

# The Homotopy analysis method for solving fuzzy system of linear equations

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## Abstract

In this paper, the Homotopy analysis method has been applied for solving Fuzzy system of linear equations (FSLE). Comparisons of the results obtained by the Homotopy analysis method (HAM) with that obtained by the Adomian decomposition method (ADM) and Jacobi iterative method suggest that both the ADM and Jacobi iterative method are special case of the HAM. The results reveal that HAM is very simple and effective. The numerical solutions are compared with the known analytical and some numerical solutions.

**Keywords:** Fuzzy system of linear equations; Homotopy analysis method; Adomian decomposition method.

## 1. Introduction

There is an inexhaustible supply of applications of linear systems. Many real-world engineering systems are too complex to be defined in precise terms, uncertainty is often involved in any engineering design process. Fuzzy sets provide a widely appreciated tool to introduce uncertain parameters into mathematical applications. In many applications, at least some of the parameters of the system should be represented by fuzzy rather than crisp numbers. Thus, it is immensely important to develop numerical procedures that would appropriately treat fuzzy linear systems and solve them. A general model for solving a fuzzy linear system whose coefficient matrix is crisp and its right column is an arbitrary fuzzy vector was first proposed by Friedman et al. [8]. Also, some iterative techniques are extended on the same system in Friman, [7, 18, 19]. Allahviranloo has proposed the ADM [2] for solving a fuzzy linear system with crisp coefficient matrix and the fuzzy right-hand side vector [19].

In 1992, Liao employed the basic ideas of the homotopy in topology to propose a general analytic method for linear and nonlinear problems, namely Homotopy analysis method (HAM), [10].

This method has been successfully applied to solve

many types of linear and nonlinear problems [6, 11, 12, 15, 16]. The HAM offers certain advantages over routine numerical methods. The HAM is better since it does not involve discretization of the variables hence is free from rounding off errors and does not require large computer memory or time. In this paper we focus on the use of the Homotopy analysis method for solving the FSLE.

## 2. Preliminaries

The concept of fuzzy numbers and arithmetic operations with this numbers were first introduced and investigated by Zadeh [13] and others. All of which observed the fuzzy number as a collection of  $\alpha$ -levels,  $0 \leq \alpha \leq 1$ , [1]. In this section the most basic notations used in fuzzy calculus are introduced.

*Definition 1.* A fuzzy number is a fuzzy map  $u : R^1 \rightarrow I = [0,1]$  which satisfies

- i.  $u$  is upper semi continuous.
- ii.  $u(x) = 0$  outside some interval  $[c, d]$ .
- iii. There exist real number  $a, b$  such that  $c \leq a \leq b \leq d$ , where
  1.  $u(x)$  is monotonic increasing on  $[c, d]$ .
  2.  $u(x)$  is monotonic decreasing on  $[b, d]$ .
  3.  $u(x) = 1, a \leq x \leq b$ .

The set of all fuzzy numbers is denoted by  $E^1$ . An alternative definition or parametric form of a fuzzy number which yields the same  $E^1$  is given by Kaleva [9] as

*Definition 2.* A fuzzy number  $u$  is a pair of functions  $(\underline{u}(r), \bar{u}(r))$ ,  $0 \leq r \leq 1$  which satisfy the following requirements:

- i.  $\underline{u}(r)$  is a bounded left-continuous non-decreasing function over  $[0,1]$ .
- ii.  $\bar{u}(r)$  is a bounded left-continuous non-increasing function over  $[0,1]$ .
- iii.  $\underline{u}(r) \leq \bar{u}(r)$ ,  $0 \leq r \leq 1$ .

By appropriate definitions the fuzzy numbers space  $\{\underline{u}(r), \bar{u}(r)\}$  becomes a convex cone  $E^1$  which is then embedded isomorphically and isometrically in to a Banach space.

*Definition 3.* The  $n \times n$  linear system of equations

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$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = y_2 \\ \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = y_n \end{cases} \quad (1)$$

where the coefficient matrix  $A = a_{ij}$ ,  $1 \leq i, j \leq n$  is a crisp  $n \times n$  matrix and  $y_i \in E^1$ ,  $1 \leq i \leq n$ , is called a fuzzy system of linear equations (FSLE).

A solution  $(x_1, x_2, \dots, x_n)^t$  to (1) one should recall that for arbitrary fuzzy numbers  $x = (\underline{x}, \bar{x})$ ,  $y = (\underline{y}, \bar{y})$  and real number  $k$ ,

1.  $x = y$  if and only if  $\underline{x}(r) = \underline{y}(r)$  and  $\bar{x}(r) = \bar{y}(r)$ .
2.  $x + y = (\underline{x}(r) + \underline{y}(r), \bar{x}(r) + \bar{y}(r))$ .
3.  $k(x) = \begin{cases} (k\underline{x}, k\bar{x}), k \geq 0 \\ (k\bar{x}, k\underline{x}), k < 0 \end{cases}$

**Definition 4.** [8] A fuzzy number vector  $(x_1, x_2, \dots, x_n)^t$  given by  $x = (\underline{x}_i(r), \bar{x}_i(r))$ ,  $1 \leq i \leq n$ ,  $0 \leq r \leq 1$ , is called a solution of the FSLE if

$$\min \left\{ \sum_{j=1}^n a_{ij} u_j \mid u_j \in [x_j]_r \right\} = \sum_{j=1}^n a_{ij} x_j = \sum_{j=1}^n \underline{a_{ij} x_j} = \underline{y}_j,$$

$$\min \left\{ \sum_{j=1}^n a_{ij} u_j \mid u_j \in [x_j]_r \right\} = \sum_{j=1}^n a_{ij} x_j = \sum_{j=1}^n \overline{a_{ij} x_j} = \bar{y}_j.$$

Consider the  $i$ th equation of the system (1):

$$a_{i1}(\underline{x}_1, \bar{x}_1) + \dots + a_{in}(\underline{x}_n, \bar{x}_n) = (\underline{y}_i(r), \bar{y}_i(r))$$

We have

$$\begin{aligned} \underline{a_{i1}x_1} + \dots + \underline{a_{in}x_n} &= \underline{y}_i, \\ \overline{a_{i1}x_1} + \dots + \overline{a_{in}x_n} &= \bar{y}_i, \quad 1 \leq i \leq n, \quad 0 \leq r \leq 1. \end{aligned} \quad (2)$$

From (2) we have two crisp  $n \times n$  linear systems for all  $i$  that there can be extended to a  $2n \times 2n$  crisp linear system as follows:

$$SX = Y \rightarrow \begin{bmatrix} S_1 \geq 0 & S_2 \leq 0 \\ S_2 \leq 0 & S_1 \geq 0 \end{bmatrix} \begin{bmatrix} \underline{X} \\ \bar{X} \end{bmatrix} = \begin{bmatrix} \underline{Y} \\ \bar{Y} \end{bmatrix}. \quad (3)$$

Thus FSLE(1) is extended to a crisp (3) where  $A = S_1 + S_2$ . Equation (3) can be written as follows:

$$\begin{cases} S_1 \underline{X} + S_2 \bar{X} = \underline{Y}, \\ S_2 \underline{X} + S_1 \bar{X} = \bar{Y}. \end{cases}$$

**Theorem 1.** [8] The matrix  $S$  is nonsingular if and only if the matrices  $A = S_1 + S_2$  and  $S_1 - S_2$  are both nonsingular.

**Definition 5.** Let  $X = \{(\underline{x}_i(r), \bar{x}_i(r)) \mid 1 \leq i \leq n\}$  denote the unique solution of  $SX = Y$ . The fuzzy number vector

$$U = \{(\underline{u}_i(r), \bar{u}_i(r)) \mid 1 \leq i \leq n\} \text{ defined by}$$

$$\begin{aligned} \underline{u}_i(r) &= \min\{\underline{x}_i(r), \bar{x}_i(r), \underline{x}_1\}, \\ \bar{u}_i(r) &= \max\{\underline{x}_i(r), \bar{x}_i(r), \underline{x}_1\} \end{aligned}$$

is called the fuzzy solution of  $SX = Y$ . If  $x = (\underline{x}_i(r), \bar{x}_i(r))$ ,  $1 \leq i \leq n$ , are all triangular fuzzy numbers then  $\underline{u}_i(r) = \underline{x}_i(r)$ ,  $\bar{u}_i(r) = \bar{x}_i(r)$ ,  $1 \leq i \leq n$ , and  $U$  is called a strong fuzzy solution. Otherwise,  $U$  is a weak fuzzy solution. See [8].

**Theorem 2.** [19] Let  $S$  be nonsingular. Then the unique solution  $X$  of (5) is always a fuzzy vector for arbitrary vector  $Y$ , if  $S^{-1}$  is nonnegative.

**Theorem 3.** [19] The matrix  $A$  with  $a_{ij} > 0$  in (1) is strictly diagonally dominant if and only if matrix  $S$  be strictly diagonally dominant.

From [5], without loss of generality, suppose that  $S_{ij} > 0$  for all  $i = 1, 2, \dots, 2n$ . Let  $S = D + L + U$  where

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_1 \end{bmatrix}, L = \begin{bmatrix} L_1 & 0 \\ S_2 & L_1 \end{bmatrix}, U = \begin{bmatrix} U_1 & S_2 \\ 0 & U_1 \end{bmatrix}$$

$(D_1)_{ii} = S_{ii}$ ,  $i = 1, \dots, n$  and suppose  $S_1 = D_1 + L_1 + U_1$ .

From  $SX = Y$ , we have

$$\begin{bmatrix} D_1 & 0 \\ 0 & D_1 \end{bmatrix} \begin{bmatrix} \underline{X} \\ \bar{X} \end{bmatrix} + \begin{bmatrix} L_1 + U_1 & S_2 \\ S_2 & L_1 + U_1 \end{bmatrix} \begin{bmatrix} \underline{X} \\ \bar{X} \end{bmatrix} = \begin{bmatrix} \underline{Y} \\ \bar{Y} \end{bmatrix}$$

Then

$$\begin{aligned} \underline{X} &= D_1^{-1} \underline{Y} - D_1^{-1} (L_1 + U_1) \underline{X} - D_1^{-1} S_2 \bar{X}, \\ \bar{X} &= D_1^{-1} \bar{Y} - D_1^{-1} (L_1 + U_1) \bar{X} - D_1^{-1} S_2 \underline{X}. \end{aligned}$$

So the Jacobi iterative technique will be

$$\begin{aligned} \underline{X}^{k+1} &= D_1^{-1} \underline{Y} - D_1^{-1} (L_1 + U_1) \underline{X}^k - D_1^{-1} S_2 \bar{X}^k, \\ \bar{X}^{k+1} &= D_1^{-1} \bar{Y} - D_1^{-1} (L_1 + U_1) \bar{X}^k - D_1^{-1} S_2 \underline{X}^k. \end{aligned}$$

The elements of  $X^{k+1} = (\underline{X}^{k+1}, \bar{X}^{k+1})^t$  are

$$\begin{aligned} \underline{x}_i^{k+1}(r) &= \frac{1}{s_{i,j}} \left[ \underline{y}_i(r) - \sum_{j=1, j \neq i}^n s_{i,j} \underline{x}_j^k(r) - \sum_{j=1}^n s_{i,n+j} \bar{x}_j^k(r) \right], \\ \bar{x}_i^{k+1}(r) &= \frac{1}{s_{i,j}} \left[ \bar{y}_i(r) - \sum_{j=1, j \neq i}^n s_{i,j} \bar{x}_j^k(r) - \sum_{j=1}^n s_{i,n+j} \underline{x}_j^k(r) \right]. \end{aligned} \quad (4)$$

$$k = 0, 1, 2, \dots \quad i = 1, \dots, n$$

The stopping criterion with tolerance  $\varepsilon > 0$  is

$$\frac{\|\bar{X}^{k+1} - \bar{X}^k\|}{\|\bar{X}^{k+1}\|} < \varepsilon, \frac{\|\underline{X}^{k+1} - \underline{X}^k\|}{\|\underline{X}^{k+1}\|} < \varepsilon, k = 0, 1, \dots$$

### 3. Homotopy analysis method for fuzzy system of linear equations

In this section we apply the HAM to the fuzzy system of linear equations. We consider (4) for any fixed  $r \in [0, 1]$

we have

$$\begin{aligned} \underline{x}_i &= \frac{y_i}{s_{i,j}} - \frac{1}{s_{i,j}} \left[ \sum_{j=1, j \neq i}^n s_{i,j} \underline{x}_j - \sum_{j=1}^n s_{i,n+j} \bar{x}_j \right], \\ \bar{x}_i &= \frac{\bar{y}_i}{s_{i,j}} - \frac{1}{s_{i,j}} \left[ \sum_{j=1, j \neq i}^n s_{i,j} \bar{x}_j - \sum_{j=1}^n s_{i,n+j} \underline{x}_j \right], \end{aligned} \quad (5)$$

To use Homotopy analysis method, let

$$\underline{L}_i(\cdot) = \bar{L}_i(\cdot) = (\cdot), \quad i = 1, 2, \dots, n.$$

are linear operators, and

$$\begin{aligned} \underline{N}_i &= \underline{\phi}_i(r; p) - \frac{y_i}{s_{i,j}} + \frac{1}{s_{i,j}} \left[ \sum_{j=1, j \neq i}^n s_{i,j} \underline{\phi}_j(r; p) - \sum_{j=1}^n s_{i,n+j} \bar{\phi}_j(r; p) \right], \\ \bar{N}_i &= \bar{\phi}_i(r; p) - \frac{\bar{y}_i}{s_{i,j}} + \frac{1}{s_{i,j}} \left[ \sum_{j=1, j \neq i}^n s_{i,j} \bar{\phi}_j(r; p) - \sum_{j=1}^n s_{i,n+j} \underline{\phi}_j(r; p) \right], \end{aligned} \quad i = 1, 2, \dots, n.$$

are nonlinear operators.

By means of generalizing the traditional homotopy method, Liao [10] constructs the so-called zero-order deformation equation

$$\begin{aligned} (1-p)\underline{L}_i[\underline{\phi}_i(r; p) - \underline{x}_{i,0}(r)] &= p \underline{h}_i \underline{H}_i(r) \underline{N}_i[\underline{\phi}_i(r; p), \bar{\phi}_i(r; p)], \\ (1-p)\bar{L}_i[\bar{\phi}_i(r; p) - \bar{x}_{i,0}(r)] &= p \bar{h}_i \bar{H}_i(r) \bar{N}_i[\underline{\phi}_i(r; p), \bar{\phi}_i(r; p)], \end{aligned} \quad i = 1, \dots, n. \quad (6)$$

where  $p \in [0,1]$  is the embedding parameter,  $\underline{h}_i \neq 0, \bar{h}_i \neq 0$  are non-zero auxiliary parameters,  $\underline{H}_i(r) \neq 0, \bar{H}_i(r) \neq 0$  are auxiliary functions,  $\underline{L}_i$  and  $\bar{L}_i$  are auxiliary linear operators,  $\underline{x}_{i,0}(r)$  and  $\bar{x}_{i,0}(r)$  are initial guesses of  $\underline{x}_i(r)$  and  $\bar{x}_i(r)$ ,  $\underline{\phi}_i(r; p)$  and  $\bar{\phi}_i(r; p)$  are unknown functions, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when  $p = 0$  and  $p = 1$ , it holds

$$\begin{aligned} \underline{\phi}_i(r; 0) &= \underline{x}_{i,0}(r), \underline{\phi}_i(r; 1) = \underline{x}_i(r), \\ \bar{\phi}_i(r; 0) &= \bar{x}_{i,0}(r), \bar{\phi}_i(r; 1) = \bar{x}_i(r). \end{aligned}$$

respectively. Thus, as  $p$  increases from 0 to 1, the solutions  $\underline{\phi}_i(r; p)$  and  $\bar{\phi}_i(r; p)$  varies from the initial guesses  $\underline{x}_{i,0}(r)$  and  $\bar{x}_{i,0}(r)$  to the solutions  $\underline{x}_i(r)$  and  $\bar{x}_i(r)$ , respectively. Expanding  $\underline{\phi}_i(r; p)$  and  $\bar{\phi}_i(r; p)$  in Taylor series with respect to  $p$ , we have

$$\underline{\phi}_i(r; p) = \underline{x}_{i,0}(r) + \sum_{m=1}^{+\infty} \underline{x}_{i,m}(r) p^m, \quad \bar{\phi}_i(r; p) = \bar{x}_{i,0}(r) + \sum_{m=1}^{+\infty} \bar{x}_{i,m}(r) p^m \quad (7)$$

where

$$\underline{x}_{i,m}(r) = \frac{1}{m!} \frac{\partial^m \underline{\phi}_i(r; p)}{\partial p^m} \Big|_{p=0}, \quad \bar{x}_{i,m}(r) = \frac{1}{m!} \frac{\partial^m \bar{\phi}_i(r; p)}{\partial p^m} \Big|_{p=0}. \quad (8)$$

If the auxiliary linear operators, the initial guesses, the auxiliary parameters, and the auxiliary function are so properly chosen, the series (7) converges at  $p = 1$ , then we have

$$\underline{x}_i = \underline{x}_{i,0}(r) + \sum_{m=1}^{+\infty} \underline{x}_{i,m}(r), \quad \bar{x}_i = \bar{x}_{i,0}(r) + \sum_{m=1}^{+\infty} \bar{x}_{i,m}(r). \quad (9)$$

which must be one of solutions of original linear equation, as proved by [16].

As  $\underline{h}_i = \bar{h}_i = -1$  and  $\underline{H}_i(r) = \bar{H}_i(r) = 1$ , (6) becomes

$$\begin{aligned} (1-p)\underline{L}_i[\underline{\phi}_i(r; p) - \underline{x}_{i,0}(r)] + p \underline{N}_i[\underline{\phi}_i(r; p), \bar{\phi}_i(r; p)] &= 0, \\ (1-p)\bar{L}_i[\bar{\phi}_i(r; p) - \bar{x}_{i,0}(r)] + p \bar{N}_i[\underline{\phi}_i(r; p), \bar{\phi}_i(r; p)] &= 0, \end{aligned} \quad i = 1, \dots, n.$$

which is used mostly in the homotopy perturbation method [4], where as the solution obtained directly, without using Taylor series [3,5]. According to the definitions (8), the governing equation can be deduced from the zero-order deformation equations (6). Define the vectors

$$\begin{aligned} \vec{\underline{x}}_{i,n} &= \{ \underline{x}_{i,0}(r), \underline{x}_{i,1}(r), \dots, \underline{x}_{i,n}(r) \}, \\ \vec{\bar{x}}_{i,n} &= \{ \bar{x}_{i,0}(r), \bar{x}_{i,1}(r), \dots, \bar{x}_{i,n}(r) \}. \end{aligned}$$

Differentiating equations (6)  $m$  times with respect to the embedding parameter  $p$  and then setting  $p = 0$  and finally dividing them by  $m!$  we have the so-called  $m$ th-order deformation equations

$$\begin{aligned} \underline{L}_i[\underline{x}_{i,m}(r) - \chi_m \underline{x}_{i,m-1}(r)] &= \underline{h}_i \underline{H}_i(r) \underline{R}_{i,m}(\vec{\underline{x}}_{i,m-1}), \\ \bar{L}_i[\bar{x}_{i,m}(r) - \chi_m \bar{x}_{i,m-1}(r)] &= \bar{h}_i \bar{H}_i(r) \bar{R}_{i,m}(\vec{\bar{x}}_{i,m-1}). \end{aligned}$$

where

$$\begin{aligned} \underline{R}_{i,m}(\vec{\underline{x}}_{i,m-1}) &= \frac{1}{(m-1)!} \frac{\partial^{m-1} \underline{N}_i[\underline{\phi}_i(r; p), \bar{\phi}_i(r; p)]}{\partial p^{m-1}} \Big|_{p=0}, \\ \bar{R}_{i,m}(\vec{\bar{x}}_{i,m-1}) &= \frac{1}{(m-1)!} \frac{\partial^{m-1} \bar{N}_i[\underline{\phi}_i(r; p), \bar{\phi}_i(r; p)]}{\partial p^{m-1}} \Big|_{p=0}. \end{aligned}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

Thus, if  $\underline{H}_i(r) = \bar{H}_i(r) = 1$  we have

$$\begin{aligned} \underline{x}_{i,m}(r) &= \chi_m \underline{x}_{i,m-1}(r) + \underline{h}_i \underline{R}_{i,m}(\vec{\underline{x}}_{i,m-1}), \\ \bar{x}_{i,m}(r) &= \chi_m \bar{x}_{i,m-1}(r) + \bar{h}_i \bar{R}_{i,m}(\vec{\bar{x}}_{i,m-1}). \end{aligned} \quad (10)$$

where

$$\begin{aligned} \underline{R}_{i,m}(\vec{\underline{x}}_{i,m-1}) &= \underline{x}_{i,m}(r) - (1-\chi_m) \frac{y_i}{s_{i,j}} + \frac{1}{s_{i,j}} \left[ \sum_{j=1, j \neq i}^n s_{i,j} \underline{x}_{j,m} - \sum_{j=1}^n s_{i,n+j} \bar{x}_{j,m} \right], \\ \bar{R}_{i,m}(\vec{\bar{x}}_{i,m-1}) &= \bar{x}_{i,m}(r) - (1-\chi_m) \frac{\bar{y}_i}{s_{i,j}} + \frac{1}{s_{i,j}} \left[ \sum_{j=1, j \neq i}^n s_{i,j} \bar{x}_{j,m} - \sum_{j=1}^n s_{i,n+j} \underline{x}_{j,m} \right], \end{aligned}$$

$i = 1, \dots, n, m = 1, 2, \dots$

The initial values for the Homotopy analysis method are as follow  $\underline{X}^0 = D_1^{-1} \underline{Y}, \bar{X}^0 = D_1^{-1} \bar{Y}$  [3]. For the convergence of the above method we refer the reader to Liao's work [10]. If (5) admits unique solution, then this method will produce the unique solution. If (5) does not possess unique solution, the HAM will give a solution among many other (possible) solutions.

### 4. Numerical results

To illustrate the method and to show the ability of the method two examples are presented.

*Example 1.* Consider the  $2 \times 2$  fuzzy system [19]

$$\begin{cases} x_1 - x_2 = (r, 2 - r), \\ x_1 + 3x_2 = (4 + r, 7 - 2r). \end{cases} \quad (11)$$

The extended  $4 \times 4$  matrix is

$$S = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 3 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The exact solutions from  $X = S^{-1}Y$  are

$$x_1 = (\underline{x}_1(r), \bar{x}_1(r)) = (1.375 + 0.625r, 2.875 - 0.875r),$$

$$x_2 = (\underline{x}_2(r), \bar{x}_2(r)) = (0.875 + 0.125r, 1.375 - 0.375r).$$

Using (10), the first few terms off HAM series (9) are:

$$\begin{cases} \underline{x}_{1,0}(r) = r, \\ \underline{x}_{1,1}(r) = \frac{h_1}{3}(-7 + 2r), \\ \underline{x}_{1,2}(r) = \frac{h_1}{3}(-7 + 2r - 7h_1 + 2h_1r - 2h_2 + h_2r), \end{cases}$$

$$\begin{cases} \bar{x}_{1,0}(r) = 2 - r, \\ \bar{x}_{1,1}(r) = \frac{\bar{h}_1}{3}(-4 - r), \\ \bar{x}_{1,2}(r) = -\frac{\bar{h}_1}{3}(4 + r + 4\bar{h}_1 + \bar{h}_1r + h_2r + h_2r), \end{cases}$$

$$\begin{cases} \underline{x}_{2,0}(r) = \frac{1}{3}(1 + r), \\ \underline{x}_{2,1}(r) = \frac{h_2}{3}r, \\ \underline{x}_{2,2}(r) = \frac{h_2}{9}(3r + 3h_2r - 7h_1 + 2h_1r), \end{cases}$$

$$\begin{cases} \bar{x}_{2,0}(r) = \frac{1}{3}(7 - 2r), \\ \bar{x}_{2,1}(r) = \frac{\bar{h}_2}{3}(2 - r), \\ \bar{x}_{2,2}(r) = -\frac{\bar{h}_2}{9}(-6 + 3r - 6\bar{h}_2 + 3\bar{h}_2r + 4\bar{h}_1 + \bar{h}_1r). \end{cases}$$

and so on. We obtained approximate solutions with chosen  $h_i = \bar{h}_i = -0.8, (i = 1, 2)$

$$\begin{cases} \underline{x}_1(r) \approx \sum_{n=0}^{10} \underline{x}_{1,n}(r) = 0.628 + 1.375r \\ \bar{x}_1(r) \approx \sum_{n=0}^{10} \bar{x}_{1,n}(r) = 2.872 - 0.873r \end{cases}$$

$$\begin{cases} \underline{x}_2(r) \approx \sum_{n=0}^{10} \underline{x}_{2,n}(r) = 0.874 + 0.126r \\ \bar{x}_2(r) \approx \sum_{n=0}^{10} \bar{x}_{2,n}(r) = 1.376 - 0.376r \end{cases}$$

Table 1, 2, 3 and 4 show comparison of errors for (11) obtained using HAM and ADM [19]. It is to be noted that when  $h_i = \bar{h}_i = -0.8, (i = 1, 2)$  HAM gives same ADM solutions [19].

Table 1. For  $\underline{x}_1$ .

r	Error(HAM)	Error(ADM)
0.0	0.000249	0.003601
0.1	0.000383	0.003652
0.2	0.000517	0.003704
0.3	0.000651	0.003755
0.4	0.000784	0.003806
0.5	0.000918	0.003858
0.6	0.001052	0.003909
0.7	0.001186	0.003960
0.8	0.001320	0.004012
0.9	0.001454	0.004064
1.0	0.001588	0.004115

Table 2. For  $\bar{x}_1$ .

r	Error(HAM)	Error(ADM)
0.0	0.003201	0.005658
0.1	0.003040	0.005504
0.2	0.002878	0.005350
0.3	0.002717	0.005195
0.4	0.002555	0.005041
0.5	0.002394	0.004887
0.6	0.002231	0.004732
0.7	0.002072	0.004578
0.8	0.001910	0.004424
0.9	0.001751	0.004269
1.0	0.001588	0.004115

Table 3. For  $\underline{x}_2$ .

r	Error(HAM)	Error(ADM)
0.0	0.000969	0.003944
0.1	0.000880	0.003824
0.2	0.000791	0.003704
0.3	0.000702	0.003584
0.4	0.000613	0.003464
0.5	0.000525	0.003344
0.6	0.000436	0.003224
0.7	0.000347	0.003104
0.8	0.000258	0.002984
0.9	0.000169	0.002864
1.0	0.000081	0.002744

Table 4. For  $\bar{x}_2$ .

r	Error(HAM)	Error(ADM)
0.0	0.000736	0.001886
0.1	0.000655	0.001972
0.2	0.000573	0.002058
0.3	0.000491	0.002143
0.4	0.000410	0.002229
0.5	0.000328	0.002315
0.6	0.000246	0.002401
0.7	0.000164	0.002486
0.8	0.000082	0.002572
0.9	0.000001	0.002658
1.0	0.000081	0.002744

Allahviranllo in [19] has shown the ADM solutions for FSLE is same the given solutions by Jacobi iterative method. Approximate solutions and exact solutions are compared in Fig. 1.

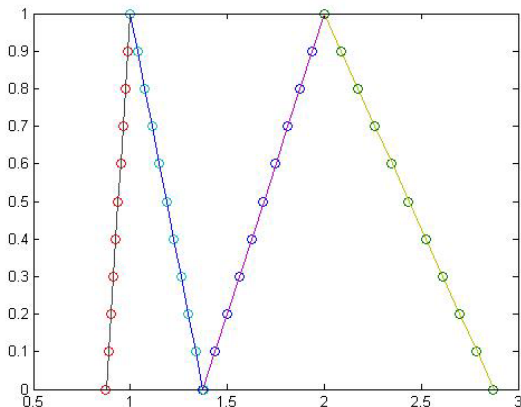


Fig1. Open circles: HAM solution (h=-0.8); Solid line: Exact solution.

Example 2. Consider the  $3 \times 3$  fuzzy system [19]

$$\begin{cases} x_1 + 0x_2 + x_3 = (r-1, 1-r) \\ x_1 + x_2 - 2x_3 = (r, 2-r) \\ -x_1 + x_2 + x_3 = (2+r, 3) \end{cases} \quad (12)$$

The extended  $6 \times 6$  matrix is

$$S = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & 1 & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

The exact solutions from  $X = S^{-1}Y$  are

$$\begin{cases} x_1 = (\underline{x}_1(r), \bar{x}_1(r)) = (0.5625r - 0.8750, -0.6875r + 0.3750) \\ x_2 = (\underline{x}_2(r), \bar{x}_2(r)) = (0.6875r + 0.1250, -0.5625r + 1.3750) \\ x_3 = (\underline{x}_3(r), \bar{x}_3(r)) = (-0.1250r + 0.7500, 0.3750r + 0.2500) \end{cases}$$

In view of (10) approximate solutions of (12) after 14

iterations with chosen  $\underline{h}_i = \bar{h}_i = -0.8, (i=1,2)$  are obtained as follows

$$\begin{cases} x_1 = (\underline{x}_1(r), \bar{x}_1(r)) \approx (0.5607r - 0.8732, -0.6857r + 0.3732) \\ x_2 = (\underline{x}_2(r), \bar{x}_2(r)) \approx (0.6857r + 0.1268, -0.5607r + 1.3732) \\ x_3 = (\underline{x}_3(r), \bar{x}_3(r)) \approx (-0.1229r + 0.7480, 0.3729r + 0.2521) \end{cases} \quad (13)$$

Table 5 compare errors of HAM solutions which are given by (13) with Adomian's solutions [19] for  $r = 1$ .

Table 5. comparison errors.

	$\underline{x}_1$	$\bar{x}_1$	$\underline{x}_2$	$\bar{x}_2$	$\underline{x}_3$	$\bar{x}_3$
HAM	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
ADM	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001

Also, in Fig. 2 approximate solutions of HAM and exact solutions are compared.

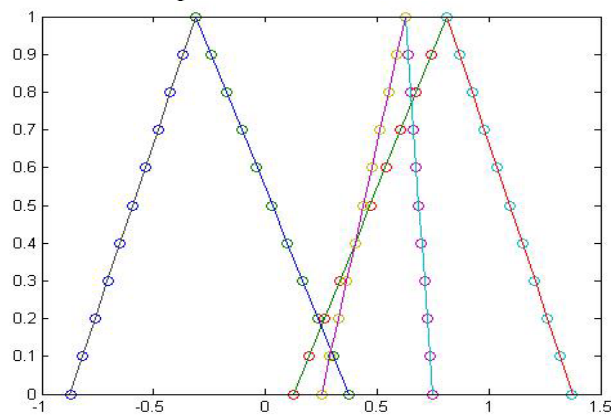


Fig. 2. Open circles: HAM solution (h=-0.7); Solid line: Exact solution.

Remark: In Figs. 1 and 2, we show the comparisons between the HAM solutions and the exact solutions. We observe that the results of the HAM is very close to the exact solutions which confirm the validity of the HAM. All the numerical results obtained by the HAM are exactly same as the ADM solutions and Jacobi iterative method solutions for special case  $\underline{h}_i = \bar{h}_i = -1$ ,  $\underline{H}_i(r) = \bar{H}_i(r) = 1$ . So it's means that the ADM and Jacobi iterative method are a special case of HAM. But HAM is more general and contains the auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region of solution series. As pointed out by Abbasbandy in [14] one had to choose a proper value of h to ensure the convergence of series solution for strongly nonlinear problems.

### 5. Conclusion

In this paper, the Homotopy analysis method has been

successfully applied to find the solution of Fuzzy system of linear equations (FSLE). It is apparently seen that HAM is a very powerful and efficient technique in finding analytical solutions for wide classes of linear problems. We have assumed that the proposed matrix  $S$  by Friedman and Ming [8] be nonsingular and  $a_{11} > 0$ , then we show the HAM provides accurate numerical solution for linear problems in comparison with the Adomian decomposition method and Jacobi iterative method. They also do not require large computer memory and discretization. The results show that HAM is powerful mathematical tool for solving (FSLE) and the ADM and Jacobi iterative method are a special case of HAM. Matlab has been used for computations in this paper.

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### References

- [1] C. V. Negoita and D. A. Ralescu, "Applications of fuzzy sets to systems analysis". *New York: Wiley*; 1975.
- [2] G. Adomian, "Solving Frontier Problems of Physics: The Decomposition Method", *Kluwer*, 1994.
- [3] H. Jafari, M. Zabihi, M. saidy, "Application of homotopy-perturbation method for solving Gas dynamics equation", *Appl. Math. Sci.*, vol. 2, no. 48, pp. 2393-2396, 2008.
- [4] J. H. He, "A coupling method of homotopy technique and perturbation technique for nonlinear problems," *Int J Nonlinear Mech*, vol. 35, no. 1, pp. 37-43, 2000.
- [5] J. H. He, "Some asymptotic methods for strongly nonlinear equations," *Int. J. Mod. Phys. B*, vol. 20, no. 10, pp. 1141-1199, 2006
- [6] M. Ayub, A. Rasheed, T. Hayat, "Exact flow of a third grade fluid past a porous plate using homotopy analysis method," *Int J Eng Sci*, vol. 41, pp. 2091-103, 2003.
- [7] M. Dehghan and B. Hashemi, "Iterative solution of fuzzy linear systems," *Appl. Math. Comput.* vol. 175, pp. 645-674, 2006.
- [8] M. Friedman, M. Ming, and A. Kandel, "Fuzzy linear systems," *FSS*, vol. 96, pp. 201-209, 1998.
- [9] O. Kaleva, "Fuzzy differential equations," *Fuzzy Sets Systems*, vol. 24, pp. 301-317, 1987.
- [10] S. J. Liao, "Beyond perturbation: introduction to the homotopy analysis method," *CRC Press, Boca Raton: Chapman & Hall*; 2003.
- [11] S. J. Liao, "A new branch of solutions of boundary-layer flows over an impermeable stretched plate," *Int J Heat Mass Transfer*, vol. 48, pp. 2529-39, 2005.
- [12] S. J. Liao and I. Pop, "Explicit analytic solution for similarity boundary layer equations," *Int J Heat Mass Transfer*, vol. 47, pp. 75-8, 2004.
- [13] S. L. Chang and L. A. Zadeh, "On fuzzy mapping and control," *IEEE. Trans. Syst. Man Cybernet.* 2:30-4, 1972.
- [14] S. Abbasbandy, "The application of homotopy analysis method to nonlinear equations arising in heat transfer," *Phys. Lett. A*, vol. 360, pp. 109-113, 2006.
- [15] T. Hayat and M. Khan, "Homotopy solutions for a generalized second-grade fluid past a porous plate," *Nonlinear Dyn*, vol. 42, pp. 395-405, 2005.
- [16] T. Hayat, M. Khan, M. Ayub, "On non-linear flows with slip boundary condition", *ZAMP*, vol. 56, pp. 1012-29, 2005.
- [17] T. Allahviranloo and M. Afshar Kermani, "Solution of a fuzzy system of linear equation," *Appl. Math. Comput.*, vol. 175, no. 1, pp. 519-531, 2006.
- [18] T. Allahviranloo, "Numerical methods for fuzzy system of linear equations," *Appl. Math. Comput*, vol. 155, no. 2, pp. 493-502, 2004.
- [19] T. Allahviranloo, "The Adomian decomposition method for fuzzy system of linear equations," *Appl. Math. Comput*, vol. 163, no. 2, pp. 553-563, 2005.