$(\in, \in \vee q)$-fuzzy Lie subalgebra and ideals

B. Davvaz and M. Mozafar

Abstract

Lie algebras are so-named in honor of Sophus Lie, a Norwegian mathematician who pioneered the study of these mathematical objects. Lie's discovery was tied to his investigation of continuous transformation groups and symmetries. The structure of the laws in physics is largely based on symmetries. The objects in Lie theory are fundamental, interesting and innovating in both mathematics and physics. It has many applications to the spectroscopy of molecules, atoms, nuclei and hadrons.

Our aim in this paper is to introduce and study a new sort of fuzzy Lie subalgebra (ideal) of a Lie algebra called $(\in, \in \vee q)$-fuzzy Lie subalgebra (ideal). These fuzzy Lie subalgebras (ideals) are characterized by their level ideals. Finally, we give a generalization of $(\in, \in \vee q)$-fuzzy Lie subalgebras (ideals).

Keywords: Fuzzy set, fuzzy point, Lie algebra, fuzzy Lie subalgebra, $(\in, \in \vee q)$-fuzzy Lie subalgebra, level set.

1. Introduction

The theory of fuzzy sets which was introduced by Zadeh [21] is applied to many mathematical branches. Rosenfeld [16] inspired the fuzzification of algebraic structures and introduced the notion of fuzzy subgroups. Das [5] characterized fuzzy subgroups by their level subgroups. In [14], Liu applied the concept of fuzzy sets to the theory of rings and introduced and examined the notion of a fuzzy ideal of a ring. A new type of fuzzy subgroup (viz, $(\in, \in \vee q)$-fuzzy subgroup) was introduced in an earlier paper of Bhakat and Das [2] by using the combined notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets. In fact, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. This concept has been studied further in [1, 3, 4, 7, 22]. Also, a generalization of Rosenfeld's fuzzy subgroup, and Bhakat and Das's fuzzy subgroup is given in [20].

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and the like. This provides sufficient motivation to researchers to review various concepts and results from the realm of abstract algebra in the broader framework of fuzzy setting.

Our aim in this paper is to introduce and study a new sort of fuzzy Lie subalgebra (ideal) of a Lie algebra called $(\in, \in \vee q)$-fuzzy Lie subalgebra (ideal). These fuzzy Lie subalgebras (ideals) are characterized by their level ideals. Finally, we give a generalization of $(\in, \in \vee q)$-fuzzy Lie subalgebras (ideals).

2. Lie algebra

Lie algebras are so-named in honor of Sophus Lie, a Norwegian mathematician who pioneered the study of these mathematical objects. Lie's discovery was tied to his investigation of continuous transformation groups and symmetries. The structure of the laws in physics is largely based on symmetries. The objects in Lie theory are fundamental, interesting and innovating in both mathematics and physics. It has many applications to the spectroscopy of molecules, atoms, nuclei and hadrons. One of the key concepts in the application of Lie algebraic methods in physics is that of spectrum generating algebras and their associated dynamic symmetries.

In order to make this paper self-sufficient we recall some basic definitions and results. The definitions may be found in references [11].

Definition 2.1: A Lie algebra is a vector space L over a field F on which a product operation $[x\ y]$ is defined satisfying the following axioms:

(i) $[x\ y]$ is bilinear for all $x, y \in L$,

(ii) $[x\ x] = 0$ for all $x \in L$,

(iii) $[[x\ y]\ z] + [[y\ z]\ x] + [[z\ x]\ y] = 0$

for all $x, y, z \in L$.

As a simple consequence of axioms (i), (ii) we have

Corresponding Author: B. Davvaz is with the Department of Mathematics, Yazd University Yazd, Iran.
E-mail: davvaz@yazduni.ac.ir and davvaz@yahoo.com
Manuscript received 22 Jan. 2009; revised 16 April 2009; accepted 19 June 2009.

$$0=[x+y\ x+y]$$
$$=[x\ x]+[x\ y]+[y\ x]+[y\ y]=[x\ y]+[y\ x].$$

Thus $[y\ x]=-[x\ y]$ and Lie multiplication is anti commutative.

Let L be a Lie algebra and M, N be subspaces of L. We define $[M\ N]$ to be the subspace of L spanned by all elements of the form $[x\ y]$ for $x \in M$, $y \in N$. Since $[y\ x]=-[x\ y]$ it is clear that $[M\ N]=[N\ M]$. Thus multiplication of subspaces is commutative.

Example 2.2: Euclidean space R^3 becomes a Lie algebra with the Lie bracket given by the cross product of vectors.

Example 2.3: Let V be a vector space over a field F such that $\dim V = 8$. If V has basis $(e_1, e_2, \cdots, e_8)$ with Lie brackets as follows;

$[e_1\ e_2]=e_5$, $[e_1\ e_3]=e_6$, $[e_1\ e_4]=e_7$,
$[e_1\ e_5]=-e_8$, $[e_2\ e_3]=e_8$, $[e_2\ e_4]=e_6$,
$[e_2\ e_6]=-e_7$, $[e_3\ e_4]=-e_5$, $[e_3\ e_5]=-e_7$,
$[e_4\ e_6]=-e_8$, and for all other $i \le j$, $[e_i\ e_j]=0$ and $[e_i\ e_j]=-[e_j\ e_i]$, then V is a Lie algebra over F.

Definition 2.4: An ideal of L is a subspace M such that $[M\ L] \subseteq M$. Since $[M\ L]=[L\ M]$ there is no distinction in the theory of Lie algebras between left ideals and right ideals. Every ideals is two-sided.

Definition 2.5: A linear transformation $\phi : L_1 \to L_2$ (L_1, L_2 Lie algebras over F) is called a homomorphism if $\phi([x\ y])=[\phi(x)\ \phi(y)]$ for all $x, y \in L$. We say that two Lie algebras L_1, L_2 over F are isomorphic if there exists a vector space isomorphism $\phi : L_1 \to L_2$ satisfying $\phi([x\ y])=[\phi(x)\ \phi(y)]$ for all x, y in L (and then ϕ is called an isomorphism of Lie algebras).

3. A survey of fuzzy Lie algebra

In our daily life , we usually want to seek opinions from professional persons with the best qualifications, for examples, the best medical doctors can provide the best diagnosis, the best pilots can provide the best navigation suggestions for airplanes etc. It is therefore desirable to incorporate the knowledge of these experts into some automatic systems so that it would become helpful for other people to make appropriate decisions which are (almost) as good as the decisions made by the top experts. With this aim in mind, our task is to design a system that would provide the best advice from the best experts in the field. However, one of the main hurdles of this incorporation is that the experts are usually unable to describe their knowledge by using precise and exact terms. For example, in order to describe the size of certain type of a tumor, a medical doctor would rarely use the exact numbers. Instead he would say something like "the size is between 1.4 and 1.6 cm". Also, an expert would usually use some words from a natural language, e.g., the size of the tumor is approximately 1.5 cm, with an error of about 0.1 cm. Thus, under such circumstances, the way to formalize the statements given by an expert is one of the main objectives of fuzzy logic.

Let X be a non-empty set. A fuzzy subset μ of X is a function $\mu : X \to [0,1]$. Let μ and λ be two fuzzy subsets of X, we say that μ is contained in λ, if $\mu(x) \le \lambda(x)$ for all $x \in X$. If $\{\mu_i\}_{i \in I}$ be a collection of fuzzy subsets of X, then we define the fuzzy subsets $\bigcap_{i \in I} \mu_i$ and $\bigcap_{i \in I} \mu_i$ by:

$$\left(\bigcap_{i \in I} \mu_i\right)(x) = \wedge_{i \in I}\{\mu_i(x)\} \text{ for all } x \in X$$

and

$$\left(\bigcup_{i \in I} \mu_i\right)(x) = \vee_{i \in I}\{\mu_i(x)\} \text{ for all } x \in X .$$

Let μ be any fuzzy subset of X. The set $\overline{\mu_t} = \{x \in X \mid \mu(x) \ge t\}$, $t \in [0,1]$, is called a level subset of μ.

Rosenfeld [16] applied the concept of fuzzy sets to the theory of groups and defined the concept of fuzzy subgroups of a group. Since then many papers concerning various fuzzy algebraic structures have appeared in the literature. Liu [14] introduced and studied the notions of fuzzy sub-rings and fuzzy ideals. In [9], Davvaz introduced the concept of fuzzy ideal of a Lie algebra and investigated the structure and properties of fuzzy ideals of a Lie algebra and he studied further properties in [8, 10], also see [12, 13, 18].

Definition 3.1 [9]: Let L be a Lie algebra and μ be a fuzzy subset of L. We say that μ is a *fuzzy Lie subalgebra* of L if for all $x, y \in L$, $\alpha \in F$

(1) $\mu(x+y) \ge \mu(x) \wedge \mu(y)$,

(2) $\mu(\alpha x) \ge \mu(x)$,

(3) $\mu([x\ y]) \ge \mu(x) \wedge \mu(y)$.

μ is called a *fuzzy ideal* of L if the condition (3) is replaced by

(4) $\mu([x\ y]) \ge \mu(x) \vee \mu(y)$.

Let L be a Lie algebra and χ_I be the characteristic function of a subset I of L. Then χ_I is a fuzzy subalgebra (ideal) if and only if I is a subalgebra(ideal), respectively.

Example 3.2[19]: Consider Example 2.2, i.e., $L=R^3$ and $[x\ y]=x\times y$, where $\times$ is cross product, for all $x,y\in L$. Define $\mu:R^3\to[0,1]$ by

$$\mu((x,y,z))=\begin{cases}1 & \text{if}\quad x=y=z=0,\\ \frac{1}{2} & \text{if}\quad x\neq 0, y=z=0,\\ 0 & \text{otherwise}\end{cases}$$

then μ is a fuzzy Lie subalgebra of *L*. But μ is not a fuzzy Lie ideal of *L*,

$$\mu([(1,0,0)(1,1,1)])=\mu((0,-1,1))=0$$

and $\mu((1,0,0))\vee\mu((1,1,1))=\frac{1}{2}\vee 0=\frac{1}{2}$.

Now, we define $\lambda:R^3\to[0,1]$ by

$$\lambda((x,y,z))=\begin{cases}1 & \text{if}\quad x=y=z=0\\ 0 & \text{otherwise}\end{cases}$$

then λ is a fuzzy Lie ideal of *L*.

Theorem 3.3 [9]:. Let *L* be a Lie algebra and μ be a fuzzy Lie subalgebra (ideal) of *L*. Then the level subset $\mu_t(\neq\phi)$ is a Lie subalgebra (ideal) of *L* for all $t\in(0,1]$ if and only if μ is a fuzzy Lie subalgebra (ideal) of *L*, respectively.

Definition 3.4 [19]: Let μ and λ be two fuzzy subsets of a Lie algebra. Then the product $\mu o\lambda$ is defined by

$$\mu o\lambda(x)=\begin{cases}\vee_{x=yz}(\mu(y)\wedge\lambda(z))\\ 0 \quad \text{if}\quad x\neq yz\end{cases}$$

4. Fuzzy Lie subalgebra and ideals redefined

For any fuzzy subset μ of *L*, the $\{x\in L\mid\mu(x)>0\}$ is called the *support* of μ, and is denoted by *supp* μ. A fuzzy set μ, on *L* which takes the value $t\in(0,1]$ at some $x\in L$ and takes the value 0 for all $y\in L$ expect *x* is called a *fuzzy point* and is denoted by x_t, where the point *x* is called its *support point* and *t* is called its *value*. A fuzzy point x_t is said to belong to (resp. be quasi-coincident with) a fuzzy set μ, written as $x_t\in\mu$ resp. $x_t q\mu$) if $\mu(x)\geq t$ (resp. $\mu(x)+t>1$). If $x_t\in\mu$ or $x_t q\mu$, then we write $x_t\in\vee q\mu$. The symbol $\overline{\in\vee q}$ means $\in\vee q$ does not hold.

*Definition 4.1***:** A fuzzy subset μ of a Lie algebra *L* is said to be an $(\in,\in\vee q)$-*fuzzy Lie subalgebra* of *L* if for all $t,r\in(0,1]$, $x,y\in L$ and $\alpha\in F$,

(ia) $x_t,y_r\in\mu$ implies $(x+y)_{t\wedge r}\in\vee q\mu$,

(ib) $x_t\in\mu$ implies $(\alpha x)_t\in\vee q\mu$,

(ii) $x_t,y_r\in\mu$ implies $([x\ y])_{t\wedge r}\in\vee q\mu$.

μ is called an $(\in,\in\vee q)$-*fuzzy ideal* of *L* if μ is an $(\in,\in\vee q)$-fuzzy Lie subalgebra of *L* and (iii) $x_t,y_r\in\mu$ implies $([x\ y])_{t\wedge r}\in\vee q\mu$.

*Theorem 4.2***:** Conditions (i)-(iii) in Definition 4.1 are equivalent to the following conditions respectively.

(1a) $\mu(x+y)\geq\mu(x)\wedge\mu(y)\wedge 0.5$ for all $x,y\in L$,

(1b) $\mu(\alpha x)\geq\mu(x)\wedge 0.5$ for all $x\in L$ and $\alpha\in F$,

(2) $\mu([x\ y])\geq\mu(x)\wedge\mu(y)\wedge 0.5$ for all $x,y\in L$,

(3) $\mu([x\ y])\geq(\mu(x)\vee\mu(y))\wedge 0.5$ for all $x,y\in L$.

Proof. (ia $\Rightarrow$ 1a): Suppose that $x,y\in L$. We consider the following cases:

(a) $\mu(x)\wedge\mu(y)<0.5$,

(b) $\mu(x)\wedge\mu(y)\geq 0.5$.

Case a: Assume that $\mu(x+y)<\mu(x)\wedge\mu(y)\wedge 0.5$, which implies $\mu(x+y)<\mu(x)\wedge\mu(y)$. Choose *t* such that $\mu(x+y)<t<\mu(x)\wedge\mu(y)$. Then $x_t,y_t\in\mu$, but $(x+y)_t\overline{\in\vee q}\mu$, which contradicts (ia).

Case b: Assume that $\mu(x+y)<0.5$. Then $x_{0.5},y_{0.5}\in\mu$, but $(x+y)_{0.5}\overline{\in\vee q}\mu$, a contradiction. Hence (la) holds.

(ib $\Rightarrow$ 1b): Suppose that $x\in L$. We consider the following cases:

(a) $\mu(x)<0.5$,

(b) $\mu(x)\geq 0.5$.

Case a: Assume that $\mu(x)=t<0.5$ and $\mu(\alpha x)=r<\mu(x)$. Choose *s* such that $r<s<t$ and $r+s<1$. Then $x_s\in\mu$ but $(\alpha x)_s\overline{\in\vee q}\mu$, which contradicts (ib). So $\mu(\alpha x)\geq\mu(x)=\mu(x)\wedge 0.5$.

Case b: Let $\mu(x)\geq 0.5$. If $\mu(\alpha x)<\mu(x)\wedge 0.5$, then $x_{0.5}\in\mu$, but $(\alpha x)_{0.5}\overline{\in\vee q}\mu$ which contradicts (ib). So $\mu(\alpha x)\geq\mu(x)\wedge 0.5$.

(ii $\Rightarrow$ 2): Suppose that $x,y\in L$. We consider the following cases:

(a) $\mu(x)\wedge\mu(y)<0.5$,

(b) $\mu(x)\wedge\mu(y)\geq 0.5$.

Case a: Assume that

$\mu([x\ y])<\mu(x)\wedge\mu(y)\wedge 0.5$, which implies $\mu([x\ y]) < \mu(x) \wedge \mu(y)$. Choose t such that $\mu([x\ y]) < t < \mu(x) \wedge \mu(y)$. Then $x_t, y_t \in \mu$, but $([x\ y])_t \overline{\in \vee q} \mu$, which contradicts (ii).

Case b: Assume that $\mu([x\ y]) < 0.5$. Then $x_{0.5}, y_{0.5} \in \mu$, but $([x\ y])_{0.5} \overline{\in \vee q} \mu$ a contradiction. Hence (2) holds.

(iii $\Rightarrow$ 3): Suppose that $x, y \in L$. We consider the following cases:

(a) $\mu(x) \vee \mu(y) < 0.5$,

(b) $\mu(x) \vee \mu(y) \geq 0.5$.

Case a: Assume that

$\mu([x\ y]) < (\mu(x) \vee \mu(y)) \wedge 0.5$,

which implies $\mu([x\ y]) < \mu(x) \vee \mu(y)$. Choose t such that $\mu([x\ y]) < t < \mu(x) \vee \mu(y)$. Then $x_t, y_t \in \mu$, but $([x\ y])_t \overline{\in \vee q} \mu$, which contradicts (iii).

Case b: Assume that $\mu([x\ y]) < 0.5$. Then $x_{0.5}, y_{0.5} \in \mu$, but $([x\ y])_{0.5} \overline{\in \vee q} \mu$ a contradiction. Hence(3)holds.

(1a $\Rightarrow$ ia): Let $x_t, y_r \in \mu$, then $\mu(x) \geq t$ and $\mu(y) \geq r$. Now, we have

$\mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge 0.5 \geq t \wedge r \wedge 0.5$.

If $t \wedge r > 0.5$, then $\mu(x+y) \geq 0.5$ which implies $\mu(x+y) + t \wedge r > 1$.

If $t \wedge r \leq 0.5$, then $\mu(x+y) \geq t \wedge r$. Therefore $(x+y)_{t \wedge r} \in \vee q \mu$.

(1b $\Rightarrow$ ib): Let $x_t \in \mu$. Then $\mu(x) \geq t$. Now, we have $\mu(\alpha x) \geq \mu(x) \wedge 0.5 \geq t \wedge 0.5$, which implies $\mu(\alpha x) \geq t$ or $\mu(\alpha x) \geq 0.5$ according as $t \leq 0.5$ or $t > 0.5$. Therefore, $(\alpha x)_t \in \vee q \mu$.

(2 $\Rightarrow$ ii): Let $x_t, y_r \in \mu$, then $\mu(x) \geq t$ and $\mu(y) \geq r$. Now, we have

$$\mu([x\ y]) \geq \mu(x) \wedge \mu(y) \wedge 0.5 \geq t \wedge r \wedge 0.5.$$

If $t \wedge r > 0.5$, then $\mu([x\ y]) \geq 0.5$ which implies

$$\mu([x\ y]) + t \wedge r > 1.$$

If $t \wedge r \leq 0.5$, then $\mu([x\ y]) \geq t \wedge r$. Therefore $([x\ y])_{t \wedge r} \in \vee q \mu$.

(3 $\Rightarrow$ iii): Let $x_t, y_r \in \mu$, then $\mu(x) \geq t$ and $\mu(y) \geq r$. Now, we have

$$\mu([x\ y]) \geq (\mu(x) \vee \mu(y)) \wedge 0.5 \geq (t \vee r) \wedge 0.5.$$

If $t \vee r > 0.5$, then $\mu([x\ y]) \geq 0.5$ which implies

$$\mu([x\ y]) + t \vee r > 1.$$

If $t \vee r \leq 0.5$, then $\mu([x\ y]) \geq t \vee r$. Therefore $([x\ y])_{t \vee r} \in \vee q \mu$. □

Corollary 4.3: Let μ be a fuzzy subset of a Lie subalgebra *L*. Then

(a)μ is an $(\in, \in \vee q)$-fuzzy Lie subalgebra of *L* if and only if the conditions (1a), (1b) and (2) in Theorem 4.2 hold.

(b)μ is an $(\in, \in \vee q)$-fuzzy ideal of *L* if and only if the conditions (1a), (1b) and (3) in Theorem 4.2 hold.

A fuzzy Lie ideal (according to Definition 3.1) is an $(\in, \in \vee q)$ -fuzzy Lie ideal. Clearly, $\mu(x) \wedge \mu(y) \geq \mu(x) \wedge \mu(y) \wedge 0.5$, $\mu(x) \geq \mu(x) \wedge 0.5$, $\mu(x) \vee \mu(y) \geq (\mu(x) \vee \mu(y)) \wedge 0.5$.

But the converse is not necessarily true. The converse is true when for every $x \in L$, $\mu(x) \leq 0.5$. If μ is an $(\in, \in \vee q)$-fuzzy Lie subalgebra of a Lie algebra *L*, then it is easy to see that $\mu(0) \geq 0.5$.

Example 4.4**:** Consider Example 2.3 and define fuzzy set μ of *V* for all $x \in V$ by

$$\mu(x) = \begin{cases} 1 & x = 0, e_8, \\ 0.7 & x = e_7, \\ 0 & \text{otherwise.} \end{cases}$$

then μ is an $(\in, \in \vee q)$-fuzzy ideal of *V*, and also is an ordinary fuzzy ideal.

*Example 4.5***:** Let *V* be a vector space over a field *F* such that $\dim V = 5$. If *V* has basis $(e_1, e_2, e_3, e_4, e_5)$ with Lie brackets as follows;

$[e_1\ e_2] = e_3$, $[e_1\ e_3] = e_5$, $[e_1\ e_4] = e_5$,

$[e_1\ e_5] = 0$, $[e_2\ e_3] = e_5$, $[e_2\ e_4] = 0$,

$[e_2\ e_5] = 0$, $[e_3\ e_4] = 0$, $[e_3\ e_5] = 0$, $[e_4\ e_5] = 0$ and $[e_i\ e_i] = 0$ and $[e_i\ e_j] = -[e_j\ e_i]$, then *V* is a Lie algebra over *F*. We define fuzzy set μ of *V* for all $x \in V$ by

$$\mu(x) = \begin{cases} 0.6 & x = 0, \\ 0.7 & x = e_3, e_5 \\ 1 & x = e_1, e_2, e_4. \end{cases}$$

then μ is an $(\in, \in \vee q)$-fuzzy ideal of *V*, but not an ordinary fuzzy ideal.

*Theorem 4.6***:** A non-empty subset *I* of *L* is a Lie subalgebra (ideal) of *L* if and only if χ_I is an $(\in, \in \vee q)$-fuzzy Lie subalgebra (ideal) of *L*.

Proof. Assume that I is an ideal of L. Then χ_I is a fuzzy ideal in the sense of Definition 3.1 and so it is an $(\in, \in \vee q)$-fuzzy ideal.

Conversely, assume that χ_I is an $(\in, \in \vee q)$-fuzzy Lie subalgebra of L. Then for every $x, y \in I$, we have

$$\chi_I(x+y) \geq \chi_I(x) \wedge \chi_I(y) \wedge 0.5 = 0.5,$$
$$\chi_I(\alpha x) \geq \chi_I(x) \wedge 0.5 = 0.5,$$
$$\chi_I([x\ y]) \geq \chi_I(x) \wedge \chi_I(y) \wedge 0.5 = 0.5.$$

So $x + y \in I$, αx, $[x\ y] \in I$. Therefore I is a Lie subalgebra of L. Assume that χ_I is an $(\in, \in \vee q)$-fuzzy Lie ideal of L. Then for every $x \in I$, $y \in L$, we have

$$\chi_I([x\ y]) \geq (\chi_I(x) \vee \chi_I(y)) \wedge 0.5 = 0.5$$

So $[x\ y] \in I$. Therefore I is a Lie ideal of L. □

Now, we characterize $(\in, \in \vee q)$-fuzzy ideals by their level ideals.

Theorem 4.7: Let μ be an $(\in, \in \vee q)$-fuzzy ideal of a Lie algebra L. Then for all $0 < t \leq 0.5$, μ_t is a non-empty set or an ideal of L. Conversely, if μ is a fuzzy subset of L such that $\mu_t(\neq \phi)$ is an ideal of L for all $0 < t \leq 0.5$, then μ is an $(\in, \in \vee q)$-fuzzy ideal of L.

Proof. Let μ be an $(\in, \in \vee q)$-fuzzy ideal of L and $0 < t \leq 0.5$. Let $x, y \in \mu_t$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. Now, we have

$$\mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge 0.5 \geq t \wedge 0.5 = t,$$
$$\mu(\alpha x) \geq \mu(x) \wedge 0.5 = t \wedge 0.5 = t,$$

and so $x + y \in \mu_t$ and $\alpha x \in \mu_t$. Now, for every $x \in \mu_t$ and $y \in L$ we have

$$\mu([x\ y]) \geq (\mu(x) \vee \mu(y)) \wedge 0.5$$
$$\geq \mu(x) \wedge 0.5 \geq t \wedge 0.5 = t$$

which implies $[x\ y] \in \mu_t$. Therefore, μ_t is a Lie ideal of L.

Conversely, let μ be a fuzzy subset of L such that $\mu_t(\neq \phi)$ be an ideal of L for all $0 < t \leq 0.5$. For every $x, y, a \in L$, we can write

$$\mu(x) \vee \mu(y) \geq (\mu(x) \vee \mu(y)) \wedge 0.5 = k_o$$

then $x \in \mu_{k_o}$ or $y \in \mu_{k_o}$. Let $x \in \mu_{k_o}$, so $[x\ y] \in \mu_{k_o}$. Therefore, $\mu([x\ y]) \geq (\mu(x) \vee \mu(y)) \wedge 0.5$. Hence, μ is an $(\in, \in \vee q)$-fuzzy ideal of L. □

Naturally, a corresponding result should be considered when $\mu_t(\neq \phi)$ is an ideal (or Lie subalgebra) of L for all $(0.5, 1]$.

Theorem 4.8: Let μ be a fuzzy subset of a Lie algebra L. Then $\mu_t(\neq \phi)$ is a Lie subalgebra of L for all $t \in (0.5, 1]$ if and only if

(1a) $\mu(x+y) \vee 0.5 \geq \mu(x) \wedge \mu(y)$ for all $x, y \in L$,

(1b) $\mu(\alpha x) \vee 0.5 \geq \mu(x)$ for all $x \in L$ and $\alpha \in F$,

(2) $\mu([x\ y]) \vee 0.5 \geq \mu(x) \wedge \mu(y)$ for all $x, y \in L$.

Moreover, $\mu_t(\neq \phi)$ is an ideal of L for all $t \in (0.5, 1]$ if and only if μ satisfies the conditions (1a), (1b) and satisfies the following condition:

(3) $\mu([x\ y]) \vee 0.5 \geq \mu(x) \vee \mu(y)$ for all $x, y \in L$.

Proof. Suppose that $\mu_t(\neq \phi)$ is a Lie subalgebra of Lie algebra L.

(1a): Suppose that for some $x, y \in L$,

$$\mu(x+y) \vee 0.5 < \mu(x) \wedge \mu(y) = t,$$

then $t \in (0.5, 1]$, $\mu(x+y) < t$, $x \in \mu_t$ and $y \in \mu_t$. Since $x, y \in \mu_t$ and μ_t is a Lie subalgebra, so $x + y \in \mu_t$ or $\mu(x+y) \geq t$, which is a contradiction with $\mu(x+y) < t$. Hence (1a) holds.

(1b): Suppose that for some $x \in L$, $\mu(\alpha x) \vee 0.5 < \mu(x) = t$, then $t \in (0.5, 1]$, $\mu(\alpha x) < t$ and $x \in \mu_t$. Since $x \in \mu_t$, we get $\alpha x \in \mu_t$ or $\mu(\alpha x) \geq t$, which is a contradiction. Hence (lb) holds.

(2): Suppose that for that some $x, y \in L$, $\mu([x\ y]) \vee 0.5 < \mu(x) \wedge \mu(y) = t$ then $t \in (0.5, 1]$, $\mu([x\ y]) < t$, $x \in \mu_t$, $y \in \mu_t$. Since $x, y \in \mu_t$ and μ_t is a Lie subalgebra, so $[x\ y] \in \mu_t$ or $\mu([x\ y]) \geq t$, which is a contradiction with $\mu([x\ y]) < t$. Hence (2) holds.

(3): If there exists $x \in L$ such that

$$\mu([x\ y]) \vee 0.5 \leq \mu(x) \vee \mu(y) = t$$

then $t \in (0.5, 1]$ and $\mu([x\ y]) < t$ and $x \in \mu_t$ or $y \in \mu_t$. Let $x \in \mu_t$. Since μ_t is an ideal, so $[x\ y] \in \mu_t$ or $\mu([x\ y]) \geq t$, which is a contradiction with $\mu([x\ y]) < t$. Hence (3) holds.

Now, suppose that conditions (1a), (1b), (2), (3) hold. We show that μ_t is an ideal of L. Assume that $t \in (0.5, 1]$, $x, y \in \mu_t$. Then

$$0.5 < t \le \mu(x) \wedge \mu(y) \le \mu(x+y) \vee 0.5$$
$$\Rightarrow \quad \mu(x+y) \ge t,$$
$$0.5 < t \le \mu(x) \le \mu(\alpha x) \vee 0.5$$
$$\Rightarrow \quad \mu(\alpha x) \ge t,$$
$$0.5 < t \le \mu(x) \wedge \mu(y) \le \mu([x\ y]) \vee 0.5$$
$$\Rightarrow \quad \mu([x\ y]) \ge t,$$

and so $x+y \in \mu_t$, $\alpha x \in \mu_t$ and $[x\ y] \in \mu_t$. Therefore, μ_t is a Lie subalgebra of *L*. Let $x \in \mu_t$ then we have

$$0.5 < t \le \mu(x) \vee \mu(y) \le \mu([x\ y]) \vee 0.5$$
$$\Rightarrow \quad \mu([x\ y]) \ge t.$$

Hence $[x\ y] \in \mu_t$. Therefore μ_t is a Lie ideal of *L*. □

Definition 4.9: Let $\alpha, \beta \in [0,1]$ and $\alpha < \beta$. Let μ be a fuzzy subset of a Lie algebra *L*. Then μ is called a *fuzzy Lie subalgebra with thresholds* of *L*, if for all $x, y \in L$ and $\gamma \in F$,

(1a) $\mu(x+y) \vee \alpha \ge \mu(x) \wedge \mu(y) \wedge \beta$,

(1b) $\mu(\gamma x) \vee \alpha \ge \mu(x) \wedge \beta$,

(2) $\mu([x\ y]) \vee \alpha \ge \mu(x) \wedge \mu(y) \wedge \beta$.

Moreover, μ is a fuzzy ideal with thresholds of *L*, if μ satisfies the conditions (1a), (1b) and satisfies the following condition:

(3) $\mu([x\ y]) \vee \alpha \ge (\mu(x) \vee \mu(y)) \wedge \beta$ for all $x, y, \gamma \in L$.

If μ is a fuzzy Lie subalgebra (ideal) with thresholds of *L*, then we can conclude that μ is an ordinary fuzzy Lie subalgebra (ideal) when $\alpha = 0$, $\beta = 1$; and μ is an $(\in, \in \vee q)$-fuzzy Lie subalgebra (ideal) when $\alpha = 0$, $\beta = 0.5$.

Now, we characterize fuzzy Lie subalgebra (ideal) with thresholds by their level Lie subalgebras (ideals).

Theorem 4.10: A fuzzy subset μ of a Lie algebra *L* is a fuzzy Lie subalgebra (ideal) with thresholds of *L* if and only if $\mu_t (\ne \phi)$ is a Lie subalgebra (ideal) of *L* for all $t \in (a, \beta]$.

Proof. Let μ is a Lie subalgebra with thresholds of *L*. Let $x, y \in \mu_t$, then $\mu(x) \ge t$ and $\mu(y) \ge t$. Now, we have

$$\mu(x+y) \vee \alpha \ge \mu(x) \wedge \mu(y) \wedge \beta \ge t \wedge \beta = t$$
$$\Rightarrow \quad \mu(x+y) \ge t,$$
$$\mu(\gamma x) \vee \alpha \ge \mu(x) \wedge \beta \ge t \wedge \beta = t$$
$$\Rightarrow \quad \mu(\gamma x) \ge t,$$
$$\mu([x\ y]) \vee \alpha \ge \mu(x) \wedge \mu(y) \wedge \beta \ge t \wedge \beta = t$$
$$\Rightarrow \quad \mu([x\ y]) \ge t$$

and so $x+y \in \mu_t$, $\gamma x \in \mu_t$ and $[x\ y] \in \mu_t$. Therefore μ_t is a Lie subalgebra of *L*. Also, for every $x \in \mu_t$, we have

$$\mu([x\ y]) \vee \alpha \ge (\mu(x) \vee \mu(y)) \wedge \beta$$
$$\ge \mu(x) \wedge \beta \ge t \wedge \beta = t \quad \Rightarrow \quad \mu([x\ y]) \ge t$$

which implies $[x\ y] \in \mu_t$. Therefore μ_t is an ideal of *L*.

Conversely, let μ be a fuzzy subset of *L* such that $\mu_t (\ne \phi)$ be a subalgebra (ideal) of *L* for all $\alpha < t \le \beta$. We show that μ is a Lie subalgebra (ideal) with thresholds of *L* or

(1a) $\mu(x+y) \vee \alpha \ge \mu(x) \wedge \mu(y) \wedge \beta$,

(1b) $\mu(\gamma x) \vee \alpha \ge \mu(x) \wedge \beta$,

(2) $\mu([x\ y]) \vee \alpha \ge \mu(x) \wedge \mu(y) \wedge \beta$.

(3) $\mu([x\ y]) \vee \alpha \ge (\mu(x) \vee \mu(y)) \wedge \beta$ for all $x, y, \gamma \in L$.

(1a): Suppose that for some $x, y \in L$, $\mu(x+y) \vee \alpha < \mu(x) \wedge \mu(y) \wedge \beta = t$, then $t \in (a, \beta]$, $\mu(x+y) < t$, $x \in \mu_t$ and $y \in \mu_t$ · Since $x, y \in \mu_t$ and μ_t is a Lie subalgebra, so $x+y \in \mu_t$ or $\mu(x+y) \ge t$, which is a contradiction with $\mu(x+y) < t$. Hence (1a) holds.

(1b): Suppose that for some $x \in L$, $\mu(\gamma x) \vee \alpha < \mu(x) \wedge \beta = t$, then $t \in (a, \beta]$, $\mu(\alpha x) < t$ and $x \in \mu_t$. Since $x \in \mu_t$, we get $\alpha x \in \mu_t$ or $\mu(\alpha x) \ge t$, which is a contradiction. Hence (lb) holds.

(2): Suppose that for some $x, y \in L$,

$$\mu([x\ y]) \vee \alpha < \mu(x) \wedge \mu(y) \wedge \beta = t,$$

then $t \in (a, \beta]$, $\mu([x\ y]) < t$ and $x \in \mu_t$ and $y \in \mu_t$. Since $x, y \in \mu_t$ and μ_t is a Lie subalgebra, so $[x\ y] \in \mu_t$ or $\mu([x\ y]) \ge t$, which is a contradiction with $\mu([x\ y]) < t$. Hence (2) holds.

Therefore μ is a Lie subalgebra with thresholds of L.

(3): If there exists $x \in L$ such that $\mu([x\ y]) \vee \alpha < (\mu(x) \vee \mu(y)) \wedge \beta = t$ then $t \in (a, \beta]$ and $\mu([x\ y]) < t$ and $x \in \mu_t$ or $y \in \mu_t$. Let $x \in \mu_t$. Since μ_t is an ideal, so $[x\ y] \in \mu_t$ or $\mu([x\ y]) \ge t$, which is a contradiction with $\mu([x\ y]) < t$. Hence (3) holds.

Therefore μ is a Lie ideal with thresholds of *L*. □

5. Conclusions

Lie algebras is one of the basic notions of mathematics. Being non-associative algebras, they are connected with many branches of mathematics. In the last several years, the relationship between mathematics and fundamental physics has reached the most significant stage at which developments in one science yield important results for the other, fuzzy group, fuzzy ring, fuzzy module, fuzzy vector space and fuzzy algebra can be given. The concepts fuzzy Lie algebras and fuzzy Lie ideals were already introduced by B. Davvaz, Y.B. Jun, K.H. Kim, E.B. Roh, C.-G. Kim, D.-S. Lee and S. El B. Yehia using fuzzy sets. In this paper, we have given the concept $(\in,\in\vee q)$-fuzzy Lie subalgebra (ideal) on Lie algebra using fuzzy points. Afterwards, we have also established correlations between these concepts and fuzzy Lie algebras(fuzzy Lie ideal). Our future work on this topic will focus on studying of adjoint representation of $(\in,\in\vee q)$-fuzzy Lie subalgebra (ideal) and the Killing form in the $(\in,\in\vee q)$-fuzzy case.

References

[1]. S. K. Bhakat, "$(\in,\in\vee q)$-fuzzy normal, quasinormal and maximal subgroups," *Fuzzy Sets Syst*, vol. 112, pp. 299-312, 2000.

[2]. S. K. Bhakat, and P. Das, "On the definition of a fuzzy subgroup," *Fuzzy Sets Syst*, vol. 51, pp. 235-241, 1992.

[3]. S. K. Bhakat, and P. Das, "$(\in,\in\vee q)$-fuzzy subgroups," *Fuzzy Sets Syst*, vol. 80, pp. 359-368, 1996.

[4]. S. K. Bhakat, and P. Das, "Fuzzy subrings and ideals redefined," *Fuzzy Sets Syst*, vol. 81, pp. 383-393, 1996.

[5]. P. Das, "Fuzzy groups and level subgroups," *J. Math. Anal. Appl*, vol. 85, pp. 264-269, 1981.

[6]. B. Davvaz, and V. Leoreanu-Fotea, "On product of fuzzy H_v -submodules," *International Journal of Fuzzy Systems*, vol. 10 , no. 2, pp. 119-122, 2008.

[7]. B. Davvaz, "$(\in,\in\vee q)$-fuzzy subnear-rings and ideals," *Soft Computing*, vol. 10, pp. 206-211, 2006.

[8]. B. Davvaz, "A note on fuzzy Lie algebras," *JP J. Algebra Number Theory Appl*, vol. 2, no. 2, pp. 131-136, 2002.

[9]. B. Davvaz, "Fuzzy Lie algebra," *Int. J. Appl. Math*, vol. 6, no. 4, pp. 449-461, 2001.

[10]. B. Davvaz, "Anti fuzzy ideals of a Lie algebra," *Pure Math. Appl*, vol. 12, pp. 131-136, 2001.

[11]. J. E. Humphreys, *Introduction to Lie Algebra and Representation Theory*, Springer-Verlag, 1972.

[12]. Y. B. Jun, K. H. Kim, and E. H. Roh, "On fuzzy ideals of Lie algebras," *J. Appl. Math. Computing*, vol. 10, pp. 251-259, 2002.

[13]. C.-G. Kim, and D.-S. Lee, "Fuzzy Lie ideals and fuzzy Lie subalgebras," *Fuzzy Sets Syst*, vol. 94, pp. 101-107, 1998.

[14]. W. J. Liu, "Fuzzy invariant subgroups and fuzzy ideals," *Fuzzy Sets Syst*, vol. 8, pp. 133-139, 1982.

[15]. X. Ma, J. Zhan, and Y.B. Jun, "Interval valued $(\in,\in\vee q)$-fuzzy ideals of pseudo-MV algebras," *International Journal of Fuzzy Systems*, vol. 10, no. 2, pp. 84-91, 2008.

[16]. A. Rosenfeld, "Fuzzy groups," *J. Math. Anal. Appl*, vol. 35, pp. 512-517, 1971.

[17]. S. Yamak, O. Kazanci, and B. Davvaz, "Divisible and pure intuitionistic fuzzy subgroups and their properties," *International Journal of Fuzzy Systems*, vol. 10, pp. 298-307, 2008.

[18]. S. El B. Yehia, "The adjoint representation of fuzzy Lie algebras," *Fuzzy Sets Syst*, vol. 119, pp. 409-417, 2001.

[19]. S. El B. Yehia, "Fuzzy ideals and fuzzy subalgebras of Lie algebras," *Fuzzy Sets Syst*, vol. 80, pp. 237-244, 1996.

[20]. X. H. Yuan, C. Zhang, and Y. H. Ren, "Generalized fuzzy groups and many valued applications," *Fuzzy Sets Syst*, vol. 138, pp. 205-211, 2003.

[21]. L. A. Zadeh, "Fuzzy sets," *Inform. Control*, vol. 8, pp. 338-353, 1965.

[22]. J. M. Zhan, B. Davvaz, and K. P. Shum, "A new view on fuzzy hypermodules," *Acta Math. Sinica, English Series*, vol. 23, pp. 1345-1356, 2007.