

The Characterization of h -semisimple Hemirings

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Abstract

In this paper, we consider the characterization of h -semisimple hemirings. First, the notions of $(\in, \in \vee q)$ -fuzzy h -ideals and $(\in, \in \vee q)$ -fuzzy h -interior ideals of a hemiring are introduced, and some of their properties are investigated. Then the concept of h -semisimple hemirings is introduced and its characterizations are established.

Keywords: Hemirings, $(\in, \in \vee q)$ -fuzzy h -ideals, $(\in, \in \vee q)$ -fuzzy h -interior ideals, h -semisimple hemirings

1. Introduction

Semirings which are regarded as a generalization of rings have been found useful for solving problems in different areas of applied mathematics and information sciences, since the structure of a semiring provides an algebraic framework for modeling and studying the key factors in these applied areas. They play an important role in studying optimization theory, graph theory, theory of discrete event dynamical systems, matrices, determinants, generalized fuzzy computation, automata theory, formal language theory, coding theory, analysis of computer programs, and so on (see [1,2] for details). Note that the ideals of semirings play a central role in the structure theory, however, they do not in general coincide with the usual ring ideals of a ring. For this reason, the usage of ideals in semirings is somewhat limited. Indeed, many results in rings apparently have no analogues in semirings using only ideals. In the literature, Henriksen [3] first defined a more restricted class of ideals in semirings, which is called k -ideals, with the property that if the semiring S is a ring then a complex in S is a k -ideal if and only if it is a ring ideal. Another more restricted class of ideals were given in hemirings by Iizula [4]. However, a definition of ideal in any addi-

tively commutative semiring S can be given which coincides with Iizula's definition provided S is a hemiring, and it is called h -ideal. The properties of h -ideals and also k -ideals of a hemiring were thoroughly investigated by Torre in [5] and by using h -ideals and k -ideals, Torre established some analogous ring theorems for hemirings.

The theory of fuzzy sets was introduced by Zadeh [6] in 1965. This theory has provided a useful mathematical tool for describing the behavior of systems that are too complex or illdefined to admit precise mathematical analysis by classical methods and tools. Extensive applications of the fuzzy set theory have been found in various fields. In recent years there has been considerable interest in the connections between semirings(hemirings) and fuzzy sets. The reader is referred to [7-17]. Recently, Yin and Li [18] introduced the concepts of fuzzy h -bi-ideals and fuzzy h -quasi-ideals of a hemiring and considered the characterization of h -hemiregular hemirings and h -intra-hemiregular hemirings. As a continuation, we introduce the concepts of $(\in, \in \vee q)$ -fuzzy h -ideals and $(\in, \in \vee q)$ -fuzzy h -interior ideals of a hemiring and study the characterization of h -semisimple hemirings.

2. Preliminaries

A semiring is an algebraic system $(S, +, \cdot)$ consisting of a non-empty set S together with two binary operations on S called addition and multiplication (denoted in the usual manner) such that $(S, +)$ and (S, \cdot) are semigroups and the following distributive laws

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c$$

are satisfied for all $a, b, c \in S$.

By zero of a semiring $(S, +, \cdot)$ we mean an element $0 \in S$ such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. A semiring $(S, +, \cdot)$ with zero is said to be a hemiring if $(S, +)$ is commutative. For the sake of simplicity, we shall omit the symbol " \cdot ", writing ab for $a \cdot b$ ($a, b \in S$).

A non-empty subset A in a hemiring S is called a left (resp. right) ideal of S if A is closed under addition and $SA \subseteq A$ (resp. $AS \subseteq A$). Further, A is called an ideal

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of S if it is both a left ideal and a right ideal of S . A subset A in a hemiring S is called an interior ideal if A is closed under addition and multiplication such that $ASA \subseteq A$.

A left ideal A of S is called a left h -ideal if $x, z \in S$, $a, b \in A$, and $x + a + z = b + z$ implies $x \in A$. Right h -ideals, h -ideals and h -interior ideals are defined similarly.

The h -closure \bar{A} of A in a hemiring S is defined as $\bar{A} = \{x \in S \mid x + a_1 + z = a_2 + z \text{ for some } a_1, a_2 \in A, z \in S\}$.

Lemma 2.1[16]: For a hemiring S , we have

- (1) $A \subseteq \bar{A}, \forall \{0\} \subseteq A \subseteq S$.
- (2) If $A \subseteq B \subseteq S$, then $\bar{A} \subseteq \bar{B}$.
- (3) $\bar{\bar{A}} = \bar{A}, \forall A \subseteq S$.
- (4) $\overline{AB} = \overline{A\bar{B}}$ and $\overline{ABC} = \overline{A\bar{B}C}, \forall A, B, C \subseteq S$.
- (5) For any left (resp. right) h -ideal or h -interior ideal A of S , we have $A = \bar{A}$.

A fuzzy subset μ in a hemiring S is defined as a mapping from S to $[0,1]$. The set of all fuzzy subsets in S is denoted by $F(S)$. For any $A \subseteq S$, the characteristic function of A , denoted by χ_A , is defined by $\chi_A = 1$ if $x \in A$ and $\chi_A = 0$ otherwise. For $\mu, \nu \in F(S)$, $\mu \subseteq \nu$ if and only if $\mu(x) \leq \nu(x)$ for all $x \in S$. And the \cup and \cap of μ and ν , denoted by $\mu \cup \nu$ and $\mu \cap \nu$, are defined as the fuzzy subsets in S by $(\mu \cup \nu)(x) = \mu(x) \vee \nu(x)$ and $(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x)$, respectively, for all $x \in S$.

A fuzzy subset μ in S of the form

$$\mu(y) = \begin{cases} r(\neq 0) & \text{if } y = x, \\ 0 & \text{otherwise} \end{cases}$$

is said to be a fuzzy point with support x and value r and is denoted by x_r . A fuzzy point x_r is said to belong to (resp. be quasi-coincident with) a fuzzy set μ , written as $x_r \in \mu$ (resp. $x_r q \mu$) if $\mu(x) \geq r$ (resp. $\mu(x) + r > 1$). If $\mu(x) \geq r$ or $\mu(x) + r > 1$, then we write $x_r \in \vee q \mu$.

Now, let us define a new order relation " $\subseteq \vee q$ " on $F(S)$, called the fuzzy inclusion or quasi-coincidence relation, as follows.

Let $\mu, \nu \in F(S)$. If $x_r \in \mu$ implies $x_r \in \vee q \nu$ for all $x \in S$ and $r \in (0,1]$, then we write $\mu \subseteq \vee q \nu$.

In the sequel, unless otherwise stated, $M(r_1, r_2, \dots, r_n)$, where n is a positive integer, will denote $r_1 \wedge r_2 \cdots \wedge r_n$ for all $r_1, r_2, \dots, r_n \in (0,1]$, $\overline{\in \vee q}$ means $\in \vee q$ does not hold and $\overline{\subseteq \vee q}$ implies $\subseteq \vee q$ is not true.

Lemma 2.2: Let $\mu, \nu \in F(S)$. Then $\mu \subseteq \vee q \nu$ if and only if $\nu(x) \geq M(\mu(x), 0.5)$ for all $x \in S$.

Proof: Assume that $\mu \subseteq \vee q \nu$. Let $x \in S$. If $\nu(x) < r = M(\mu(x), 0.5)$, then $x_r \in \mu$ but $x_r \notin \vee q \nu$, a contradiction. Hence $\nu(x) \geq M(\mu(x), 0.5)$.

Conversely, assume that $\nu(x) \geq M(\mu(x), 0.5)$ for all $x \in S$. If $\overline{\mu \subseteq \vee q \nu}$, then there exists $x_r \in \mu$ but $x_r \notin \vee q \nu$, and so $\mu(x) \geq r, \nu(x) < r$ and $\nu(x) < 0.5$, which contradicts $\nu(x) \geq M(\mu(x), 0.5)$.

Lemma 2.3: Let $\mu, \nu, \omega \in F(S)$ be such that $\mu \subseteq \vee q \nu \subseteq \vee q \omega$. Then $\mu \subseteq \vee q \omega$.

Proof: It is straightforward by Lemma 2.2.

We have a question as follows. If $\mu \subseteq \vee q \nu$ and $\nu \subseteq \vee q \mu$, then $\mu = \nu$? The following example gives a negative answer.

Example 2.4: Let $S = \{0, a, b\}$ be a set with a addition operation (+) and a multiplication operation (\cdot) as follows:

+	0	a	b	and	.	0	a	b
0	0	a	b		0	0	0	0
a	a	0	b		a	0	0	0
b	b	b	0		b	0	0	b

Then S is a hemiring. Define two fuzzy subsets μ and ν in S by $\mu(0) = 0.6, \mu(a) = 0.6, \mu(b) = 0.5$ and $\nu(0) = 0.5, \nu(a) = 0.5, \nu(b) = 0.6$, respectively. Then $\mu \subseteq \vee q \nu$ and $\nu \subseteq \vee q \mu$, but $\mu \neq \nu$.

Let S be a hemiring. We define a relation " \approx " on $F(S)$ as follows:

$\mu \approx \nu$ if and only if $\mu \subseteq \vee q \nu$ and $\nu \subseteq \vee q \mu$ for all $\mu, \nu \in F(S)$.

Then Lemmas 2.2 and 2.3 give that " \approx " is an equivalence relation on $F(S)$.

Next, we will introduce the definition of the h -sum of two fuzzy subsets in a hemiring S .

Definition 2.5: Let μ and ν be fuzzy subsets in a hemiring S . Then the h -sum of μ and ν is defined by

$$(\mu +_h \nu)(x) = \sup_{x+a_1+b_1+z=a_2+b_2+z} M(\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2))$$

if there exist $a_1, a_2, b_1, b_2, z \in S$ such that $x + a_1 + b_1 + z = a_2 + b_2 + z$ and $(\mu +_h \nu)(x) = 0$ otherwise.

In [18], Yin and Li introduced the h -intrinsic product of two fuzzy subsets in a hemiring S as follows.

Definition 2.6[18]: Let μ and ν be fuzzy subsets in a hemiring S . Then the h -intrinsic product of μ and ν is defined by

$$(\mu \odot_h \nu)(x) = \sup_{x+\sum_{i=1}^m a_i b_i+z=\sum_{j=1}^n a'_j b'_j+z} M(\mu(a_i), \mu(a'_j), \nu(b_i), \nu(b'_j))$$

for all $i = 1, \dots, m; j = 1, \dots, n$ and $(\mu \odot_h \nu)(x) = 0$ if

$$x \text{ cannot be expressed as } x + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z.$$

By directly calculation we obtain immediately the following results.

Lemma 2.7: Let S be a hemiring and $\mu_1, \mu_2, \nu_1, \nu_2 \in F(S)$ such that $\mu_1 \subseteq \vee q \mu_2$ and $\nu_1 \subseteq \vee q \nu_2$. Then

- (1) $\mu_1 \odot_h \nu_1 \subseteq \vee q \mu_2 \odot_h \nu_2$.
- (2) $\mu_1 \cap \nu_1 \subseteq \vee q \mu_2 \cap \nu_2$.

Lemma 2.8: Let S be a hemiring and $\mu, \nu, \omega \in F(S)$. Then

- (1) $\mu \odot_h (\nu \cup \omega) = \mu \odot_h \nu \cup \mu \odot_h \omega$.
- (2) $\mu \odot_h (\nu \cap \omega) \subseteq \vee q \mu \odot_h \nu \cap \mu \odot_h \omega$.

Lemma 2.9[18]: Let S be a hemiring and $A, B \subseteq S$. Then we have

- (1) $A \subseteq B$ if and only if $\chi_A \subseteq \vee q \chi_B$.
- (2) $\chi_A \cap \chi_B = \chi_{A \cap B}$.
- (3) $\chi_A \odot_h \chi_B = \chi_{\overline{AB}}$.
- (4) $\chi_A +_h \chi_B = \chi_{\overline{A+B}}$.

Proof: We only show (4). Let $x \in S$. If $x \in \overline{A+B}$, then

$(\chi_A +_h \chi_B)(x) = 1$ and $x + a'_1 + b'_1 + z = a'_2 + b'_2 + z$ for some $a'_1, a'_2 \in A, b'_1, b'_2 \in B$ and $z \in S$. Thus we have

$$\begin{aligned} (\chi_A +_h \chi_B)(x) &= \sup_{x+a_1+b_1+z=a_2+b_2+z} M(\chi_A(a_1), \chi_A(a_2), \chi_B(b_1), \chi_B(b_2)) \\ &\geq M(\chi_A(a'_1), \chi_A(a'_2), \chi_B(b'_1), \chi_B(b'_2)) = 1, \end{aligned}$$

and so $(\chi_A +_h \chi_B)(x) = 1 = \chi_{\overline{A+B}}(x)$.

If $x \notin \overline{A+B}$, then $\chi_{\overline{A+B}}(x) = 0$ and there does not exist $a_1, a_2 \in A, b_1, b_2 \in B$ and $z \in S$ such that $x + a_1 + b_1 + z = a_2 + b_2 + z$. Thus $(\chi_A +_h \chi_B)(x) = 0 = \chi_{\overline{A+B}}(x)$.

In any case, we have $\chi_A +_h \chi_B = \chi_{\overline{A+B}}$. Therefore (4) is satisfied.

3. $(\in, \in \vee q)$ -fuzzy h -ideals in a hemiring

In this section, using the new order relation, we define and investigate $(\in, \in \vee q)$ -fuzzy h -ideals and $(\in, \in \vee q)$ -fuzzy h -interior ideals of a hemiring.

Definition 3.1: A fuzzy subset μ in a hemiring S is called an $(\in, \in \vee q)$ -fuzzy left (resp. right) h -ideal if it satisfies:

- (i) $\mu +_h \mu \subseteq \vee q \mu$,
- (ii) $\chi_S \odot_h \mu \subseteq \vee q \mu$ (resp. $\mu \odot_h \chi_S \subseteq \vee q \mu$),
- (iii) $x + a + z = b = z, a_r, b_s \in \mu \rightarrow x_{M(r,s)} \in \vee q \mu$ for all $a, b, x, z \in S$ and $r, s \in (0, 1]$.

A fuzzy subset μ in a hemiring S is called an $(\in, \in \vee q)$ -fuzzy h -ideal of S if it is both an $(\in, \in \vee q)$ -fuzzy left h -ideal and an $(\in, \in \vee q)$ -fuzzy right h -ideal of S .

Definition 3.2: A fuzzy subset μ in a hemiring S is called an $(\in, \in \vee q)$ -fuzzy left (resp. right) h -ideal if it satisfies:

- (i) $\mu +_h \mu \subseteq \vee q \mu$,
- (ii) $\mu \odot_h \mu \subseteq \vee q \mu$,
- (iii) $\chi_S \odot_h \mu \odot_h \chi_S \subseteq \vee q \mu$,
- (iv) $x + a + z = b = z, a_r, b_s \in \mu \rightarrow x_{M(r,s)} \in \vee q \mu$ for all $a, b, x, z \in S$ and $r, s \in (0, 1]$.

Note that if μ is an $(\in, \in \vee q)$ -fuzzy left h -ideal (right h -ideal, h -interior), then $\mu(0) \geq M(\mu(x), 0.5)$ for all $x \in S$ since $0 + x + 0 = x + 0$.

Theorem 3.3: A fuzzy subset μ in a hemiring S is an $(\in, \in \vee q)$ -fuzzy left (resp. right) h -ideal of S if and only if it satisfies the following conditions: $\forall a, b, x, z \in S$

- (1) $\mu(x + y) \geq M(\mu(x), \mu(y), 0.5)$;
- (2) $\mu(xy) \geq M(\mu(y), 0.5)$ (resp. $\mu(xy) \geq M(\mu(x), 0.5)$);
- (3) $x + a + z = b + z \rightarrow \mu(x) \geq M(\mu(a), \mu(b), 0.5)$.

Proof: Let μ be any $(\in, \in \vee q)$ -fuzzy left h -ideal of S . Now, if possible, let $\mu(x + y) < r = M(\mu(x), \mu(y), 0.5)$ for some $x, y \in S$. Then $\mu(x) \geq r, \mu(y) \geq r$ and $\mu(x + y) < r \leq 0.5$, that is, $(x + y)_r \in \vee q \mu$. On the other hand, we have

$$(\mu +_h \nu)(x + y) = \sup_{x+y+a_1+b_1+z=a_2+b_2+z} M(\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2)) \geq M(\mu(0), \mu(x), \mu(y)) \geq M(\mu(x), \mu(y), 0.5) \geq r.$$

Hence $(x + y)_r \in \mu +_h \mu$, which contradicts $\mu +_h \mu \subseteq \vee q\mu$. Therefore condition (1) is satisfied. The proof of condition (2) is similar to that of (1). For condition (3), if there exist $a, b, x, z \in S$ with $x + a + z = b + z$ and $s \in (0, 1]$ such that $\mu(x) < s = M(\mu(a), \mu(b), 0.5)$, this implies that $a_s, b_s \in \mu$ and $x_s \in \vee q\mu$, which contradicts the condition (iii) of Definition 3.1. Hence condition (3) is valid.

Conversely, assume that the given conditions hold. For $x_r \in \mu +_h \mu$, if possible, let $x_r \in \vee q\mu$. Then $\mu(x) < r$ and $\mu(x) < 0.5$. If there exist $a_1, a_2, b_1, b_2, x, z \in S$ with $x + a_1 + b_1 + z = a_2 + b_2 + z$, then by conditions (1) and (3), we have

$$0.5 > \mu(x) \geq M(\mu(a_1 + b_1), \mu(a_2 + b_2), 0.5) \geq M(\mu(a_1), \mu(a_2), \mu(b_1), \mu(b_2), 0.5),$$

which implies $\mu(x) \geq M(\mu(a_1), \mu(a_2), \mu(b_1), \mu(b_2))$.

Hence we have

$$r \leq (\mu +_h \nu)(x) = \sup_{x+a_1+b_1+z=a_2+b_2+z} M(\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2)) \leq \sup_{x+a_1+b_1+z=a_2+b_2+z} \mu(x) = \mu(x),$$

a contradiction. Hence condition (i) of Definition 3.1 is satisfied. The proof of condition (ii) is similar to that of condition (i). For condition (iii), if there exist $a, b, x, z \in S$ and $r, s \in (0, 1]$ with $x + a + z = b + z$ and $a_r, b_s \in \mu$ such that $x_{M(r,s)} \in \vee q\mu$, then $\mu(a) \geq r$, $\mu(b) \geq s$, $\mu(x) < M(r, s)$ and $\mu(x) < 0.5$, which contradicts $\mu(x) \geq M(\mu(a), \mu(b), 0.5)$. Thus condition (iii) is valid. Therefore, μ is an $(\in, \in \vee q)$ -fuzzy left h -ideal of S .

The case for $(\in, \in \vee q)$ -fuzzy right h -ideals can be similarly proved.

As a directly consequence of Theorem 3.3, we have the following result.

Theorem 3.4: A fuzzy subset μ in a hemiring S is an $(\in, \in \vee q)$ -fuzzy h -ideal of S if and only if it satisfies the following conditions: $\forall a, b, x, z \in S$

- (1) $\mu(x + y) \geq M(\mu(x), \mu(y), 0.5)$;
- (2) $\mu(xy) \geq M(\mu(x) \vee \mu(y), 0.5)$;
- (3) $x + a + z = b + z \rightarrow \mu(x) \geq M(\mu(a), \mu(b), 0.5)$.

Theorem 3.5: A fuzzy subset μ in a hemiring S is an $(\in, \in \vee q)$ -fuzzy h -interior ideal of S if and only if it

satisfies the following conditions: $\forall a, b, x, z \in S$

- (1) $\mu(x + y) \geq M(\mu(x), \mu(y), 0.5)$;
- (2) $\mu(xy) \geq M(\mu(x), \mu(y), 0.5)$;
- (3) $\mu(xyz) \geq M(\mu(y), 0.5)$;
- (4) $x + a + z = b + z \rightarrow \mu(x) \geq M(\mu(a), \mu(b), 0.5)$.

Proof: Let μ be any $(\in, \in \vee q)$ -fuzzy h -interior ideal of S . The proof of conditions (1), (2) and (4) is similar to that of Theorem 3.3. We show (3). If possible, let $\mu(xyz) < r = M(\mu(y), 0.5)$ for some $x, y, z \in S$.

Then $\mu(y) \geq r$ and $\mu(xyz) < r \leq 0.5$, that is, $(xyz)_r \in \vee q\mu$. On the other hand, we have

$$\begin{aligned} & (\chi_S \odot_h \mu \odot_h \chi_S)(xyz) \\ &= \sup_{xyz+\sum_{i=1}^m a_i b_i + z' = \sum_{j=1}^n a'_j b'_j + z'} M((\chi_S \odot_h \mu)(a_i), (\chi_S \odot_h \mu)(a'_j)) \\ &\geq M((\chi_S \odot_h \mu)(0), (\chi_S \odot_h \mu)(xy)) \\ &\geq M((\chi_S \odot_h \mu)(xy), 0.5) \\ &= M \left(\sup_{xy+\sum_{i=1}^m a_i b_i + z' = \sum_{j=1}^n a'_j b'_j + z'} M(\mu(b_i), \mu(b'_j)), 0.5 \right) \\ &\geq M(M(\mu(0), \mu(y)), 0.5) \geq M(\mu(y), 0.5) \geq r. \end{aligned}$$

Hence $(xyz)_r \in \chi_S \odot_h \mu \odot_h \chi_S$, which contradicts

$$\chi_S \odot_h \mu \odot_h \chi_S \subseteq \vee q\mu.$$

Conversely, assume that the given conditions hold. The proof the conditions (i), (ii) and (iv) of Definition 3.2 is similar to that of Theorem 3.3. We show (iii). For $x_r \in \chi_S \odot_h \mu \odot_h \chi_S$, if possible, let $x_r \in \vee q\mu$. Then $\mu(x) < r$ and $\mu(x) < 0.5$. On the other hand, we have

$$\begin{aligned} & (\chi_S \odot_h \mu \odot_h \chi_S)(x) \\ &= \sup_{x+\sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z} M((\chi_S \odot_h \mu)(a_i), (\chi_S \odot_h \mu)(a'_j)) \\ &= \sup_{x+\sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z} M \left(\sup_{a_i + \sum_{k=1}^{m_i} a_{ik} b_{ik} + z_1 = \sum_{l=1}^{n_i} a'_{il} b'_{il} + z_1} M(\mu(b_{ik}), \mu(b'_{il})), \right. \\ &\quad \left. \sup_{a'_j + \sum_{p=1}^{m_j} a_{jp} b_{jp} + z_2 = \sum_{q=1}^{n_j} a'_{jq} b'_{jq} + z_2} M(\mu(b_{jp}), \mu(b'_{jq})) \right) \end{aligned}$$

$$= \sup_{x + \sum_{i=1}^{m'} a_i b_i c_i + z' = \sum_{j=1}^{n'} a'_j b'_j c'_j + z'} M(\mu(b_i), \mu(b'_j)).$$

For $x + \sum_{i=1}^{m'} a_i b_i c_i + z' = \sum_{j=1}^{n'} a'_j b'_j c'_j + z'$, by conditions (1),

(2) and (3), we have

$$\begin{aligned} 0.5 > \mu(x) &\geq M\left(\mu\left(\sum_{i=1}^{m'} a_i b_i c_i\right), \mu\left(\sum_{j=1}^{n'} a'_j b'_j c'_j\right), 0.5\right) \\ &\geq M(M(\mu(a_i b_i c_i), 0.5), M(\mu(a'_j b'_j c'_j), 0.5), 0.5) \\ &\geq M(\mu(b_i), \mu(b'_j), 0.5), \end{aligned}$$

this implies $\mu(x) \geq M(\mu(b_i), \mu(b'_j))$. Thus we have

$$\begin{aligned} r &\leq (\chi_S \odot_h \mu \odot_h \chi_S)(x) \\ &= \sup_{x + \sum_{i=1}^{m'} a_i b_i c_i + z' = \sum_{j=1}^{n'} a'_j b'_j c'_j + z'} M(\mu(b_i), \mu(b'_j)) \\ &\leq \sup_{x + \sum_{i=1}^{m'} a_i b_i c_i + z' = \sum_{j=1}^{n'} a'_j b'_j c'_j + z'} \mu(x) = \mu(x), \end{aligned}$$

a contradiction. Hence $x_r \in \mu$, which gives $\chi_S \odot_h \mu \odot_h \chi_S \subseteq \vee q \mu$. Therefore the condition (iii) of Definition 3.2 is valid. This completes the proof.

Lemma 3.6: Let S be a hemiring and $A \subseteq S$. Then the following conditions hold.

(1) A is a left (resp. right) h -ideal of S if and only if χ_A is an $(\in, \in \vee q)$ -fuzzy left (resp. right) h -ideal of S .

(2) A is an h -interior ideal of S if and only if χ_A is an $(\in, \in \vee q)$ -fuzzy h -interior ideal of S .

Proof: The proof is straightforward.

4. H -semisimple hemiring

Definition 4.1: A subset A in a hemiring S is called weak idempotent if $A = \overline{AA}$. A fuzzy subset μ in a hemiring S is called $(\in, \in \vee q)$ -fuzzy weak idempotent if $\mu \approx \mu \odot_h \mu$.

Example 4.2: Consider Example 2.4. Let $A = \{0, a\}$. Evidently $AA = \{0\}$ and $\overline{AA} = \{0, a\} = A$. Let μ be a fuzzy subset in S such that $\mu(0) = 0.6$, $\mu(a) = 0.5$, $\mu(b) = 0.6$. Then $\mu \approx \mu \odot_h \mu$.

Definition 4.3: A hemiring S is called h -semisimple if every h -ideal of S is weak idempotent.

Lemma 4.4: A hemiring S is h -semisimple if and only if one of the following conditions holds:

(1) There exist $c_i, d_i, e_i, f_i, c'_j, d'_j, e'_j, f'_j, z \in S$ such that $x + \sum_{i=1}^m c_i x d_i e_i x f_i + z = \sum_{j=1}^n c'_j x d'_j e'_j x f'_j + z$ for all $x \in S$.

(2) $x \in \overline{SxSxS}$ for all $x \in S$.

(3) $A \subseteq \overline{SASAS}$ for all $A \subseteq S$.

Proof: Assume that S is an h -semisimple hemiring. Let x be any element of S . Then the set $\overline{Sx + xS + SxS + Nx}$, where $N = \{0, 1, 2, \dots\}$, is the principal h -ideal of S generated by x . By Lemma 2.1 and assumption, we have $A = \overline{A} = \overline{AA}$ and so

$$\begin{aligned} x = 0 + x &\in \overline{Sx + xS + SxS + Nx} \\ &= \overline{(Sx + xS + SxS + Nx)(Sx + xS + SxS + Nx)} \\ &= \overline{SxSx + SxxS + SxSxS + SxNx + xSSx + xSxS + xSSxS} \\ &\quad + \overline{xSNx + SxSSx + SxSxS + SxSSxS + SxSNx} \\ &\quad + \overline{NxSx + NxSxS + NxSxS + NxNx} \\ &\subseteq \overline{SxSSxS}, \end{aligned}$$

this implies that there exist $c_i, d_i, e_i, f_i, c'_j, d'_j, e'_j, f'_j, z \in S$ such that $x + \sum_{i=1}^m c_i x d_i e_i x f_i + z = \sum_{j=1}^n c'_j x d'_j e'_j x f'_j + z$. Hence (1) holds. (1) \Leftrightarrow (2) \Leftrightarrow (3) is clear. Assume that (3) holds. Let A be any h -ideal of S , by assumption and Lemma 2.1, we have $A \subseteq \overline{SASAS}$. The converse inclusion always holds, and so we have $A = \overline{AA}$. Therefore S is h -semisimple. This completes the proof.

As is easily seen, every $(\in, \in \vee q)$ -fuzzy h -ideal of a hemiring S is an $(\in, \in \vee q)$ -fuzzy h -interior ideal of S . But if S is h -semisimple, we have the following theorem.

Theorem 4.5: Every $(\in, \in \vee q)$ -fuzzy h -interior ideal of an h -semisimple hemiring S is an $(\in, \in \vee q)$ -fuzzy h -ideal of S .

Proof: Assume that S is an h -semisimple hemiring. Let μ be an $(\in, \in \vee q)$ -fuzzy h -interior ideal of S and let x and y be any elements of S . Since S is h -semisimple, there exist $c_i, d_i, e_i, f_i, c'_j, d'_j, e'_j, f'_j, z \in S$ such that

$$x + \sum_{i=1}^m c_i x d_i e_i x f_i + z = \sum_{j=1}^n c'_j x d'_j e'_j x f'_j + z, \text{ and so}$$

$$xy + \sum_{i=1}^m c_i x d_i e_i x f_i y + zy = \sum_{j=1}^n c'_j x d'_j e'_j x f'_j y + zy. \text{ Then}$$

we have

$$\begin{aligned} \mu(xy) &\geq M(\mu(\sum_{i=1}^m c_i x d_i e_i x f_i y), \mu(\sum_{j=1}^n c'_j x d'_j e'_j x f'_j y), 0.5) \\ &\geq M(M(\mu(c_i x d_i e_i x f_i y), 0.5), M(\mu(c'_j x d'_j e'_j x f'_j y), 0.5), 0.5) \\ &= M(\mu((c_i x d_i e_i) x (f_i y)), \mu((c'_j x d'_j e'_j) x (f'_j y)), 0.5) \\ &\geq M(M(\mu(x), 0.5), M(\mu(y), 0.5), 0.5) = M(\mu(x), 0.5). \end{aligned}$$

This implies that μ is an $(\in, \in \vee q)$ -fuzzy right h -ideal of S . In a similar way we may prove that μ is an $(\in, \in \vee q)$ -fuzzy left h -ideal of S . Thus μ is an $(\in, \in \vee q)$ -fuzzy h -ideal of S .

Theorem 4.6: A hemiring S is h -semisimple if and only if for any $(\in, \in \vee q)$ -fuzzy h -interior ideals μ and ν of S we have $\mu \cap \nu \approx \mu \odot_h \nu$.

Proof: Assume that S is h -semisimple. Let μ and ν be any $(\in, \in \vee q)$ -fuzzy h -interior ideals of S . Then, by Theorem 4.5, both μ and ν are $(\in, \in \vee q)$ -fuzzy h -ideals of S , and so $\mu \odot_h \nu \subseteq \vee q \mu \odot_h \chi_S \subseteq \vee q \mu$ and $\mu \odot_h \nu \subseteq \vee q \chi_S \odot_h \nu \subseteq \vee q \nu$. Thus $\mu \odot_h \nu \subseteq \vee q \mu \cap \nu$. Now let x be any element of S . Then, since S is h -semisimple, there exist $c_i, d_i,$

$$e_i, f_i, c'_j, d'_j, e'_j, f'_j, z \in S \text{ such that } x + \sum_{i=1}^m c_i x d_i e_i x f_i$$

$$+ z = \sum_{j=1}^n c'_j x d'_j e'_j x f'_j + z. \text{ Thus we have}$$

$$\begin{aligned} (\mu \odot_h \nu)(x) &= \sup_{x + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z} M(\mu(a_i), \mu(a'_j), \nu(b_i), \nu(b'_j)) \\ &\geq M(\mu(c_i x d_i), \mu(c'_j x d'_j), \nu(e_i x f_i), \nu(e'_j x f'_j)) \\ &\geq M(\mu(x), \nu(x), 0.5) = M((\mu \cap \nu)(x), 0.5). \end{aligned}$$

This implies $\mu \cap \nu \subseteq \vee q \mu \odot_h \nu$. So $\mu \cap \nu \approx \mu \odot_h \nu$.

Conversely, assume that the given condition holds. Let A be any h -ideal of S . Then it is clear that A is an h -interior ideal of S . By Lemma 3.6, the characteristic function χ_A of A is an $(\in, \in \vee q)$ -fuzzy h -interior ideal of S . Now by assumption and Lemma 2.9, we have

$$\chi_A = \chi_A \cap \chi_A \approx \chi_A \odot_h \chi_A = \underline{\chi_{AA}}.$$

Using Lemma 2.9 again, we have $A = \underline{AA}$. Thus S is h -semisimple.

Combining Theorems 4.5 and 4.6, we obtain directly the following theorem.

Theorem 4.7: Let S be a hemiring. Then the following conditions are equivalent.

- (1) S is h -semisimple.
- (2) Every $(\in, \in \vee q)$ -fuzzy h -ideal is $(\in, \in \vee q)$ -fuzzy weak idempotent.
- (3) Every $(\in, \in \vee q)$ -fuzzy h -interior ideal is $(\in, \in \vee q)$ -fuzzy weak idempotent.
- (4) $\mu \cap \nu \approx \mu \odot_h \nu$ for every $(\in, \in \vee q)$ -fuzzy h -ideals μ and ν of S .
- (5) $\mu \cap \nu \approx \mu \odot_h \nu$ for every $(\in, \in \vee q)$ -fuzzy h -ideal μ and every $(\in, \in \vee q)$ -fuzzy h -interior ideal ν of S .
- (6) $\mu \cap \nu \approx \mu \odot_h \nu$ for every $(\in, \in \vee q)$ -fuzzy h -interior ideal μ and every $(\in, \in \vee q)$ -fuzzy h -ideal ν of S .
- (7) The set of all $(\in, \in \vee q)$ -fuzzy h -ideals of S is a semilattice under the multiplication of fuzzy subsets and the relation " \approx " on $F(S)$, that is, $\mu \odot_h \nu \approx \nu \odot_h \mu$ and $\mu \odot_h \mu \approx \mu$ for all $(\in, \in \vee q)$ -fuzzy h -ideals μ and ν of S .
- (8) The set of all $(\in, \in \vee q)$ -fuzzy h -interior ideals of S is a semilattice under the multiplication of fuzzy subsets and the relation " \approx " on $F(S)$, that is, $\mu \odot_h \nu \approx \nu \odot_h \mu$ and $\mu \odot_h \mu \approx \mu$ for all $(\in, \in \vee q)$ -fuzzy h -interior ideals μ and ν of S .

5. Conclusions

In this paper, our aim is to promote the research and development of fuzzy technology by studying the fuzzy hemirings. We gave the notions of $(\in, \in \vee q)$ -fuzzy h -ideals and $(\in, \in \vee q)$ -fuzzy h -interior ideals of a hemiring and investigated some of their properties. We provided the concept of h -semisimple hemirings and gave its characterizations in term of $(\in, \in \vee q)$ -fuzzy h -ideals and $(\in, \in \vee q)$ -fuzzy h -interior ideals. Our future work on this topic will focus on studying of intuitionistic or interval-valued fuzzy sets in hemirings and other algebraic structures of hemirings.

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