

Table 1. The Comparison with Traditional Methods.

Method / Solving Effect	A [4]	B [1]	C [2]	D [5]	E [6]	F
Solved the maximum	Yes	Yes	Yes	Yes	Yes	Yes
Solved the minimum	U	U	No	No	No	Yes
Less computing quantity	No	No	No	No	Yes	Yes
Simplicity of programming	No	No	No	No	Yes	Yes

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This paper explains the method of solution in the following sections. First of all, the key concept of a new definition for decomposition is explained. Then the region of solution will explain the occasion of intersection operation. In addition, mapping analysis will explain the principle of decomposition. After that, the critical matrix will explain how to regulate according to the status of unique, overlapped, redundant, and together mappings, as well as the whole procedure of decomposition. In addition, more examples are illustrated in the next section. The last section includes the conclusion and some application cases.

2. Solution Region of Decomposition

A. *New Definition of Decomposition Solution*

The key concept of this method is a new decomposition definition regulated according to the third rule of Tsukamoto's method [7].

For solving the vector x_{1j} of fuzzy relation equation: $x_{1j} \circ r_{jk} = b_{1k}$, three rules of Tsukamoto's method are,

1. If $r_{jk} > b_{1k}$, then $\max(x_{1j}) = b_{1k}$ and $\min(x_{1j}) = b_{1k}$.
2. If $r_{jk} = b_{1k}$, then $\max(x_{1j}) = 1.0$ and $\min(x_{1j}) = b_{1k}$.
3. If $r_{jk} < b_{1k}$, then the equation is unsolved.

This paper regulates the third rule as the new definition [8],

3. If $r_{jk} < b_{1k}$, then $\max(x_{1j}) = 1.0$ and $\min(x_{1j}) = 0.0$.

This also signifies regulating the unsolved condition into the uncertain region of the bounded zone between 1.0 and 0.0. After the third rule is regulated, the decomposition procedures of fuzzy relation equation become easy to program for computer analysis.

Some papers have defined the solution of unsolved equation as an empty set (Φ) [1,2,5]; however, this will induce the complexity of programming and increase the quantity of computing.

B. *Intersection for Decomposition Solution*

As in the discussion on Figure 1, the decomposition solution is a region bounded by one maximum and probably more than one minimum [1]. The three rules of the new solution method point out clearly that the edges of solution region are determined by elements of $\underset{\sim}{B}$, 1.0, and 0.0 [8].

Example A:

If $\underset{\sim}{X} \circ \underset{\sim}{R} = \underset{\sim}{B}$, $\underset{\sim}{X} = [X_{11}]$, $\underset{\sim}{R} = [0.8 \quad 0.5 \quad 0.3 \quad 0.6]$ and $\underset{\sim}{B} = [0.6 \quad 0.5 \quad 0.3 \quad 0.6]$, then the possible maximum solution of X_{11} is 0.6, 1.0, 1.0, or 1.0. The possible minimum solution of X_{11} is 0.6, 0.5, 0.3, or 0.6. The possible region of solution is shown as Figure 2 [9].

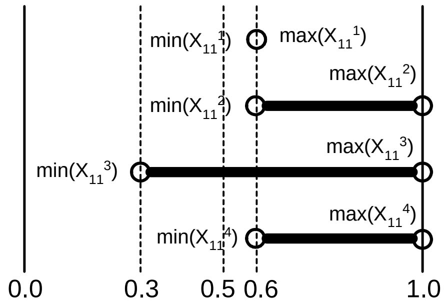

Figure 2. Solution Region of Example A.

Example A shows element X_{11} was determined by the pairs of (R_{11}, B_{11}), (R_{12}, B_{12}), (R_{13}, B_{13}), and (R_{14}, B_{14}). So, the solution of X_{11} will be the intersection of these four individual solutions of X_{11}^1 to X_{11}^4. The maximum solution of X_{11} will be the smallest of all possible maximum solutions; the minimum solution of X_{11} is the biggest of all minimum solutions [8-9]:

$$MaxX_{11} = \min\left(\max(X_{11}^1) \quad \max(X_{11}^2) \quad \max(X_{11}^3) \quad \max(X_{11}^4)\right)$$

$$MinX_{11} = \max\left(\min(X_{11}^1) \quad \min(X_{11}^2) \quad \min(X_{11}^3) \quad \min(X_{11}^4)\right)$$

The maximum solution of example A is 0.6 and the minimum solution of example A is 0.6. The above explanation shows the procedure is slightly while the input vector $\underset{\sim}{X}$ just has a single element.

Example B:

If $\underset{\sim}{X} \circ \underset{\sim}{R} = \underset{\sim}{B}$, $\underset{\sim}{X} = [X_{11}]$, $\underset{\sim}{R} = [0.7 \quad 0.2 \quad 0.3 \quad 0.5]$ and $\underset{\sim}{B} = [0.7 \quad 0.6 \quad 0.3 \quad 0.6]$, the possible region of solution is shown in Figure 3.

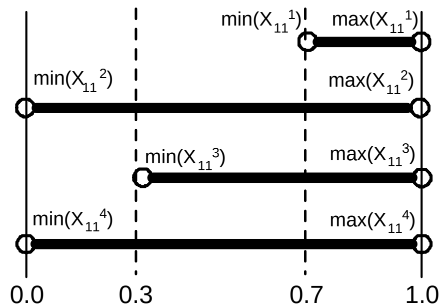

Figure 3. Solution Region of Example B.

The maximum solution of example B is 1.0 and the minimum is 0.7. There are two elements (0.6 of column 2 & 4) of output vector $\underset{\sim}{B}$ are couldn't satisfied the solution. This means that the input vector $\underset{\sim}{X}$ wouldn't be just a single element as in example B.

3. Mapping Analysis of Solution

When vector $\underset{\sim}{X}$ has more than one element, the analysis will be much more complex. In Section 3, this paper explains the logical mapping analysis principle for such a complicated situation. Logical mapping linkage shows the operation procedure of the composition and decomposition of the fuzzy relation equation. A composition of fuzzy relation equation list is shown below:

$$\underset{\sim}{X} = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1j} \end{bmatrix} \quad (2)$$

$$\underset{\sim}{R} = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1k} \\ R_{21} & R_{22} & \cdots & R_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ R_{j1} & R_{j2} & \cdots & R_{jk} \end{bmatrix} \quad (3)$$

$$\underset{\sim}{B} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1k} \end{bmatrix} \quad (4)$$

$$\underset{\sim}{X} \circ \underset{\sim}{R} = \underset{\sim}{B} \quad (5)$$

$$B_{1k} = \max[\min(X_{11}, R_{1k}), \ldots, \min(X_{1j}, R_{jk})] \quad (6)$$

Equation (6) shown that B_{1k} is determined by all elements of $\underset{\sim}{X}_{1j}$ and same column elements of $\underset{\sim}{R}_{jk}$. It also points out that the extreme value of $\underset{\sim}{X}_{1j}$ and $\underset{\sim}{R}_{jk}$ will determine the solution B_{1k}.

A. *Mapping Diagram of Composition*

The purpose of the mapping diagram is to show every related element that deals with the same solution [9]. A mapping diagram will concentrate every related linkage together. The mapping diagram of (6) is shown in Figure 4.

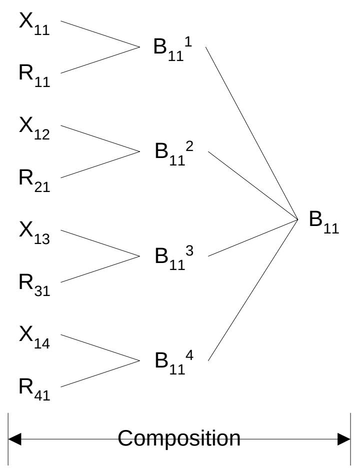

Figure 4. Mapping diagram of composition.

B. *Mapping Diagram of Decomposition*

Like the mapping diagram of composition, the linkage of every related element concerning decomposition will be shown in a similar diagram [9]. The mapping diagrams of maximum and minimum decomposition of X_{11} are shown in Figures 5 and 6.

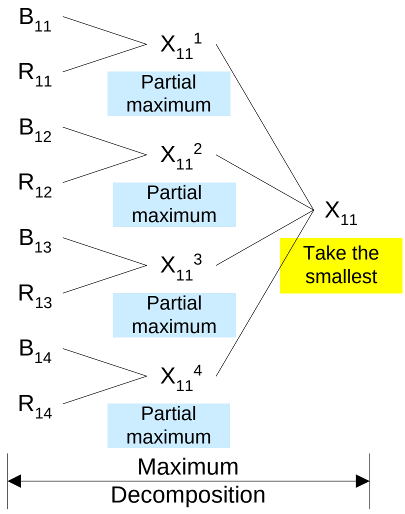

Figure 5. Mapping diagram of maximum decomposition.

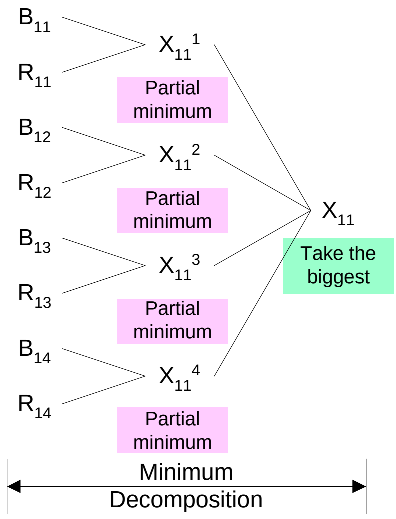

Figure 6. Mapping diagram of minimum decomposition.

As discussed in Section 2, the decomposition solution of X_{11} will be the intersection of $X_{11}^1 \circ R_{11} = B_{11}$, $X_{11}^2 \circ R_{12} = B_{12}$, $X_{11}^3 \circ R_{13} = B_{13}$, and $X_{11}^4 \circ R_{14} = B_{14}$. The maximum solution of X_{11} will be the smallest one of the four above equations, and the minimum solution will be the greatest one of them.

Figure 7 shows the whole mapping linkage from composition to decomposition. As seen, only $\underset{\sim}{B}$ affects the decomposition solution of $\underset{\sim}{X}$. It also confirms that the solution value of vector $\underset{\sim}{X}$ will be just each element's value of $\underset{\sim}{B}$ and the uncertainty value of 1.0 and 0.0. Such a situation was discussed in Section 1, where a new decomposition solution definition was proposed.

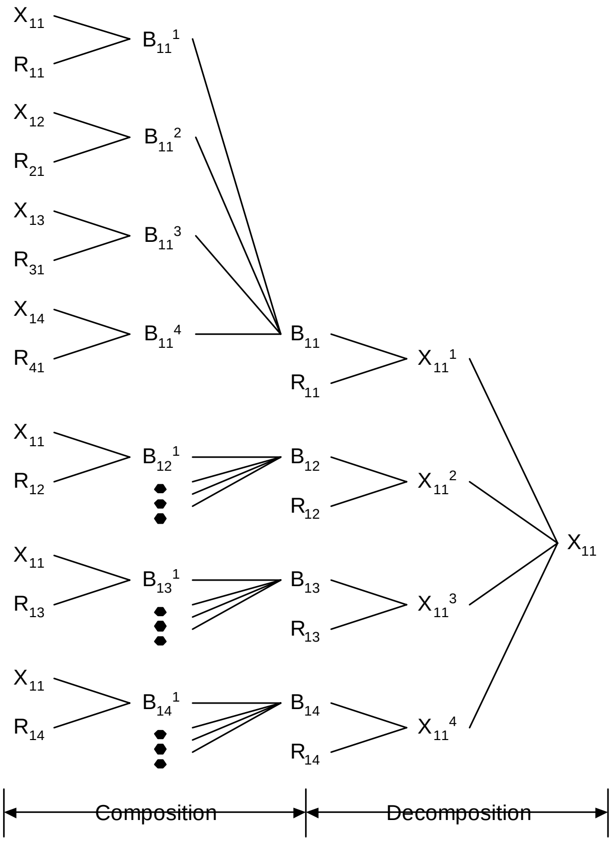

Figure 7. Mapping diagram from composition to decomposition.

4. Regulate the Possible Critical Matrix m for Deriving Minimum Solution

This section explains the advanced logical mapping analysis of complex situations. Some mapping statuses, such as unique, overlapped, together, and redundant are explained here.

In order to gather every possible value of $\underset{\sim}{X}$ together for solution mapping analysis, this paper proposes two critical matrixes: **M** and **m**. Critical matrix **M** consists of every partial maximum value of $\underset{\sim}{X}$. Critical matrix **m** consists of every partial minimum value of $\underset{\sim}{X}$ [8]. Figures 8 and 9 show the intersection to **M** and **m** for solving the solutions.

$$M=\begin{bmatrix}\max(X_{11}^1) & \max(X_{12}^1) & \max(X_{13}^1) & \max(X_{14}^1)\\ \max(X_{11}^2) & \max(X_{12}^2) & \max(X_{13}^2) & \max(X_{14}^2)\\ \max(X_{11}^3) & \max(X_{12}^3) & \max(X_{13}^3) & \max(X_{14}^3)\\ \max(X_{11}^4) & \max(X_{12}^4) & \max(X_{13}^4) & \max(X_{14}^4)\end{bmatrix}$$

minimum intersection

$$MaxX=\begin{bmatrix}X_{11} & X_{12} & X_{13} & X_{14}\end{bmatrix}$$

Fig. 8. Take the smallest one as the maximum solution.

$$m=\begin{bmatrix}\min(X_{11}^1) & \min(X_{12}^1) & \min(X_{13}^1) & \min(X_{14}^1)\\ \min(X_{11}^2) & \min(X_{12}^2) & \min(X_{13}^2) & \min(X_{14}^2)\\ \min(X_{11}^3) & \min(X_{12}^3) & \min(X_{13}^3) & \min(X_{14}^3)\\ \min(X_{11}^4) & \min(X_{12}^4) & \min(X_{13}^4) & \min(X_{14}^4)\end{bmatrix}$$

maximum intersection

$$MinX=\begin{bmatrix}X_{11} & X_{12} & X_{13} & X_{14}\end{bmatrix}$$

Fig. 9. Take the biggest one as the minimum solution.

The maximum solution is making an intersection to every column of critical matrix **M**. In this operation, critical matrix **M** needs no regulation because of maximum solution's width first and most possibility characteristic [3].

Before takes the intersection operation to critical matrix **m.** The critical matrix **m** should be regulated by some procedure [8-10] because of its depth first and least necessity characteristic. Before regulation, critical matrix **m** is not credible, just a possible critical matrix.

A. *Regulate the Bigger Elements in Possible Matrix*

Sometimes in the mapping diagram, there are some possible minimum values of **m** bigger than the maximum solution's value of the same column (figure 10).

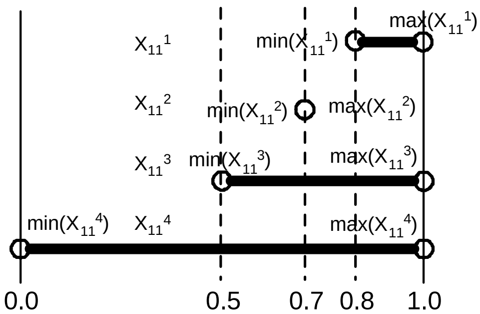

Figure 10. Region of possible minimum.

In Figure 10, the maximum solution of X_{11} is 0.7, but the possible minimum of X_{11}^1 is 0.8. It is a contradictory status. So, the bigger possible minimum element should be regulated first. This situation may happen when vector $\underset{\sim}{X}$ has more than one element.

This regulatory procedure is used to eliminate the contradictory element of matrix **m** by setting the value as 0.0. It also means that the element lost the freedom (opportunity) to determine the final solution of X_{11}. The regulated region of possible minimum is shown in figure 11 (dotted line),

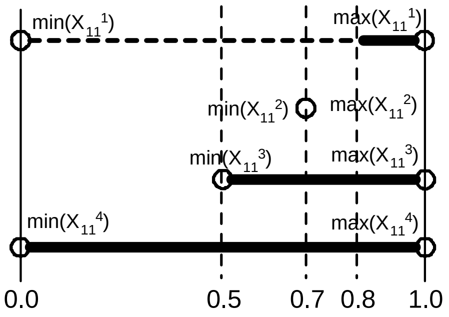

Figure 11. Regulated region of possible minimum.

Below is a decomposition example of fuzzy relation.
Example C: $\underset{\sim}{X} \circ \underset{\sim}{R} = \underset{\sim}{B}$, given $\underset{\sim}{R}$ and $\underset{\sim}{B}$

$$R = \begin{bmatrix} .2 & .7 & .1 & 0 \\ 0 & .4 & .5 & .1 \\ .2 & .3 & .4 & .0 \end{bmatrix} \tag{7}$$

$$B = \begin{bmatrix} .2 & .3 & .4 & .1 \end{bmatrix} \tag{8}$$

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \end{bmatrix} \tag{9}$$

$$M = \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.3 & 0.3 & 1.0 \\ 1.0 & 0.4 & 1.0 \\ 1.0 & 1.0 & 1.0 \end{bmatrix} \tag{10}$$

In (10), the smallest one of every column is taken; then the maximum solution is derived as:

$$MaxX = \begin{bmatrix} 0.3 & 0.3 & 1.0 \end{bmatrix} \tag{11}$$

$$m = \begin{bmatrix} 0.2 & 0.0 & 0.2 \\ 0.3 & 0.3 & 0.3 \\ 0.0 & 0.4 & 0.4 \\ 0.0 & 0.1 & 0.0 \end{bmatrix} \tag{12}$$

According to the principle of regulate bigger possible minimum, critical matrix **m** has an element at row 3 and column 2, which is bigger than the column 2 value of $MaxX$. So, the value 0.4 should be eliminated as 0.0, as shown in (13),

$$m = \begin{bmatrix} 0.2 & 0.0 & 0.2 \\ 0.3 & 0.3 & 0.3 \\ 0.0 & 0.0 & 0.4 \\ 0.0 & 0.1 & 0.0 \end{bmatrix} \begin{matrix} B_{11} \\ B_{12} \\ B_{13} \\ B_{14} \end{matrix} \tag{13}$$
$$\quad X_{11} \quad X_{12} \quad X_{13}$$

B. *Find out the Redundant Mapping*

This subsection explains the redundant possible element of **m**. The mapping diagram of (13) is shown in figure 12. In this figure, some elements' value is 0.0; it means they lost the opportunity to determine the final solution [8-10]. In (13), when the value of X_{1k}^{j} is 0.0, it is because of the uncertainty value 0.0 (when $r_{jk} < b_{1k}$) and lost freedom in the above procedure.

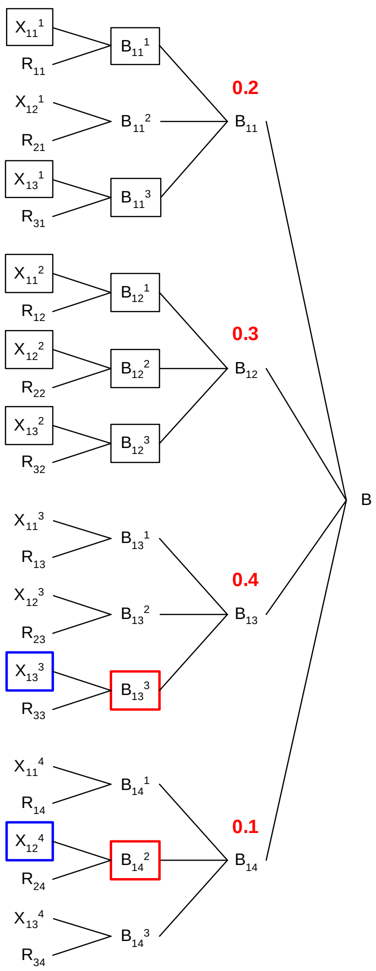

Figure 12. Mapping diagram of (13).

The mapping diagram can be shown as a compact diagram. It is called a linkage diagram [8,9].

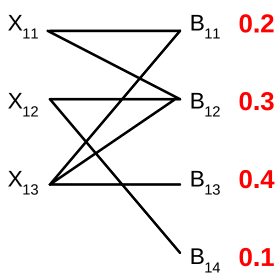

Figure 13. Compact linkage diagram of (13).

In Figures 12 and 13, two elements of $\underset{\sim}{B}$ can be exactly figured out; they are all just determined by a single element of $\underset{\sim}{X}$. They are $X_{12}^{4} \leftrightarrow B_{14}^{2}$ and $X_{13}^{3} \leftrightarrow B_{13}^{3}$. These statuses are called "Unique Mapping" status.

Element B_{12} is determined by X_{11}^2, X_{12}^2 and X_{13}^2 together. In this paper, the above statuses are referred to as "Together Mapping". If together mappings are without "Overlapped Mapping", they are reasonable. And they could be retained for determining the final solution.

In Figure 12, element X_{11} has more opportunity to determine B_{11} and B_{12}. In the same situation, element X_{12} could determine B_{11} and B_{14}. And element X_{13} could determine B_{11}, B_{12} and B_{13}. These multiple mapping statuses are called "Overlapped Mapping". Because a single element of $\underset{\sim}{X}$ could not determine difference elements with difference values of $\underset{\sim}{B}$, some "Redundant mapping" should be eliminated. The elimination will be executed according to the analysis of unique mapping and overlapped mapping.

C. *Redundant Elimination by Unique Mapping Analysis*

Since the unique mapping will monopolize the opportunity to determine solution, every overlapped mapping with unique mapping is redundant mapping. They should all be eliminated. In the compact diagram of Figure 13, there are two unique mappings. According to the principle of making the intersection operation to critical matrix **m**, the biggest unique mapping takes the elimination basis above all others.

The first step of this elimination is to determine the unique mapping status, then to figure out whether there is more than one unique mapping status. If there is, then we take the biggest one and proceed to the next step.

The next step is to find out whether any mapping linkage is overlapped with this unique mapping. If so, then we make the intersection to them. It means taking the biggest value as the representation value of unique mapping. Then the redundant overlapped mappings are eliminated.

After that, one more procedure has to be done; finding out whether any eliminated mapping was having the linkage of together mapping status with another mapping. If so, then we eliminate the residual linked together mappings because the redundant linked together mappings lost their opportunity to determine the solution.

In (13), unique mappings are $X_{13}^3 \leftrightarrow B_{13}$ of value 0.4 and $X_{12}^4 \leftrightarrow B_{14}$ of value 0.1. We start from the bigger mapping one $X_{13}^3 \leftrightarrow B_{13}$, and make the intersection to the mappings of $X_{13}^3 \leftrightarrow B_{13}$, $X_{13}^2 \leftrightarrow B_{12}$, and $X_{13}^1 \leftrightarrow B_{11}$. After that, the mappings of $X_{13}^2 \leftrightarrow B_{12}$ and $X_{13}^1 \leftrightarrow B_{11}$ are eliminated. Some residual linked together mappings such as $X_{11}^1 \leftrightarrow B_{11}$, $X_{11}^2 \leftrightarrow B_{12}$, and $X_{12}^2 \leftrightarrow B_{12}$ are eliminated either. The elimination diagram is shown as Figure 14. After the repeated procedure of elimination, the unique mapping status will no longer remain.

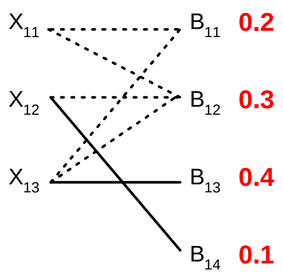

Figure 14. Eliminated linkage diagram (13).

D. *Redundant Elimination by Overlapped Mapping Analysis*

The next procedure is the elimination of overlapped mapping. The first step is to find any overlapped mapping pair in matrix **m**. If there is, then make the intersection to the biggest overlapped pairs first. Then the redundant overlapped mapping is eliminated in the same column. The element that has biggest value will remain [8-10].

The next step is to find out whether any eliminated mapping has linked together mapping with another mapping. If so, then we eliminate the redundant linked together mappings. Each eliminated mapping also means that it has lost its opportunity to determine the solution.

Figure 15 is a critical matrix **m** of (14) with three overlapped mapping statuses. The biggest overlapped mapping pair is column one of **m**.

$$m = \begin{bmatrix} 0.2 & 0.2 & 0.0 & 0.0 \\ 0.4 & 0.0 & 0.4 & 0.0 \\ 0.6 & 0.6 & 0.6 & 0.0 \\ 0.7 & 0.7 & 0.7 & 0.7 \end{bmatrix} \tag{14}$$

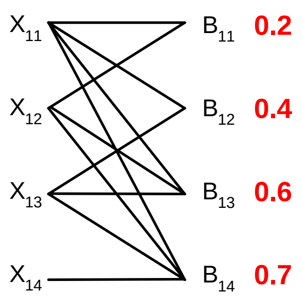

Figure 15. Compact linkage diagram of (14).

In (14), there are three overlapped mapping pairs. The first overlapped mapping pair is $X_{11}^1 \leftrightarrow B_{11}$, $X_{11}^2 \leftrightarrow B_{12}$, $X_{11}^3 \leftrightarrow B_{13}$, $X_{11}^4 \leftrightarrow B_{14}$. The second pair is $X_{12}^1 \leftrightarrow B_{11}$,

$X_{12}^3 \leftrightarrow B_{13}$, $X_{12}^4 \leftrightarrow B_{14}$. And the third pair is $X_{13}^2 \leftrightarrow B_{12}$, $X_{13}^3 \leftrightarrow B_{13}$, $X_{13}^4 \leftrightarrow B_{14}$. So, take the biggest pair and make an intersection to these mappings of $X_{11}^1 \leftrightarrow B_{11}$, $X_{11}^2 \leftrightarrow B_{12}$, $X_{11}^3 \leftrightarrow B_{13}$, $X_{11}^4 \leftrightarrow B_{14}$. Then the mappings of $X_{11}^1 \leftrightarrow B_{11}$, $X_{11}^2 \leftrightarrow B_{12}$, $X_{11}^3 \leftrightarrow B_{13}$ are eliminated.

After that, all redundant mappings will be eliminated in the next procedure; the mappings of $X_{12}^1 \leftrightarrow B_{11}$, $X_{12}^3 \leftrightarrow B_{13}$, $X_{12}^4 \leftrightarrow B_{14}$, $X_{13}^2 \leftrightarrow B_{12}$, $X_{13}^3 \leftrightarrow B_{13}$, $X_{13}^4 \leftrightarrow B_{14}$, and $X_{14}^4 \leftrightarrow B_{14}$ are eliminated. The elimination diagram is shown in Figure 16.

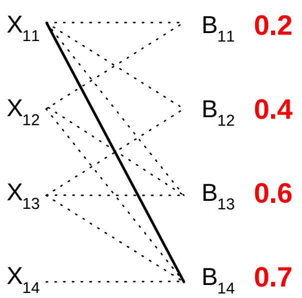

Figure 16. Eliminated linkage diagram of (14).

After the repeated procedure of elimination, the overlapped mapping status will no longer exist. The only probable residual status is together mapping.

As the possible critical matrix **m** was regulated by the above procedure, elements of **m** are no longer possible minimum value. They are all credible elements of **m** to determine the final minimum solution of $\underset{\sim}{X}$. So, matrix **m** is a credible critical matrix from now on.

E. *Separate All Residual Together Mapping*

The first step is to find all of the residual linked together mapping pairs in **m**. The second step is to separate them into single elements represented at the final solution at the same time [8-10]. So, if there are two pairs of linked together mapping, one has three elements and the other has two elements. Then the number of final solutions will be six ($2 \times 3 = 6$). In (15), it has one residual linked together mapping pair. They are elements of row 5, having the same value of 0.35. Three linkages of $X_{12}^5 \leftrightarrow B_{15}$, $X_{13}^5 \leftrightarrow B_{15}$, and $X_{15}^5 \leftrightarrow B_{15}$ are the residual linked together mappings. The linkage diagram is shown as Figure 17. Figure 18 is the separated diagram of (15).

$$m = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.54 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.35 & 0.35 & 0.0 & 0.35 \end{bmatrix} \quad (15)$$

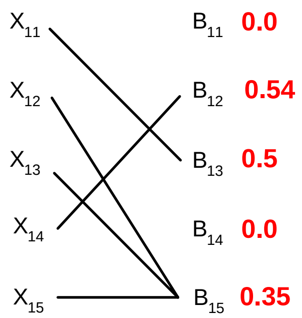

Figure 17. Together mapping linkage diagram of (15).

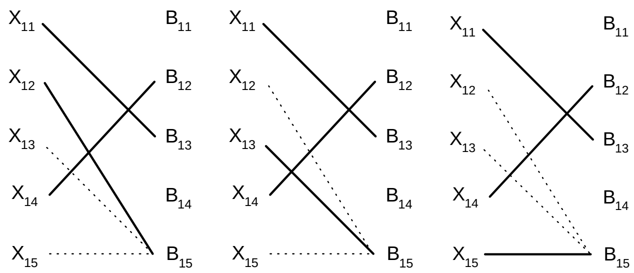

Figure 18. Separated residual together mappings of (15).

5. Solving Procedure of Relation Equation

A. *Solving the Maximum Solution*

In the paper, there are two steps for solving the maximum solution, and six steps for the minimum solution. The critical matrix **M** and **m** are the key components for the decomposition operation. Below are the two procedures for solving the maximum solution.

I. *Generate the Critical Matrix **M***

Example D [1]: Fuzzy relation of $\underset{\sim}{X} \circ \underset{\sim}{R} = \underset{\sim}{B}$, given the matrix $\underset{\sim}{R}$ and vector $\underset{\sim}{B}$ as below,

$$\underset{\sim}{R} = \begin{bmatrix} 0.1 & 0.4 & 0.5 & 0.1 \\ 0.9 & 0.7 & 0.2 & 0.0 \\ 0.8 & 1.0 & 0.5 & 0.0 \\ 0.1 & 0.3 & 0.6 & 0.0 \end{bmatrix} \quad (16)$$

$$\underset{\sim}{B} = \begin{bmatrix} 0.8 & 0.7 & 0.5 & 0.0 \end{bmatrix} \quad (17)$$

$$M=\begin{bmatrix}1.0 & 0.8 & 1.0 & 1.0\\ 1.0 & 1.0 & 0.7 & 1.0\\ 1.0 & 1.0 & 1.0 & 0.5\\ 0.0 & 1.0 & 1.0 & 1.0\end{bmatrix} \quad (18)$$

II. *Make Intersection to Critical Matrix **M***

$$Max(X_{11})=\min(\max(X_{11}^{1}),\ldots,\max(X_{11}^{4})) \quad (19)$$

According (18), $Max(X_{11})=0.0$, $Max(X_{12})=0.8$, $Max(X_{13})=0.7$, and $Max(X_{14})=0.5$. The maximum decomposition solution of example D is,

$$Max(X)=\begin{bmatrix}0.0 & 0.8 & 0.7 & 0.5\end{bmatrix} \quad (20)$$

B. *Solving the Minimum Solution*

I. *Generate the Possible Critical Matrix **m***

According to the new definition of the decomposition method in Section 1, a critical matrix **m** is generated as (21). Before facing further regulation, elements' values of matrix **m** are still at the situation of possible.

$$m=\begin{bmatrix}0.0 & 0.8 & 0.8 & 0.0\\ 0.0 & 0.7 & 0.7 & 0.0\\ 0.5 & 0.0 & 0.5 & 0.5\\ 0.0 & 0.0 & 0.0 & 0.0\end{bmatrix} \quad (21)$$

II. *Regulate the Bigger Value in Possible Matrix **m***

Some elements of **m** are regulated by the value of 0.0; it means that when the element's value of **m** is bigger than the same column element's value of $Max(X)$, its value is set as 0.0. Matrix (22) is the regulated **m** of (21),

$$m=\begin{bmatrix}0.0 & 0.8 & 0.0 & 0.0\\ 0.0 & 0.7 & 0.7 & 0.0\\ 0.0 & 0.0 & 0.5 & 0.5\\ 0.0 & 0.0 & 0.0 & 0.0\end{bmatrix} \quad (22)$$

III. *Redundant Elimination by Unique Mapping Analysis*

- **Determine the unique mapping status**: In the matrix of (22), there is only one unique mapping, so the next step is not needed; then go to the third step of this procedure.
- **Determine biggest unique mapping status**: If there is more than one unique mapping status in matrix **m**, then we find the biggest and treat it first.
- **Eliminate the redundant overlapped mapping in the same column**: The elimination takes the biggest value as the new value of unique mapping. Then set all the other elements as the value 0.0. In matrix (22), the redundant overlapped mapping element is 0.7 (row 2, column 2). The regulated matrix is shown as (23).

$$m=\begin{bmatrix}0.0 & 0.8 & 0.0 & 0.0\\ 0.0 & 0.0 & 0.7 & 0.0\\ 0.0 & 0.0 & 0.5 & 0.5\\ 0.0 & 0.0 & 0.0 & 0.0\end{bmatrix} \quad (23)$$

- **Eliminate the redundant linked together mapping in the related row**: The elimination sets all of the other elements as the value 0.0; that was the linked together mapping relation with the eliminated value in the above procedure. In matrix (23), the redundant linked together mapping element 0.7 (row 2, column 3) needs to be eliminated. The regulated matrix is shown as (24).

$$m=\begin{bmatrix}0.0 & 0.8 & 0.0 & 0.0\\ 0.0 & 0.0 & 0.0 & 0.0\\ 0.0 & 0.0 & 0.5 & 0.5\\ 0.0 & 0.0 & 0.0 & 0.0\end{bmatrix} \quad (24)$$

IV. *Redundant Elimination by Overlapped Mapping Analysis*

- **Determine the Overlapped Mapping Status:** In another matrix case of (25), there are four overlapped mapping pairs.

$$m=\begin{bmatrix}0.4 & 0.0 & 0.4 & 0.4 & 0.4\\ 0.9 & 0.0 & 0.9 & 0.0 & 0.9\\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3\\ 0.5 & 0.0 & 0.5 & 0.5 & 0.5\end{bmatrix} \quad (25)$$

- **Determine the biggest Overlapped Mapping Pairs:** In the matrix of (25), the biggest overlapped mapping pairs are columns 1, 3 and 5.
- **Eliminate the redundant overlapped mapping in the same column**: The elimination makes the intersection operation to every biggest overlapped mapping pair. Then the biggest value will remain in the column. The other elements will be eliminated. The eliminated matrix of (25) is shown in (26).

$$m=\begin{bmatrix}0.0 & 0.0 & 0.0 & 0.4 & 0.0\\ 0.9 & 0.0 & 0.9 & 0.0 & 0.9\\ 0.0 & 0.3 & 0.0 & 0.3 & 0.0\\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0\end{bmatrix} \quad (26)$$

- **Eliminate the redundant together mapping in the related row**: The elimination sets all of the other elements as the value 0.0; that was the linked together mapping relation with eliminated value in the above procedure. In matrix (26), the redundant linked together mapping elements are in row one, three, and four. The regulated matrix is shown as (27).

$$m = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.9 & 0.0 & 0.9 & 0.0 & 0.9 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \quad (27)$$

V. *Separate All Residual Together Mapping*

- **Determine the residual together mapping pairs:** In the original case of (24), critical matrix **m** has a residual linked together mapping pair in row 3. So, it needs to proceed to the next step; otherwise jump to the last step.
- **Separate the linked together mapping pairs**: This step is to separate every linked together mapping pair into single elements represented in the final solution at one time. Critical matrix **m** of (24) will be separated as (28). The element number of this linked together mapping pair is two. Then the number of matrix **m** will be two.

$$m = \begin{bmatrix} 0.0 & 0.8 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \text{ or } \begin{bmatrix} 0.0 & 0.8 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \quad (28)$$

VI. *Make an Intersection to Every Critical Matrix* ***m***

After the above steps have regulated the critical matrix. Then the final solution of decomposition could be figured out by the intersection to every column of critical matrix **m**. The matrixes of (28) will figure out two minimum solutions, as (29) and (30).

$$MinX_1 = [0.0 \quad 0.8 \quad 0.5 \quad 0.0] \quad (29)$$

$$MinX_2 = [0.0 \quad 0.8 \quad 0.0 \quad 0.5] \quad (30)$$

VII. *Flow Chart of Solving Minimum Solution*

Finally, the flow chart of the whole procedure of the decomposition for minimum solution can be illustrated as in Figure 19.

As in Figure 19, these steps all consist of logical mapping analyzing techniques. So, the computing quantity is low, and the algorithm is simple to program for computer usage.

6. Illustrative Examples

Here are some addition examples to illustrate the new method.

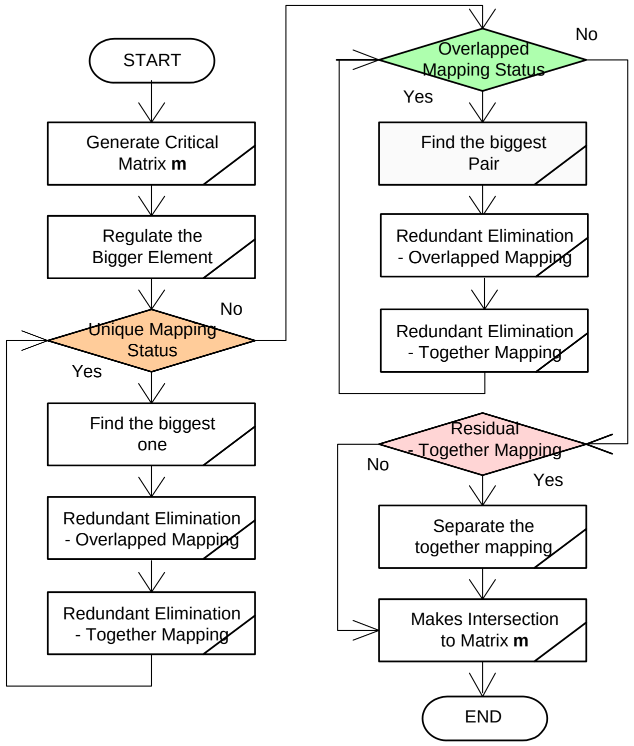

Figure 19. Flow chart for solving minimum solution.

- **Example I** [5]: Fuzzy relation $\underset{\sim}{X} \circ \underset{\sim}{R} = \underset{\sim}{B}$, given

$$\underset{\sim}{R} = \begin{bmatrix} 0.2 & 0.7 & 0.1 & 0.0 \\ 0.0 & 0.4 & 0.5 & 0.1 \\ 0.2 & 0.3 & 0.4 & 0.0 \end{bmatrix} \quad (31)$$

$$\underset{\sim}{B} = [0.2 \quad 0.5 \quad 0.3 \quad 0.1] \quad (32)$$

$$M = \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.5 & 1.0 & 1.0 \\ 1.0 & 0.3 & 0.3 \\ 1.0 & 1.0 & 1.0 \end{bmatrix} \quad (33)$$

$$MaxX = [0.5 \quad 0.3 \quad 0.3] \quad (34)$$

$$m = \begin{bmatrix} 0.2 & 0.0 & 0.2 \\ 0.5 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.3 \\ 0.0 & 0.1 & 0.0 \end{bmatrix} \quad (35)$$

No element of (35) is bigger than (34), and there are two unique mappings in (35). After elimination by unique mapping analysis,

$$m = \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} \quad (36)$$

There was no element of linked together mapping in (36); make an intersection to it, and the minimum solution came out as (37).

$$MinX = [0.5 \quad 0.3 \quad 0.0] \tag{37}$$

- **Example II** [11]: Fuzzy relation $\underset{\sim}{X} \circ \underset{\sim}{R} = \underset{\sim}{B}$, given

$$\underset{\sim}{R} = \begin{bmatrix} 0.6 & 0.6 & 0.6 \\ 0.6 & 0.6 & 0.6 \\ 0.6 & 0.6 & 0.6 \end{bmatrix} \tag{38}$$

$$\underset{\sim}{B} = [0.5 \quad 0.5 \quad 0.5] \tag{39}$$

$$M = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{bmatrix} \tag{40}$$

$$MaxX = [0.5 \quad 0.5 \quad 0.5] \tag{41}$$

$$m = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{bmatrix} \tag{42}$$

There was no element of (42) bigger than (41), but **m** has three overlapped mapping pairs. Also (42) has three biggest overlapped mapping pairs. So, take these biggest pairs to make intersections, as shown in (43). After the intersection operation there was no need for elimination due to the redundant status.

$$m = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} \tag{43}$$

There was one linked together mapping pair in (43); then they were separated and intersections to them were made. The minimum solution came out as (44).

$$\begin{aligned} MinX_1 &= [0.5 \quad 0.0 \quad 0.0] \\ MinX_2 &= [0.0 \quad 0.5 \quad 0.0] \\ MinX_3 &= [0.0 \quad 0.0 \quad 0.5] \end{aligned} \tag{44}$$

- **Example III**: Fuzzy relation $\underset{\sim}{X} \circ \underset{\sim}{R} = \underset{\sim}{B}$, given

$$\underset{\sim}{R} = \begin{bmatrix} 0.35 & 0.39 & 0.22 & 0.04 \\ 0.17 & 0.35 & 0.39 & 0.09 \\ 0.00 & 0.30 & 0.43 & 0.26 \\ 0.09 & 0.22 & 0.30 & 0.39 \\ 0.43 & 0.35 & 0.22 & 0.00 \end{bmatrix} \tag{45}$$

$$\underset{\sim}{B} = [0.30 \quad 0.30 \quad 0.30 \quad 0.10] \tag{46}$$

$$M = \begin{bmatrix} 0.3 & 1.0 & 1.0 & 1.0 & 0.3 \\ 0.3 & 0.3 & 1.0 & 1.0 & 0.3 \\ 1.0 & 0.3 & 0.3 & 1.0 & 1.0 \\ 1.0 & 1.0 & 0.1 & 0.1 & 1.0 \end{bmatrix} \tag{47}$$

$$MaxX = [0.30 \quad 0.30 \quad 0.10 \quad 0.10 \quad 0.30] \tag{48}$$

$$m = \begin{bmatrix} 0.30 & 0.00 & 0.00 & 0.00 & 0.30 \\ 0.30 & 0.30 & 0.30 & 0.00 & 0.30 \\ 0.00 & 0.30 & 0.30 & 0.30 & 0.00 \\ 0.00 & 0.00 & 0.10 & 0.10 & 0.00 \end{bmatrix} \tag{49}$$

There were three elements of (49) bigger than (48). Then (49) is regulated as (50),

$$m = \begin{bmatrix} 0.30 & 0.00 & 0.00 & 0.00 & 0.30 \\ 0.30 & 0.30 & 0.0 & 0.00 & 0.30 \\ 0.00 & 0.30 & 0.0 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.10 & 0.10 & 0.00 \end{bmatrix} \tag{50}$$

There was one unique mapping in (50). After, the bigger one was taken and eliminated by unique mapping analysis,

$$m = \begin{bmatrix} 0.30 & 0.00 & 0.00 & 0.00 & 0.30 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.30 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.10 & 0.10 & 0.00 \end{bmatrix} \tag{51}$$

There were two linked together mapping pairs in (51); they were then separated and intersections to them were made. The minimum solution came out as (52).

$$\begin{aligned} MinX_1 &= [0.3 \quad 0.3 \quad 0.1 \quad 0.0 \quad 0.0] \\ MinX_2 &= [0.0 \quad 0.3 \quad 0.1 \quad 0.0 \quad 0.3] \\ MinX_3 &= [0.3 \quad 0.3 \quad 0.0 \quad 0.1 \quad 0.0] \\ MinX_4 &= [0.0 \quad 0.3 \quad 0.0 \quad 0.1 \quad 0.3] \end{aligned} \tag{52}$$

- **Example IV** [4]: Fuzzy relation $\underset{\sim}{X} \circ \underset{\sim}{R} = \underset{\sim}{B}$, given

$$\underset{\sim}{R} = \begin{bmatrix} 0.5 & 0.2 & 0.8 & 0.1 \\ 0.8 & 0.2 & 0.8 & 0.1 \\ 0.9 & 0.1 & 0.4 & 0.1 \\ 0.3 & 0.95 & 0.1 & 0.1 \\ 0.85 & 0.1 & 0.1 & 0.1 \\ 0.4 & 0.8 & 0.1 & 0.0 \end{bmatrix} \tag{53}$$

$$\underset{\sim}{B} = [0.85 \quad 0.6 \quad 0.5 \quad 0.1] \tag{54}$$

$$M = \begin{bmatrix} 1.0 & 1.0 & 0.85 & 1.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 1.0 & 0.6 & 1.0 & 0.6 \\ 0.5 & 0.5 & 1.0 & 1.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \end{bmatrix} \tag{55}$$

$$MaxX = [0.5 \quad 0.5 \quad 0.85 \quad 0.6 \quad 1.0 \quad 0.6] \tag{56}$$

$$m = \begin{bmatrix} 0.0 & 0.0 & 0.85 & 0.0 & 0.85 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.6 & 0.0 & 0.6 \\ 0.5 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.0 \end{bmatrix} \tag{57}$$

There were no elements of (57) bigger than those in

(56), and no unique mapping pair, but there were four overlapped mapping pairs. So, we took the biggest pairs of column 3 and 5, and made intersections to these pairs. After that, the redundant elements were eliminated. The eliminated matrix is shown as (58),

$$m = \begin{bmatrix} 0.0 & 0.0 & 0.85 & 0.0 & 0.85 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.6 & 0.0 & 0.6 \\ 0.5 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \quad (58)$$

There were three linked together mapping pairs in; (58) they were then separated and intersections made to them. The minimum solution came out as (59).

$$\begin{aligned} MinX_1 &= [0.5 \quad 0.0 \quad 0.85 \quad 0.6 \quad 0.0 \quad 0.0] \\ MinX_2 &= [0.0 \quad 0.5 \quad 0.85 \quad 0.6 \quad 0.0 \quad 0.0] \\ MinX_3 &= [0.5 \quad 0.0 \quad 0.85 \quad 0.0 \quad 0.0 \quad 0.6] \\ MinX_4 &= [0.0 \quad 0.5 \quad 0.85 \quad 0.0 \quad 0.0 \quad 0.6] \\ MinX_5 &= [0.5 \quad 0.0 \quad 0.0 \quad 0.0 \quad 0.85 \quad 0.6] \\ MinX_6 &= [0.0 \quad 0.5 \quad 0.0 \quad 0.0 \quad 0.85 \quad 0.6] \\ MinX_7 &= [0.5 \quad 0.0 \quad 0.0 \quad 0.6 \quad 0.85 \quad 0.0] \\ MinX_8 &= [0.0 \quad 0.5 \quad 0.0 \quad 0.6 \quad 0.85 \quad 0.0] \end{aligned} \quad (59)$$

- **Example V**: Fuzzy relation $\underset{\sim}{X} \circ \underset{\sim}{R} = \underset{\sim}{B}$, given

$$\underset{\sim}{R} = \begin{bmatrix} 0.4 & 0.9 & 0.3 & 0.5 \\ 0.4 & 0.5 & 0.5 & 0.5 \\ 0.4 & 0.9 & 0.3 & 0.5 \\ 0.4 & 0.4 & 0.3 & 0.5 \\ 0.4 & 0.9 & 0.3 & 0.5 \end{bmatrix} \quad (60)$$

$$\underset{\sim}{B} = [0.4 \quad 0.9 \quad 0.3 \quad 0.5] \quad (61)$$

$$M = \begin{bmatrix} 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 1.0 & 0.3 & 1.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \end{bmatrix} \quad (62)$$

$$MaxX = [1.0 \quad 0.3 \quad 1.0 \quad 1.0 \quad 1.0] \quad (63)$$

$$m = \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.4 & 0.4 \\ 0.9 & 0.0 & 0.9 & 0.0 & 0.9 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix} \quad (64)$$

There were two elements of (64) bigger than those in (63). Then (64) was regulated as (65),

$$m = \begin{bmatrix} 0.4 & 0.0 & 0.4 & 0.4 & 0.4 \\ 0.9 & 0.0 & 0.9 & 0.0 & 0.9 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\ 0.5 & 0.0 & 0.5 & 0.5 & 0.5 \end{bmatrix} \quad (65)$$

Matrix ***m*** has no unique mapping status, but **m** has four overlapped mapping pairs. Also (65) has three biggest overlapped mapping pairs, so we took these biggest pairs to make the intersection operation, and eliminated the redundant elements. The eliminated matrix is shown as (66),

$$m = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.9 & 0.0 & 0.9 & 0.0 & 0.9 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \quad (66)$$

There was one linked together mapping pair in (66); they were separated and intersections made to them. The minimum solution came out as (68).

$$\begin{aligned} MinX_1 &= [0.9 \quad 0.0 \quad 0.0 \quad 0.0 \quad 0.0] \\ MinX_2 &= [0.0 \quad 0.0 \quad 0.9 \quad 0.0 \quad 0.0] \\ MinX_3 &= [0.0 \quad 0.0 \quad 0.0 \quad 0.0 \quad 0.9] \end{aligned} \quad (67)$$

- **Example VI** [6]: Fuzzy relation $\underset{\sim}{X} \circ \underset{\sim}{R} = \underset{\sim}{B}$, given

$$\underset{\sim}{R} = \begin{bmatrix} 1.0 & 0.9 & 0.8 & 0.6 & 0.4 & 0.3 & 0.1 \\ 0.4 & 0.7 & 0.7 & 0.4 & 0.4 & 0.2 & 0.2 \\ 0.6 & 0.8 & 0.7 & 0.5 & 0.3 & 0.3 & 0.2 \\ 0.5 & 0.4 & 0.7 & 0.4 & 0.3 & 0.2 & 0.1 \\ 0.9 & 0.8 & 0.6 & 0.3 & 0.1 & 0.3 & 0.2 \\ 0.8 & 0.9 & 0.6 & 0.5 & 0.1 & 0.3 & 0.1 \end{bmatrix} \quad (68)$$

$$\underset{\sim}{B} = [0.9 \quad 0.8 \quad 0.7 \quad 0.5 \quad 0.4 \quad 0.3 \quad 0.2] \quad (69)$$

$$M = \begin{bmatrix} 0.9 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 0.8 & 1.0 & 1.0 & 1.0 & 1.0 & 0.8 \\ 0.7 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 0.5 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \end{bmatrix} \quad (70)$$

$$MaxX = [0.5 \quad 1.0 \quad 1.0 \quad 1.0 \quad 1.0 \quad 0.8] \quad (71)$$

$$m = \begin{bmatrix} 0.9 & 0.0 & 0.0 & 0.0 & 0.9 & 0.0 \\ 0.8 & 0.0 & 0.8 & 0.0 & 0.8 & 0.8 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.5 & 0.0 & 0.0 & 0.5 \\ 0.4 & 0.4 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.3 & 0.0 & 0.3 & 0.0 & 0.3 & 0.3 \\ 0.0 & 0.2 & 0.2 & 0.0 & 0.2 & 0.0 \end{bmatrix} \quad (72)$$

There were three elements of (72) bigger than those in (71). Then (72) is regulated as (73),

$$m=\begin{bmatrix}0.0&0.0&0.0&0.0&0.9&0.0\\0.0&0.0&0.8&0.0&0.8&0.8\\0.0&0.7&0.7&0.7&0.0&0.0\\0.5&0.0&0.5&0.0&0.0&0.5\\0.4&0.4&0.0&0.0&0.0&0.0\\0.3&0.0&0.3&0.0&0.3&0.3\\0.0&0.2&0.2&0.0&0.2&0.0\end{bmatrix} \quad (73)$$

There was one unique mapping in (73). After taking the bigger one and eliminated by unique mapping analysis, (74) is shown as,

$$m=\begin{bmatrix}0.0&0.0&0.0&0.0&0.9&0.0\\0.0&0.0&0.0&0.0&0.0&0.0\\0.0&0.7&0.7&0.7&0.0&0.0\\0.5&0.0&0.5&0.0&0.0&0.5\\0.4&0.4&0.0&0.0&0.0&0.0\\0.0&0.0&0.0&0.0&0.0&0.0\\0.0&0.0&0.0&0.0&0.0&0.0\end{bmatrix} \quad (74)$$

There are three pairs of overlapped mapping in (74). After taking the biggest one and eliminating them by overlapped mapping analysis,

$$m=\begin{bmatrix}0.0&0.0&0.0&0.0&0.9&0.0\\0.0&0.0&0.0&0.0&0.0&0.0\\0.0&0.0&0.7&0.0&0.0&0.0\\0.0&0.0&0.0&0.0&0.0&0.5\\0.4&0.4&0.0&0.0&0.0&0.0\\0.0&0.0&0.0&0.0&0.0&0.0\\0.0&0.0&0.0&0.0&0.0&0.0\end{bmatrix} \quad (75)$$

There was one linked together mapping pair in (75), then separated them and made intersections to them. The minimum solution came out as (76).

$$\begin{aligned}MinX_1&=\begin{bmatrix}0.4&0.0&0.7&0.0&0.9&0.0\end{bmatrix}\\MinX_2&=\begin{bmatrix}0.0&0.4&0.7&0.0&0.9&0.0\end{bmatrix}\end{aligned} \quad (76)$$

♦ **Example VII**: Fuzzy relation $\underset{\sim}{X}\circ\underset{\sim}{R}=\underset{\sim}{B}$, given

$$\underset{\sim}{R}=\begin{bmatrix}0.5&0.8&1.0&0.8&0.7\\0.5&0.9&1.0&0.8&0.2\\0.4&0.9&1.0&0.6&0.7\\0.5&0.95&0.7&0.3&0.6\\0.45&0.6&1.0&0.8&0.7\end{bmatrix} \quad (77)$$

$$\underset{\sim}{B}=\begin{bmatrix}0.5&0.9&1.0&0.8&0.7\end{bmatrix} \quad (78)$$

$$M=\begin{bmatrix}1.0&1.0&1.0&1.0&1.0\\1.0&1.0&1.0&0.9&1.0\\1.0&1.0&1.0&1.0&1.0\\1.0&1.0&1.0&1.0&1.0\\1.0&1.0&1.0&1.0&1.0\end{bmatrix} \quad (79)$$

$$MaxX=\begin{bmatrix}1.0&1.0&1.0&0.9&1.0\end{bmatrix} \quad (80)$$

$$m=\begin{bmatrix}0.5&0.5&0.0&0.5&0.0\\0.0&0.9&0.9&0.9&0.0\\1.0&1.0&1.0&0.0&1.0\\0.8&0.8&0.0&0.0&0.8\\0.7&0.0&0.7&0.0&0.7\end{bmatrix} \quad (81)$$

There were no elements of (81) bigger than those in (80), and no unique mapping status, but there were five overlapped mapping pairs. So, the biggest pair in column 2 is taken to make intersections and eliminated the redundant elements of (81). The eliminated matrix is shown as (82),

$$m=\begin{bmatrix}0.0&0.0&0.0&0.0&0.0\\0.0&0.0&0.0&0.0&0.0\\0.0&1.0&0.0&0.0&0.0\\0.0&0.0&0.0&0.0&0.0\\0.7&0.0&0.7&0.0&0.7\end{bmatrix} \quad (82)$$

There was one linked together mapping pair in (82); then they were separated and intersections made to them. The minimum solution came out as (83).

$$\begin{aligned}MinX&=\begin{bmatrix}0.7&1.0&0.0&0.0&0.0\end{bmatrix}\\MinX&=\begin{bmatrix}0.0&1.0&0.7&0.0&0.0\end{bmatrix}\\MinX&=\begin{bmatrix}0.0&1.0&0.0&0.0&0.7\end{bmatrix}\end{aligned} \quad (83)$$

♦ **Example VIII**: Fuzzy relation $\underset{\sim}{X}\circ\underset{\sim}{R}=\underset{\sim}{B}$, given

$$\underset{\sim}{R}=\begin{bmatrix}0.5&0.4&1.0&0.3&0.3\\0.2&0.3&0.3&1.0&0.6\\0.7&0.6&0.3&0.0&0.5\\1.0&0.6&0.1&0.2&0.1\\0.2&0.1&0.9&0.8&0.7\end{bmatrix} \quad (84)$$

$$\underset{\sim}{B}=\begin{bmatrix}0.9&0.54&0.5&0.4&0.35\end{bmatrix} \quad (85)$$

$$M=\begin{bmatrix}1.0&1.0&1.0&0.9&1.0\\1.0&1.0&0.54&0.54&1.0\\0.5&1.0&1.0&1.0&0.5\\1.0&0.4&1.0&1.0&0.4\\1.0&0.35&0.35&1.0&0.35\end{bmatrix} \quad (86)$$

$$MaxX=\begin{bmatrix}0.5&0.35&0.35&0.54&0.35\end{bmatrix} \quad (87)$$

$$m=\begin{bmatrix}0.0 & 0.0 & 0.0 & 0.9 & 0.0\\ 0.0 & 0.0 & 0.54 & 0.54 & 0.0\\ 0.5 & 0.0 & 0.0 & 0.0 & 0.5\\ 0.0 & 0.4 & 0.0 & 0.0 & 0.4\\ 0.0 & 0.35 & 0.35 & 0.0 & 0.35\end{bmatrix} \quad (88)$$

There were four elements of (88) bigger than those in (87). The eliminated matrix is shown as (89),

$$m=\begin{bmatrix}0.0 & 0.0 & 0.0 & 0.0 & 0.0\\ 0.0 & 0.0 & 0.0 & 0.54 & 0.0\\ 0.5 & 0.0 & 0.0 & 0.0 & 0.0\\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0\\ 0.0 & 0.35 & 0.35 & 0.0 & 0.35\end{bmatrix} \quad (89)$$

Although in (89) there are two unique mappings, they need no redundant elimination. There was one linked together mapping pair in (89); then they were separated into individual representations. The minimum solution came out as (90).

$$\begin{aligned}MinX_1&=[0.5 \quad 0.35 \quad 0.0 \quad 0.54 \quad 0.0]\\ MinX_2&=[0.5 \quad 0.0 \quad 0.35 \quad 0.54 \quad 0.0]\\ MinX_3&=[0.5 \quad 0.0 \quad 0.0 \quad 0.54 \quad 0.35]\end{aligned} \quad (90)$$

♦ **Example IX** [2]: Fuzzy relation $\underset{\sim}{X}\circ\underset{\sim}{R}=\underset{\sim}{B}$, given

$$\underset{\sim}{R}=\begin{bmatrix}.3391 & .3682 & .6702 & .8195\\ .4757 & .3823 & .9954 & .6934\\ .4403 & .6001 & .6981 & .3742\\ .5857 & .4295 & .4027 & .0096\\ .4329 & .1488 & .8493 & .7798\end{bmatrix} \quad (91)$$

$$\underset{\sim}{B}=[.5456 \quad .5244 \quad .8987 \quad .7544] \quad (92)$$

$$M=\begin{bmatrix}1.0 & 1.0 & 1.0 & .5456 & 1.0\\ 1.0 & 1.0 & .5244 & 1.0 & 1.0\\ 1.0 & .8987 & 1.0 & 1.0 & 1.0\\ .7544 & 1.0 & 1.0 & 1.0 & .7544\end{bmatrix} \quad (93)$$

$$MaxX=[.7544 \quad .8987 \quad .5244 \quad .5456 \quad .7544] \quad (94)$$

$$m=\begin{bmatrix}0 & 0 & 0 & .5456 & 0\\ 0 & 0 & .5244 & 0 & 0\\ 0 & .8987 & 0 & 0 & 0\\ .7544 & 0 & 0 & 0 & .7544\end{bmatrix} \quad (95)$$

There was no element of (95) bigger than those in (94), no unique mapping status, and no overlapped mapping pairs. But (95) has one linked together mapping pair; they were then separated them and intersections made to them. The minimum solution came out as (96).

$$\begin{aligned}MinX_1&=[.7544 \quad .8987 \quad .5244 \quad .5456 \quad 0]\\ MinX_2&=[0 \quad .8987 \quad .5244 \quad .5456 \quad .7544]\end{aligned} \quad (96)$$

♦ **Example X** [12]: Fuzzy relation $\underset{\sim}{X}\circ\underset{\sim}{R}=\underset{\sim}{B}$, given

$$\underset{\sim}{R}=\begin{bmatrix}0.5 & 0.7 & 0.0 & 0.0\\ 0.4 & 0.6 & 0.4 & 0.0\\ 0.2 & 0.4 & 0.5 & 0.6\\ 0.1 & 0.2 & 0.0 & 0.8\end{bmatrix} \quad (97)$$

$$\underset{\sim}{B}=[0.5 \quad 0.5 \quad 0.4 \quad 0.0] \quad (98)$$

$$M=\begin{bmatrix}1.0 & 1.0 & 1.0 & 1.0\\ 0.5 & 0.5 & 1.0 & 1.0\\ 1.0 & 1.0 & 0.4 & 1.0\\ 1.0 & 1.0 & 0.0 & 0.0\end{bmatrix} \quad (99)$$

$$MaxX=[0.5 \quad 0.5 \quad 0.0 \quad 0.0] \quad (100)$$

$$m=\begin{bmatrix}0.5 & 0.0 & 0.0 & 0.0\\ 0.5 & 0.5 & 0.0 & 0.0\\ 0.0 & 0.4 & 0.4 & 0.0\\ 0.0 & 0.0 & 0.0 & 0.0\end{bmatrix} \quad (101)$$

There was one element of (101) bigger than (100) those in (100). The eliminated matrix is shown as (102),

$$m=\begin{bmatrix}0.5 & 0.0 & 0.0 & 0.0\\ 0.5 & 0.5 & 0.0 & 0.0\\ 0.0 & 0.4 & 0.0 & 0.0\\ 0.0 & 0.0 & 0.0 & 0.0\end{bmatrix} \quad (102)$$

There was one unique mapping status in (102). After elimination by unique mapping analysis,

$$m=\begin{bmatrix}0.5 & 0.0 & 0.0 & 0.0\\ 0.0 & 0.0 & 0.0 & 0.0\\ 0.0 & 0.4 & 0.0 & 0.0\\ 0.0 & 0.0 & 0.0 & 0.0\end{bmatrix} \quad (103)$$

There was no linked together mapping pair in (103) and the intersection was made; the minimum solution comes out as (104).

$$MinX=[0.5 \quad 0.4 \quad 0.0 \quad 0.0] \quad (104)$$

The above examples are including many cases of decomposition that proposed in the other papers. They could summarize in a compact table as below.

Table 2. The Summarization of Illustrate Examples.

Number / Example	Bigger Element	Unique Mapping	Overlap. Mapping: Mapping Pair	Overlap. Mapping: Biggest Pair	Together Mapping	Minimum Solution
1	-	2	-	-	-	1
2	-	-	3	3	3×1	3
3	3	1	-	-	2×2	4
4	-	-	4	2	2×2×2	8
5	2	-	4	3	3×1	3
6	3	1	3	1	2×1	2
7	2	-	5	1	3×1	3
8	4	2	-	-	3×1	3
9	-	-	-	-	2×1	2
10	1	2	-	-	-	1

7. Conclusions

By the algorithm of logical mapping analysis, the new method proposed by this paper will find the solution of fuzzy relation equations, not only the unique maximum solution but also the real, and every minimum solution. The performance of the new method, compared to the series examples in papers employing traditional methods, has been clearly confirmed.

The major operation of the new method is intersecting, so it will just need small computing quantity, and is easy to program for web analysis, even for mobile phone analyzing usage. Besides, as in the example of fuzzy relation equation, the new method could be used for solving the input vector, as well as the relation matrix.

Because of the necessity character of minimum solution, this method could be used for solving the approximated optimization of nonlinear problems. Such situation happens frequently in social science field studies.

Appendix

A. *Application for Web-page Analysis*

This study accomplished a web page for decomposition analysis at the Electric Laboratory of National Taiwan Normal University. The web page provides some enhanced functions such as the setting of row, column, item names, picture upload for analysis, Delphi interaction analysis, etc. Besides, this study invited 31 skill profession teachers of industrial vocational high schools to assess the weighting analysis on website. The decomposition web page is shown in Figure 20 [13].

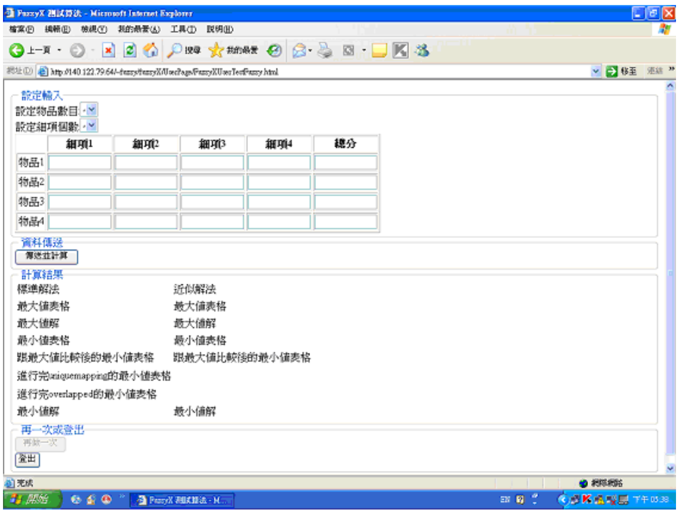

Figure 20. Web page for fuzzy decomposition.

B. *Application for Cell Phone Analysis*

This study also accomplished a program (mobile 6) for cell phone analysis usage. It attested to the fact that the proposed method has the desired performance abilities: small computing quantity and simple to program for computer usage. Here is an example of fuzzy decomposition using mobile phone, as shown in Figure 21.

C. *Application for Skill Assessment Factor Analysis*

An expert teacher in regard to the assessment of a new skill, can always determine a suitable total score without consideration of every sub-item. Besides, while performing the skill assessment in school, there was always the formal weighting arrangement of every sub-item. Is the arrangement suitable? The new method proposed in this paper could be used as the technique for analyzing the arrangement of weighting [10].

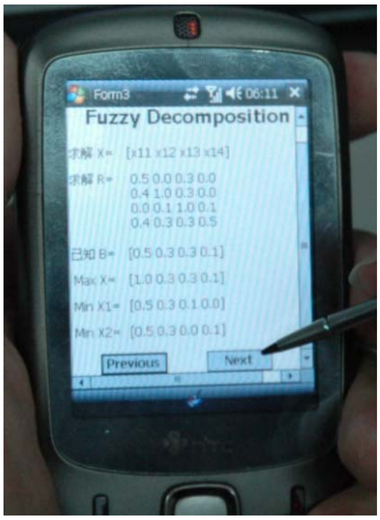

Figure 21. Fuzzy decomposition using mobile phone.

This method could probe the weighting arrangement of expert's cognizance. And the method could also be used for self-analysis by inexperienced teachers, or even to evaluate whether a teacher is an expert or a beginner.

D. *Application for Credit Evaluation Factor Analysis*

In social science research, nonlinear relation equation is an ongoing requirement. For example, the method of Analytic Hierarchy Process (AHP) is always used for determining the management factors, such as the 3F of financial factor, economic factor, and management factor. While examining the weighting of each factor, nonlinear equations are frequently employed. The new method could used to determine the possible region of every evaluation factor.

E. *Programming Sample*

The following lists some programming samples of the major computing procedure using MTHLAB language.

```
R=[ 0.1   0.4   0.5   0.1
    0.9   0.7   0.2   0.0
    0.8   1.0   0.5   0.0
    0.1   0.3   0.6   0.0   ];
B=[ 0.8   0.7   0.5   0.0   ];
R=R'; B=B'; c=size(R);
% create critical matrix M & m
for j=1 : c(2);
for i=1 : c(1);
    if R(i,j)>B(i);   M(i,j)=B(i); m(i,j)=B(i); end;
    if R(i,j)==B(i); M(i,j)=1; m(i,j)=B(i); end;
    if R(i,j)<B(i);   M(i,j)=1; m(i,j)=0; end;
end;
    MaxV(j)=min(M(:,j));   %     Maxv solved
end;
% make matrix FD & only for assist solving unique status
for j=1 : c(2);
for i=1 : c(1);
    if m(i,j)>MaxV(j); m(i,j)=0; end;
    if m(i,j)>0; FD(i,j)=1; else FD(i,j)=0; end;
end;
end;
for i=1 : c(1);
for j=1 : c(2);
    if FD(i,j)==1 & sum(FD(i,:)==1; only(j)=m(i,j); end;
end;
end;
% detect & regulate critical matrix m by "Unique Status"
MinV=MaxV-MaxV;
for p=1 : c(2);
for i=1 : c(1);
for j=1 : c(2);
    if m(i,j)==max(only) & FD(i,j)==1 &
sum(FD(i,: ))==1;
    MinV(j)=max(m(:,j));
    for a=1 : c(1);
        if a~=i & m(a,j)>0
        for d=1 : c(2); m(a,d)=0; FD(a,d)=0; end;
        end;
    end;
    FD(i,j)=0;
    only(j)=0;
    end;
end;
end;
end;
% Designed by Huan-Wen Tzeng %
```

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References

[1] G. j. Klir and B. Yuan, "*Fuzzy Sets and Fuzzy Logic : Theory and Applications*," International Ed., Pearson Education Taiwan, Taipei, Taiwan, 2005.

[2] J. Lu and S. C. Fang, "Solving Nonlinear Optimization Problems With Fuzzy Relation Equation Constraints," *Fuzzy Sets Syst*., vol. 119, issue 1, pp. 1-20, 2001.

[3] H. J. Zimmermann, "*Fuzzy Set Theory And Its Application*," Second Edition, Kluwer Academic Publishers, Boston, 1991.

[4] W. F. Wu, "*Fuzzy Mathematics and Application*," Electronic Industry Press, Beijing, China, 1986.

[5] Y. K. Wu, S. M. Guu, and Y. C. Liu, "An Accelerated Approach for Solving Fuzzy Relation Equations With a Linear Objective Function," *IEEE Trans. on Fuzzy System*, vol. 10, no. 4, pp. 552-558, 2002.

[6] L. Luoh, W. J. Wang, and Y. K. Liaw, "Matrix-Pattern-Based Computer Algorithm for Solving Fuzzy Relation Equations," *IEEE Trans. on Fuzzy System*, vol. 11, no. 1, pp. 552-558, 2003.

[7] Y. Tsukamoto, "An approach to fuzzy reasoning method," *Advances in fuzzy set theory and applications*, pp. 137-149, 1973.

[8] H. W. Tzeng, "Fuzzy Decomposition Method by the Elimination Analysis of Freedom," In *Proc. of IEEE International Conference on System Man & Cybernetic*, pp. 818-822, 2007.

[9] H. W. Tzeng, "Using Sagittal Mapping Analysis for Fuzzy Decomposition," In *Proc. of IEEE International Conference on Granular Computing*, pp. 597-601, 2008.

[10] H. W. Tzeng, "The Fuzzy Decomposition Technique for Weighting Analysis of Skill Assessment," In *Proc. of 39th ASEE/IEEE Frontiers in Education Conference*, pp. T3C1-T3C7, 2009.

[11] F. C. Tang, "Perturbation Techniques for Fuzzy Matrix Equations," *Fuzzy Sets Syst*., vol. 109, issue 3, pp. 363-369, 2000.

[12] L. Luoh, W. J. Wang, and Y. K. Liaw, "New Algorithm for Solving Fuzzy Relation Equations," *Mathematics and Computers in Simulation*, vol. 59, issue 4, pp. 329-333, 2002.

[13] http://140.122.79.64/~fuzzy/fuzzyX/UserPage/FuzzyXUserLogin.html

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