

# Two Techniques to Process Conflicting Subjective Probabilities in Individual Decision Making

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## Abstract

**Two techniques for resolving conflicts in elicited subjective probabilities for individual decision making are proposed. The first one adjusts the fuzzy sets that represent the subjective probabilities in order to eliminate the conflicting parts, whereas the second one accomplishes the same task by expanding those sets. While the first technique is proposed as a necessary pre-processing step when the Laplace criterion is applied, the second one extends the results from Wald and Maximax criteria and proposes even more extreme outcomes. The application of both techniques to a problem of betting on sport games with conflicting elicited probabilities shows that in contrast to the proposal, the conventional pessimistic and optimistic analysis do not cover all the possibilities implied by the decision maker uncertainty. These techniques are intended for applications where probabilities are subjectively elicited and therefore more likely to include conflicting values.**

**Keywords:** *Decision making, elicitation technique, fuzzy set, imprecision, subjective probability.*

## 1. Introduction

Frequently, decision makers (DM) need to make individual decisions under complex and uncertain environments. Their knowledge and expertise of the situation give them the ability to evaluate the probability of the outcomes that certain actions might give. Although probability theory states that the sum of the probabilities of all the events in the outcome space must be equal to one, elicited subjective values usually do not satisfy this restriction. Therefore, it can be said that they constitute a set of conflicting probabilities.

However, most of the techniques for individual

decision making under uncertainty assume that the elicited values satisfy the addition to one. Therefore, two techniques that adjust the subjective probabilities in order to resolve the aforementioned conflict are proposed. The first one achieves this by reducing or eliminating a portion in the values which represent the probabilities, whereas the second one accomplishes the task by expanding those sets.

The reason for using fuzzy sets in the elicitation of subjective probabilities from non-ideal DMs, is explained in 2, as well as four criteria of individual decision making under strict uncertainty. Then two techniques for adjustment of conflicting subjective probabilities are proposed in 3 and their utilization in conjunction with the aforementioned criteria is discussed in 4. Finally, the application of both techniques to a problem of betting on sport games is presented in 5 as an illustration of their usage.

## 2. Non-Ideal DM and Decision Analysis under Strict Uncertainty

### A. Elicited Subjective Probabilities of Non-Ideal DMs

One of the motivations of fuzzy decision analysis (FDA) is to allow people to express their beliefs in a natural language, by associating fuzzy values to a set of linguistic labels such as “very likely” or “around 20% of the cases” [1]-[7]. In this way, FDA acknowledges two facts: first, that only ideal DMs can provide singletons as values representing their beliefs and second, that non-ideal DMs’ (NIDM) beliefs need a different kind of values, namely fuzzy values, for a better representation. Thus, an important advantage of FDA is its flexibility, and hence the comfort gained by DMs when expressing their beliefs. In addition, fuzzy values can model the imprecision inherent to NIDMs’ beliefs.

On the other hand, the relation between fuzzy values and linguistic labels is not unique. It mainly depends on the choices made by the decision analysts (DA), whose role is just to help the DM to analyze his or her beliefs in order to make a decision. An undesirable consequence of this fact is that the result of the analysis can be influenced by the DA’s choice regarding the relationship “fuzzy values - linguistic labels”. Since the decision should be based only on the DM’s beliefs, it becomes

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clear that it is desirable to have methods that help DMs to uniquely define fuzzy values for their beliefs.

One of such elicitation methods is analyzed in [8]-[10] where NIDMs' beliefs are elicited with the triple bisection method. This is done through an interview between the DA and the DM in which the latter must solve a set of preferential equations consisting of two fictitious lotteries. For example, a preferential equation verbally proposed to the DM may be: "which lottery do you prefer: lottery1, in which if A happens you receive \$100 and if A does not happen you receive nothing; or lottery2, in which if B happens you receive \$100 and if B does not happen you receive nothing?". In this way, the DM's belief on the likelihood of event A is elicited by solving this equation where the event B is a simple event previously explained to the DM. Event B is usually defined as the drawing of a black ball from an urn containing (m) black balls and (n-m) white balls. The number of black balls is changed until the DM expresses indifference between both lotteries, which means that the subjective probability of event A is (m/n). The corresponding mathematical formulation is:

$$p(B) = m/n, \quad (1)$$

$$\langle \$100(A)\$ \rangle \sim \langle \$100(B)\$ \rangle \Rightarrow p(A) = m/n,$$

where the symbol " $\sim$ " indicates indifference.

Unlike ideal DMs, NIDMs express indifference for more than one value of (m) because they are unable to turn from indifference to strict preference when the value of (m) only slightly changes. Therefore, NIDM's elicited beliefs are not represented by singletons but by fuzzy values which are called "indifference functions". In the interval where the DM expresses indifference, the indifference function has the value 1, whereas in the intervals where he turns to strict preference it has the value 0. Intermediate values are given in the intervals where the DM is unable to give a clear answer. In order to avoid complex calculations, these values can be assumed to change linearly and hence, the indifference function has a trapezoidal shape. The fact that the DM is only required to provide qualitative and not quantitative information makes the elicited data to be closer to the DM's actual belief since DMs feel more comfortable and find it easier to provide qualitative dichotomized assessments than quantitative ones [11], [12].

In other words, the indifference function is the membership function of the fuzzy sets that represent the NIDM's subjective probabilities. They have a trapezoidal shape and model how NIDMs estimate probabilities. In the following analysis, all the fuzzy sets representing probabilities are in fact indifference functions elicited from the NIDM.

#### B. Four Criteria of Decision Analysis under Strict Uncertainty

One of the usually followed approaches in FDA is to reduce the fuzzy values to singletons and then to apply conventional crisp Decision Analysis techniques. It is argued in [13], [14] that individual decision making with fuzzy probabilities elicited as explained in (2.A.) can be considered as a problem of decision making under strict uncertainty.

Therefore they propose to reduce the fuzzy values to singletons by applying techniques inspired by criteria for decision making under strict uncertainty, namely the Laplace criterion, Wald or Maximin criterion, Maximax criterion and Hurwicz criterion.

Let  $[p_{id}, p_{iu}] \subset [0,1]$  be the support of the subjective fuzzy probability  $P_i$  elicited from the DM for the state  $s_i \in S = \{s_1, s_2, \dots, s_n\}$ . The expressions for the techniques that calculate the singletons corresponding to the fuzzy probabilities are given as follows [13].

##### 1) Laplace criterion

According to the Laplace principle of insufficient reason, if no information is available regarding a set of hypotheses, then all of them can be assumed equally probable [15]. By applying this criterion to the elicited fuzzy probabilities, the problem becomes equivalent to assuming a uniform probability density function in the interval equal to the support of the fuzzy value. The reduced probability  $p_i$  for the interval  $[p_{id}, p_{iu}]$  is calculated with:

$$[p_{id}, p_{iu}] \rightarrow p_i = (1 - \alpha)p_{id} + \alpha p_{iu},$$

$$\alpha = \frac{1 - \sum_{j=1}^n p_{jd}}{\sum_{j=1}^n p_{ju} - \sum_{j=1}^n p_{jd}}. \quad (2)$$

If the denominator in the expression for  $\alpha$  is zero, then it is a degenerate case where all the fuzzy sets are actually singletons and therefore conventional Decision Analysis suffices. This expression coincides with the one given in [16] which is obtained from different assumptions.

##### 2) Wald or Maximin criterion

Wald criterion maximizes the benefit of the worst possible outcomes. If the sub-indices for the states  $s_i$  are given so that  $s_1 > s_2 > \dots > s_n$ , (" $x > y$ " denoting strict preference of x over y) then:

$$[p_{id}, p_{iu}] \rightarrow p_i = \begin{cases} p_{id} & \text{if } i < \beta, \\ (1 - \alpha_i)p_{id} + \alpha_i p_{iu} & \text{if } i = \beta, \\ p_{iu} & \text{if } i > \beta, \end{cases}$$

$$\alpha_i = \begin{cases} \frac{1 - \sum_{j=1}^i p_{jd} - \sum_{j=i+1}^n p_{ju}}{p_{iu} - p_{id}} & \text{if } p_{iu} - p_{id} \neq 0, \\ 0 & \text{if } p_{iu} - p_{id} = 0, \end{cases} \quad (3)$$

$$\beta = \arg\{\alpha_i \mid \alpha_i \in (0,1)\}.$$

3) Maximax criterion

This criterion maximizes the benefit of the best possible outcome and therefore it is similar to Wald criterion with the difference that the sub-indices of the states  $s_i$  should be given so that  $s_n > s_{n-1} > \dots > s_1$ .

4) Hurwicz criterion

The application of this criterion gives a reduced probability which is the weighted average of those given by Wald and Maximax.

$$[p_{id}, p_{iu}] \rightarrow p_i = (1 - \alpha)p_{i, \text{Maximax}} + \alpha p_{i, \text{Wald}}, \quad (4)$$

where  $\alpha \in [0,1]$  is a pessimistic index.

The reduced probabilities  $p_i$  for three fuzzy probabilities ( $n=3$ ) given by Laplace, Wald and Maximax criteria are shown as black dots in Fig. 1. The cube in Fig. 1. is defined by the set of values  $C = \{(x,y,z) | x \in [p_{1d}, p_{1u}], y \in [p_{2d}, p_{2u}], z \in [p_{3d}, p_{3u}]\}$ , where the probabilities values  $x, y, z$  of states  $s_1, s_2, s_3$ , are expressed in the axes X, Y, and Z respectively. The shaded plane is the set of values in  $X \times Y \times Z$  that belongs to the cube and whose sum is equal to one. It is interesting to notice that the reduced probability  $p_i$  depends not only on their corresponding fuzzy probabilities  $P_i$  alone, but on their whole set  $\{P_1, P_2, P_3\}$ . Moreover, contrary to the assumptions in [16], they also depend on the benefits  $c_i$ . For example, the reduced probability values given by Wald, Maximax and Hurwicz criteria change if the DM's preference on the states satisfies  $s_1 > s_2 > s_3$ , rather than  $s_3 > s_2 > s_1$ .

### 3. Techniques to Process Conflicting Subjective Probabilities in Individual Decision Making

Besides the fact that elicited probabilities are better expressed with fuzzy values when NIDMs are considered, there are other consequences to be accounted. One of them is the fact that depending on the complexity of the situation being analyzed, it is possible that NIDMs provide values which are not consistent, i.e. there is no combination of probability values with membership greater than zero which adds to one.

*Definition 1:* A finite set of interval probabilities  $P_1, P_2, \dots, P_n$  is consistent if there exists values  $p_1 \in P_1, p_2 \in P_2, \dots, p_n \in P_n$ , such that  $p_1 + p_2 + \dots + p_n = 1$ .

*Definition 2:* A finite set of fuzzy probabilities  $P_1, P_2, \dots, P_n$  is consistent if there exist values  $p_1, p_2, \dots, p_n$  such that  $\mu_{P_1}(p_1) > 0, \mu_{P_2}(p_2) > 0, \dots, \mu_{P_n}(p_n) > 0$ , and  $p_1 + p_2 + \dots + p_n = 1$ .

*Preposition 1:* A finite set of interval probabilities  $P_1 = [p_{1d}, p_{1u}], P_2 = [p_{2d}, p_{2u}], \dots, P_n = [p_{nd}, p_{nu}]$  is consistent, if they satisfy

$$\sum_{j=1}^n p_{jd} \leq 1 \leq \sum_{j=1}^n p_{ju}. \quad (5)$$

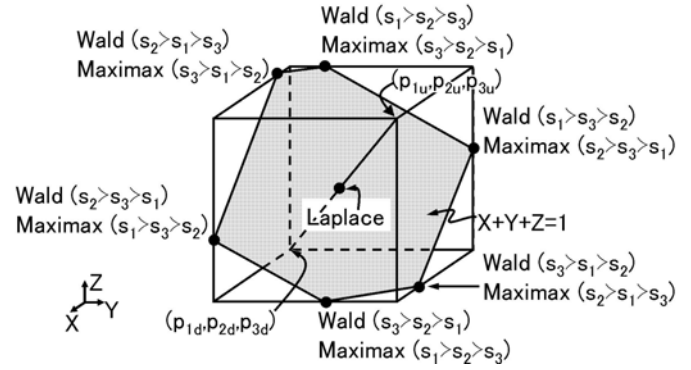


Figure 1. Location of the reduced probabilities for the three state case ( $n=3$ ) given by Laplace, Wald and Maximax criteria. The probabilities of the states  $s_1, s_2$ , and  $s_3$  are displayed along the axes X, Y, and Z. The cube is defined by the support  $[p_{1d}, p_{1u}]$  of the three elicited fuzzy probabilities, and the shaded plane is the set of probability values within the cube that satisfy addition to unity.

*Preposition 2:* A finite set of fuzzy probabilities  $P_1, P_2, \dots, P_n$  is consistent if their support is consistent.

*Preposition 3:* If the support of the  $\alpha$ -cut of a finite set of fuzzy probabilities  $P_1, P_2, \dots, P_n$  is consistent for  $\alpha=1$ , then their support is consistent for any  $\alpha$ .

The proofs of the above prepositions are straightforward and are omitted in order to keep the paper within proper length.

Other consequence of considering NIDMs is that the elicited fuzzy probabilities can include values which violate the constraint of addition to one. For example, if only two states are analyzed and their fuzzy probabilities' support are  $\text{Supp}(P_1) = [0.2, 0.4]$  and  $\text{Supp}(P_2) = [0.7, 0.9]$ , it is clear that if  $P_1$  takes values greater than 0.3, then  $P_2$  goes beyond the limits of  $\text{Supp}(P_2)$ . The same can be stated for values greater than 0.8 for  $P_2$ . DMs are usually requested to refine their assessments when they provide this kind of conflicting values during the elicitation process. However, it is possible that in some complex situations NIDMs are unable to provide values which do not conflict with each other even after thinking over the situation, and in consequence it is necessary to use methods which adjust the elicited values so that (2)-(4) can be applied.

In the following, two techniques for adjusting conflicting elicited subjective probabilities are proposed. The first one is necessary in order to make (2) valid, whereas the second considers extreme cases by extending the uncertainty expressed by the DM.

#### A. "Adjustment by Contraction" Technique

The adjustment by contraction technique "contracts" or reduces the membership values of the fuzzy probabilities in order to ensure that all the values in the intervals defined by  $\alpha$ -cuts can be taken without forcing

other values beyond their boundaries.

Procedure:

*Step 1:* Discretize the range of membership [0,1] into a finite number of values  $\alpha_1, \alpha_2, \dots, \alpha_m$ .

*Step 2:* For an  $\alpha_i$ , determine the support of the  $\alpha$ -cuts of the fuzzy probabilities  $P_1, P_2, \dots, P_n$ , namely  $[p_{1d,i}, p_{1u,i}], [p_{2d,i}, p_{2u,i}], \dots, [p_{nd,i}, p_{nu,i}]$ . As explained in Subsection 2.A., all the fuzzy probabilities are normal and therefore the  $\alpha$ -cuts produce non-null intervals.

*Step 3:* The adjusted values  $[q_{jd,i}, q_{ju,i}]$  are:

$$q_{jd,i} = \max\{p_{jd,i}, 1 - \sum_{\substack{k=1, \\ k \neq j}}^n p_{ku,i}\},$$

$$q_{ju,i} = \min\{p_{ju,i}, 1 - \sum_{\substack{k=1, \\ k \neq j}}^n p_{kd,i}\}.$$
(6)

*Step 4:* Go back to Step2 and repeat the process for the rest of the  $\alpha$  values.

The “contracted” intervals obtained in Step3 are the support of the  $\alpha$ -cuts of the adjusted fuzzy probabilities. The degree of accuracy of the result increases as the number of values of  $\alpha$  does.

Fig. 2. shows an example of the possible result for the case of three fuzzy probabilities. The bigger cube is defined by the support  $[p_{id}, p_{iu}]$  ( $j=1,2,3$ ) of the  $\alpha$ -cut of the fuzzy probabilities’ original values whereas the smaller cube drawn with thicker lines is defined by the intervals  $[q_{jd}, q_{ju}]$  ( $j=1,2,3$ ) calculated with the adjusted by contraction technique.

### B. “Adjustment by Expansion” Technique

Unlike the previous technique, the adjustment by expansion technique allows some of the probability values to go beyond their boundaries. This is done in order to account on the extremely pessimistic and/or optimistic results implied by the conflicting elicited probabilities. For instance, in the example at the beginning of this Section 3, if the DM prefers the state  $s_1$  to the state  $s_2$ , and the Wald criterion is applied, the worse outcome occurs when  $P_2$  takes the maximum possible value (0.8) whereas  $P_1$  takes its minimum (0.2). However, a more pessimistic outcome is to consider that, since the DM assesses that  $s_2$  can occur with a probability higher than 0.8, he somehow overestimates  $P_1$  and  $s_1$  can occur with probability lower than 0.2, thus allowing  $s_2$  to reach its maximum at 0.9. We call this extreme pessimistic outcome “pessimistic maximin outcome”. In a similar fashion, the “optimistic maximax outcome” is defined by allowing in the analysis that the state most favorable for the DM occurs with its maximum probability.

This technique is only applied with the Maximax criterion if the fuzzy probability of the most (least) favorable state includes upper (lower) values that

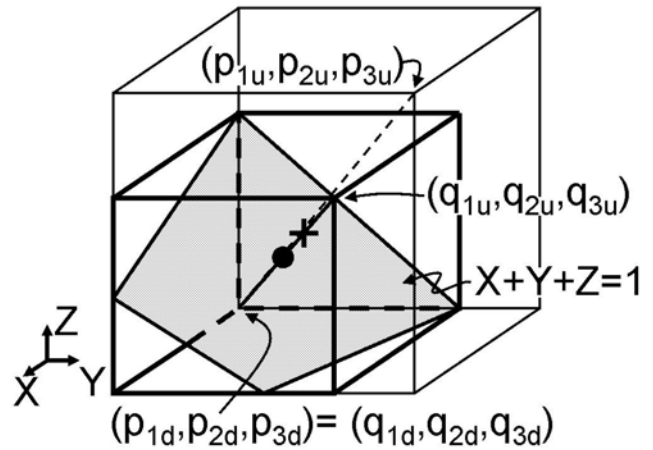


Figure 2. Example of the result given by the adjustment by contraction technique for the case of three fuzzy probabilities. The bigger cube is defined by the support  $[p_{id}, p_{iu}]$  of the fuzzy probabilities’  $\alpha$ -cuts, whereas the smaller cube drawn in thicker lines is defined by the adjusted intervals  $[q_{id}, q_{iu}]$ . The corresponding reduced probabilities calculated with the Laplace criterion are shown as a cross and a black circle, respectively.

provoke other probabilities to take values beyond their lower (upper) limits. If Wald criterion is applied, the condition is similar with the most favorable state replaced by the least one, and vice versa.

Procedure:

*Step 1:* Rearrange the order of the states so that, if action  $A_j$  is performed the sub-indices of the states satisfy  $s_{1j} > s_{2j} > \dots > s_{nj}$ .

*Step 2:* Discretize the range of membership [0,1] into a finite number of values  $\alpha_1, \alpha_2, \dots, \alpha_{1m}$ .

*Step 3:* For an  $\alpha_i$ , determine the support of the  $\alpha$ -cuts of the fuzzy probabilities  $P_1, P_2, \dots, P_n$ , namely  $[p_{1d,i}, p_{1u,i}], [p_{2d,i}, p_{2u,i}], \dots, [p_{nd,i}, p_{nu,i}]$ . As explained in Subsection 2.A., all the fuzzy probabilities are normal and therefore the  $\alpha$ -cuts produce non-null intervals.

*Step 4:* If Maximax (Wald) criterion is considered, define the intervals  $[p_{kd,new}, p_{ku,new}]$  for  $k=1,2, \dots, n$  (sub-indices “i” have been removed to keep the following expressions simple) according to the following rules:

$$\text{(Maximax):} \left\{ \begin{array}{l} p_{1u} + \sum_{s=2}^n p_{sd} > 1 \text{ (case=1)} \rightarrow \\ \rightarrow \left\{ \begin{array}{l} [p_{1d,new}, p_{1u,new}] = [p_{1d}, p_{1u}] \\ [p_{kd,new}, p_{ku,new}] = [0, p_{kd}], \end{array} \right. \quad k=2,3, \dots, n \\ p_{nd} + \sum_{s=1}^{n-1} p_{su} < 1 \text{ (case=2)} \rightarrow \\ \rightarrow \left\{ \begin{array}{l} [p_{kd,new}, p_{ku,new}] = [p_{ku}, 1], \\ [p_{nd,new}, p_{nu,new}] = [p_{nd}, p_{nu}] \end{array} \right. \quad k=1,2, \dots, n-1 \end{array} \right.$$

$$\begin{aligned}
 \text{(Wald):} & \left\{ \begin{aligned} & p_{nu} + \sum_{s=1}^{n-1} p_{sd} > 1 \text{ (case=3)} \rightarrow \\ & \rightarrow \begin{cases} [p_{kd,new}, p_{ku,new}] = [0, p_{kd}], & k=1, 2, \dots, n-1 \\ [p_{nd,new}, p_{nu,new}] = [p_{nd}, p_{nu}] \end{cases} \\ & p_{1d} + \sum_{s=2}^n p_{su} < 1 \text{ (case=4)} \rightarrow \\ & \rightarrow \begin{cases} [p_{1d,new}, p_{1u,new}] = [p_{1d}, p_{1u}] & k=2, 3, \dots, n \\ [p_{kd,new}, p_{ku,new}] = [p_{ku}, 1], \end{cases} \end{aligned} \right. \quad (7)
 \end{aligned}$$

If no rule applies, then go to Step7 because the adjustment by expansion technique is not necessary for this particular  $\alpha$  value.

Step 5: Depending on the criterion being considered, apply (3) or the equivalent for the Maximax criterion to the new intervals  $[p_{kd,new}, p_{ku,new}]$ . Let us denote the result as  $\{p_k, k=1, 2, \dots, n\}$ .

Step 6: The adjusted values  $[q_{kd}, q_{ku}]$  are:

$$\begin{aligned}
 \text{(Maximax):} & \left\{ \begin{aligned} & \text{(case=1):} \\ & \rightarrow \begin{cases} [q_{1d}, q_{1u}] = [p_{1d}, p_{1u}] \\ [q_{kd}, q_{ku}] = [p_k, p_{ku}], & k=2, 3, \dots, n \end{cases} \\ & \text{(case=2):} \\ & \rightarrow \begin{cases} [q_{kd}, q_{ku}] = [p_{kd}, p_k], & k=1, 2, \dots, n-1 \\ [q_{nd}, q_{nu}] = [p_n, p_{nu}] \end{cases} \end{aligned} \right. \\
 \text{(Wald):} & \left\{ \begin{aligned} & \text{(case=3):} \\ & \rightarrow \begin{cases} [q_{kd}, q_{ku}] = [p_k, p_{ku}], & k=1, 2, \dots, n-1 \\ [q_{nd}, q_{nu}] = [p_{nd}, p_n] \end{cases} \\ & \text{(case=4):} \\ & \rightarrow \begin{cases} [q_{1d}, q_{1u}] = [p_1, p_{1u}] \\ [q_{kd}, q_{ku}] = [p_{kd}, p_k], & k=2, 3, \dots, n \end{cases} \end{aligned} \right. \quad (8)
 \end{aligned}$$

Step 7: Go back to Step3 and repeat the process for the rest of the  $\alpha$  values.

The resulting expanded fuzzy probabilities depend on the action chosen  $A_j$ . Different actions lead to different expanded fuzzy probabilities and, similar to the case of reduced probabilities, mentioned at the end of Subsection 2.B., the expanded probability  $[q_{kd}, q_{ku}]$  do not only depend on its corresponding fuzzy values  $P_k$  but on the whole set  $\{P_1, P_2, \dots, P_n\}$  in addition to the DM's preferences regarding the states  $\{s_{1j}, s_{2j}, \dots, s_{nj}\}$  as specified in Step1.

Step4 allows the probabilities of the states different from the most or least favorable ones to go beyond either their upper or lower boundaries by setting them as 1 or 0 respectively.

Cases 1 and 3 in Step4 and Step6 cover the situations illustrated in the examples given at the beginning of this Subsection 3.B. Case 4 covers the situation where the Wald criterion is considered in addition to the possibility

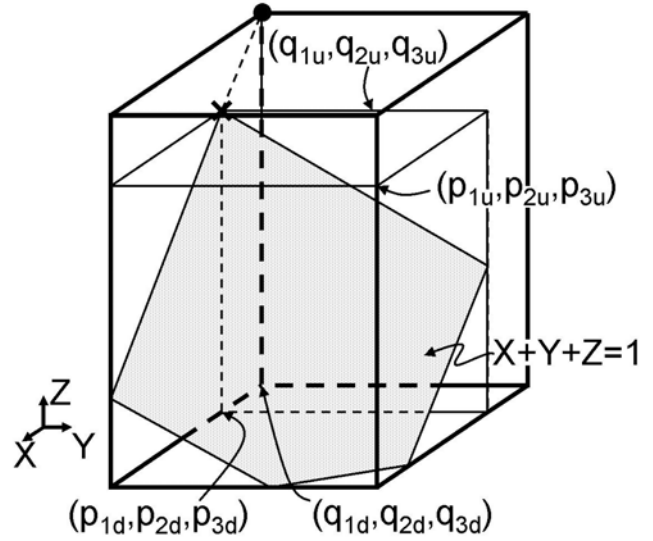


Figure 3. Example of the result given by the adjustment by expansion technique for the case of three fuzzy probabilities. The probabilities of states  $s_1, s_2,$  and  $s_3$  are displayed along the axes X, Y, and Z. The smaller cube is defined by the support  $[p_{id}, p_{iu}]$  of the fuzzy probabilities'  $\alpha$ -cuts, whereas the bigger cube drawn in thicker lines is defined by the adjusted intervals  $[q_{id}, q_{iu}]$  when the Wald criterion is applied and  $s_1 > s_2 > s_3$  is assumed. The corresponding reduced probabilities are shown as a cross and a black circle, respectively.

that the DM does not overestimate the probability  $P_1$  of the most favorable state but underestimates the probability  $P_2$  of the least favorable one. For example, if  $P_1=[0.2, 0.4]$  and  $P_2=[0.5, 0.7]$  ( $s_1 > s_2$ ), an outcome more pessimistic than conventional Wald's outcome is to consider that  $P_2$  is underestimated and that  $s_2$  -the least desirable state- can occur with probability higher than 0.7 because  $P_1$  can reach a value as low as 0.2. Case 2 covers a similar situation when Maximax criterion is considered.

Although it is possible to extend this technique to allow in the analysis the case when the next best (worst) state also happens with its maximum probability and so on, there exists the risk that the uncertainty added by the technique increases to the point of making the elicited information meaningless.

Fig. 3. shows an example of the possible result for the case of three states  $s_1, s_2,$  and  $s_3$  whose fuzzy probabilities are displayed in the axes X, Y, and Z respectively. The smaller cube is defined by the support of the  $\alpha$ -cut of the fuzzy probabilities' original values whereas the bigger cube drawn with thicker lines is defined by the intervals  $[q_{jd}, q_{ju}]$  ( $j=1, 2, 3$ ) calculated with the adjusted by expansion technique. In the application of the technique, the Wald criterion is considered and  $s_1 > s_2 > s_3$  assumed.

#### 4. Utilization of the Techniques with the Four Criteria

Although the proposed techniques follow the same idea, they have a different approach for adjusting the conflicting probabilities. As a consequence, when the techniques are applied together with the criteria for decision making under strict uncertainty, some of the criteria show more clearly the advantages of the proposals.

##### A. Utilization of the Adjustment by Contraction Technique

###### 1) Utilization with the Laplace criterion

Figure 1 shows a geometrical interpretation of the reduced probabilities according to the Laplace criterion and (2), for the three-state case. The reduced probability values are given by the coordinates of the intersection point between the hypercube's main diagonal with the hyperplane defined by the sum of the probabilities equal to one. Therefore, the solution is sensitive to the values of the corners of the hypercube, or in other words, it is sensitive to the extreme values of the intervals defined by the  $\alpha$ -cuts. In [13], (2) is given assuming that the elicited probabilities do not conflict with each other. If the elicited fuzzy probabilities present conflicting values, Step3 in Section 3.A. reduces the length of the support of the  $\alpha$ -cuts. Thus, (2) must be applied after the adjustment by contraction technique.

For Individual Decision Analysis with fuzzy probabilities  $P_i$  for states  $s_i$ , and prizes  $c_{ij}$  (expressed as utility) obtained if  $s_i$  happens after performing action  $A_j$ , the action that the DM should perform in order to maximize his satisfaction (according to the preferences expressed by the utility of  $c_{ij}$ ) is determined as follows.

Procedure:

*Step 1:* Obtain an  $\alpha$ -level for the analysis from the DM.

*Step 2:* Determine the support of the  $\alpha$ -cuts of  $P_i$ :  $[p_{id}, p_{iu}]$ .

*Step 3:* Apply Step3 of the adjustment by contraction technique on the intervals  $[p_{id}, p_{iu}]$ . Denote the adjusted values as  $[q_{id}, q_{iu}]$ .

*Step 4:* Apply Laplace criterion (2) to  $[q_{id}, q_{iu}]$  and find the reduced probabilities  $q_i$ .

*Step 5:* Select the action  $A_j$ , where

$$j = \arg \left\{ \max_j \left\{ \sum_{i=1}^n q_i c_{ij} \right\} \right\}. \quad (9)$$

The action that the analysis recommends the DM to perform is  $A_j$ .

The point indicating the reduced probabilities  $q_i$  given by the Laplace criterion in Step4 is shown in Fig. 2. as a black circle, whereas the cross indicates the reduced probabilities that result if the adjustment by contraction

technique is not applied.

###### 2) Utilization with other criteria

Different from the Laplace criterion, Wald, Maximax and Hurwicz criteria implicitly ignore the conflicting portions of the fuzzy probabilities and, therefore, (3) can be used directly. The result does not change if the adjustment by contraction technique is previously applied.

The action recommended by the analysis is usually chosen for a given  $\alpha$ -cut and by applying (9). If the DM requires the comparison of the results obtained for several values of  $\alpha$ , it is common to follow a fuzzy ranking method [5] or an averaging technique [17].

##### B. Utilization of the Adjustment by Expansion Technique

###### 1) Utilization with Wald and Maximax criteria

For Individual Decision Analysis with fuzzy probabilities  $P_i$  for states  $s_i$ , and prizes  $c_{ij}$  (expressed as utility) obtained if  $s_i$  happens after performing action  $A_j$ , the action that the DM should perform in order to maximize his satisfaction (according to the preferences expressed by the utility of  $c_{ij}$ ) is determined as follows.

Procedure:

*Step 1:* Obtain an  $\alpha$ -level for the analysis from the DM.

*Step 2:* Determine the support of the  $\alpha$ -cuts of  $P_i$ :  $[p_{id}, p_{iu}]$ .

*Step 3:* Rearrange the order of states so that, if action  $A_j$  is performed the sub-indices of the states satisfy  $s_{1j} > s_{2j} > \dots > s_{nj}$ .

*Step 4:* Apply Step4 to Step6 of the adjustment by expansion technique to the intervals  $[p_{id}, p_{iu}]$ . Denote the adjusted values as  $[q_{id}, q_{iu}]$ .

*Step 5:* If Wald criterion is considered, apply (3) to the values  $[q_{id}, q_{iu}]$  and calculate the reduced probabilities  $q_i$ . If the Maximax criterion is considered, calculate  $q_i$  by using the equivalent expression as explained in Subsection 2.B.3.

*Step 6:* Calculate the expected benefit  $b_j = \sum_{k=1}^n q_k c_{kj}$  for action  $A_j$ .

*Step 7:* Go back to Step3 and repeat the process for the rest of actions  $A_j$ .

*Step 8:* The DM should perform the action  $A_j$  with the highest expected benefit  $b_j$ .

The point indicating the reduced probabilities  $q_i$  given by the Wald criterion in Step5 ("pessimistic maximin") is shown in Fig. 3. as a black circle, whereas the cross indicates the reduced probabilities given by the conventional Wald criterion.

###### 2) Utilization with other criteria

The adjustment by expansion technique is based on the Wald or Maximax criteria and therefore the

technique is intended to be applied together with them. The Laplace criterion can also be applied to the resulting expanded fuzzy probabilities if the DM considers it appropriate and sensible to do so, though.

### 5. Application to Betting on Sport Games

The proposed techniques are applied to a betting problem in sport games to illustrate their application. The betting problem is based on the one presented in [13] and [14].

#### A. Definition of the Problem

Bets on the result of a soccer match between team A (host) and team B (visitor) are offered by a bookmaker house. The DM can bet a non-negative amount of dollars on each result, namely home win, draw or away win, or on a combination of them. The bookmaker house pays 1:5 for away win, that is, a profit of \$5 per dollar bet which amounts to a payoff of \$6 including the original bet of \$1. Home win and draw pay 3:1 and 2:5, respectively.

Let  $s_1$ ,  $s_2$ , and  $s_3$ , be the states home win, draw and away win, respectively. Therefore, the payoff coefficients are  $k_1=(1+1/3)=4/3$ ,  $k_2=(1+5/2)=7/2$ , and  $k_3=(1+5/1)=6$ . The DM owns a maximum of \$100 for betting and his goal is to maximize the profit obtained from the bets. The analysis is similar if the DM's goal is to maximize the expected utility of the profit and it is assumed that the DM's utility function is approximately linear in the range of money being considered.

The subjective probabilities of the events  $s_1$ ,  $s_2$ , and  $s_3$ , elicited from the NIDM are:  $P_1=0.18,0.19,0.21,0.22$ ,  $P_2=(0.45,0.47,0.53,0.55)$ , and  $P_3=(0.21,0.23,0.27,0.29)$  respectively, where  $P=(a,b,c,d)$  is defined to be a trapezoidal fuzzy value with the following membership function:

$$\mu_{P=(a,b,c,d)} = \begin{cases} \frac{(x-a)}{(b-a)} & x \in [a, b], \\ 1 & x \in [b, c], \\ \frac{(d-x)}{(d-c)} & x \in [c, d], \\ 0 & \text{otherwise.} \end{cases} \quad (10)$$

#### B. Optimal Bet using the Adjustment by Contraction Technique

The support of the  $\alpha$ -cuts of  $P_1$ ,  $P_2$ , and  $P_3$ , for  $\alpha=1$  is  $[0.19,0.21]$ ,  $[0.47,0.53]$ , and  $[0.23,0.27]$  respectively. Proposition1 shows that they are consistent whereas Proposition3 shows that the support of the  $\alpha$ -cut for any value of  $\alpha$  are also consistent. By applying (6), it becomes clear that the elicited fuzzy probabilities have conflicting values. The adjusted by contraction fuzzy pr-

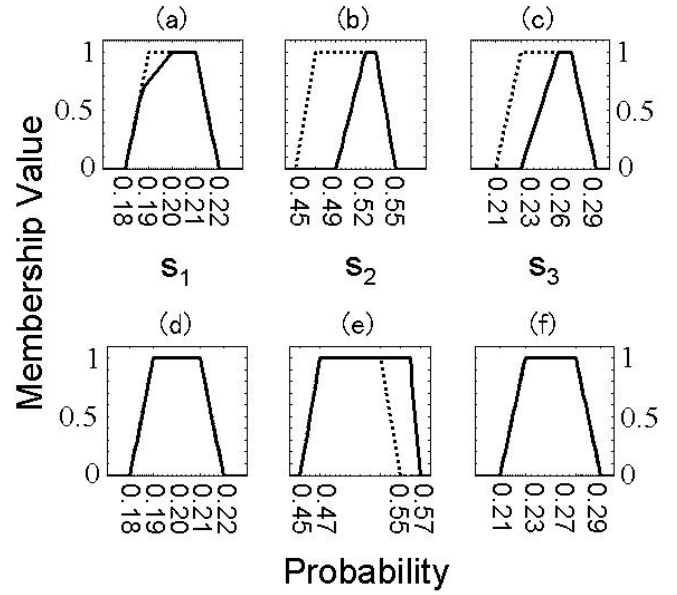


Figure 4. (a)-(c) Original and adjusted by contraction probabilities of  $s_1$ ,  $s_2$ , and  $s_3$  respectively. (d)-(f) Original and adjusted by expansion probabilities of  $s_1$ ,  $s_2$ , and  $s_3$ , with Wald criterion and  $s_3 > s_1 > s_2$  being assumed. The original fuzzy sets are drawn with dotted lines whereas the “contracted” and “expanded” fuzzy probabilities are drawn with thicker lines.

obabilities are shown in Fig. 4(a-c), where  $\alpha$  is taken at intervals of 0.1.

For example, for  $\alpha=0$  the intervals defined by the  $\alpha$ -cut and the corresponding adjusted intervals are:

Original:

$$P_{1,0}=[0.18,0.22], P_{2,0}=[0.45,0.55], P_{3,0}=[0.21,0.29],$$

Adjusted:

$$Q_{1,0}=[0.18,0.22], Q_{2,0}=[0.49,0.55], Q_{3,0}=[0.23,0.29],$$

and the reduced probabilities according to the Laplace criterion (2) are  $q_1=0.205$ ,  $q_2=0.5275$ , and  $q_3=0.2675$ . Therefore, the optimal bet can be calculated by formulating an optimization problem to maximize the profit:

$$\begin{aligned} \text{Maximize } z &= q_1(k_1x_1 - x_1 - x_2 - x_3) + \\ &+ q_2(k_2x_2 - x_1 - x_2 - x_3) + \\ &+ q_3(k_3x_3 - x_1 - x_2 - x_3) = \\ &= \sum_{i=1}^3 q_i k_i x_i - \sum_{i=1}^3 x_i \end{aligned} \quad (11)$$

$$\begin{aligned} \text{subject to: } &\sum_{i=1}^3 x_i \leq 100 \\ &x_i \geq 0, \end{aligned}$$

where  $x_1$ ,  $x_2$ , and  $x_3$ , are the bets placed on events  $s_1$ (home win),  $s_2$ (draw), and  $s_3$ (away win) with payoff coefficients  $k_1$ ,  $k_2$ , and  $k_3$ , respectively. Since the values of  $q_i$  and  $k_i$  are known, the optimization problem is linear and it can be solved with any linear programming technique. For instance, MATLAB's function “linprog” finds the optimal bet results to be  $x_1=0$ ,  $x_2=100$ , and  $x_3=0$ ,

which gives an expected profit of \$84.625.

If the Wald criterion is considered, the formulation of the optimization problem is:

$$\begin{aligned} & \text{Maximize}_{x_i} \min_{p_i} \left\{ \sum_{i=1}^3 p_i k_i x_i - \sum_{i=1}^3 x_i \right\} \\ & \text{subject to: } \sum_{i=1}^3 p_i = 1, \quad (p_i \in P_i) \\ & \sum_{i=1}^3 x_i \leq 100 \\ & x_i \geq 0, \end{aligned} \tag{12}$$

which is not linear, since  $p_i$  are unknown and they take values that minimize the profit (worst possible outcome) as seen in the objective function. The problem can be formulated as a linear optimization, however, since the objective function is linear with respect to  $p_i$  when values  $x_i$  are considered to be constant. Therefore, the minimum value takes place at one of the extremes of the combination of values of  $p_i$  that add to one. In general, there are at most six possible combination of extreme values, but only four for the given values:  $(p_1, p_2, p_3) \in \{(0.18, 0.53, 0.29), (0.18, 0.55, 0.27), (0.22, 0.49, 0.29), (0.22, 0.55, 0.23)\}$ .

The formulation of the resulting linear optimization problem is:

$$\begin{aligned} & \text{Maximize}_{x_1, x_2, x_3} z \\ & \text{subject to:} \\ & z \leq (0.18k_1 - 1)x_1 + (0.53k_2 - 1)x_2 + (0.29k_3 - 1)x_3 \\ & z \leq (0.18k_1 - 1)x_1 + (0.55k_2 - 1)x_2 + (0.27k_3 - 1)x_3 \\ & z \leq (0.22k_1 - 1)x_1 + (0.49k_2 - 1)x_2 + (0.29k_3 - 1)x_3 \\ & z \leq (0.22k_1 - 1)x_1 + (0.55k_2 - 1)x_2 + (0.23k_3 - 1)x_3 \\ & x_1 + x_2 + x_3 \leq 100 \\ & x_1, x_2, x_3 \geq 0. \end{aligned} \tag{13}$$

After solving the optimization problem, the results given by MATLAB's function "linprog" show that the optimal bet is  $x_1=0$ ,  $x_2=63.1579$ , and  $x_3=36.8421$ , which gives an expected profit of \$72.4211 or \$72 with 42 cents.

Finally, the solution of the optimization problem when the Maximax criterion is considered gives optimal bets equal to  $x_1=0$ ,  $x_2=100$ , and  $x_3=0$ , which produces an expected profit of \$92.5.

### C. Optimal Bet using the Adjustment by Expansion Technique

The values for the profit given by the analysis in Section 5.B. based on the conventional Wald and Maximax criteria define the extreme values of the interval where the value of the profit can be. However, a more pessimistic as well as a more optimistic result can be calculated in order to take into account the

uncertainty of the DM shown by his elicited probabilities. These even more extreme values are calculated through adjustment by expansion technique. A set of particular adjusted by expansion fuzzy probabilities are shown in Fig. 4(d-f), where  $\alpha$  is taken at intervals of 0.1, Wald criterion is considered and  $s_3 > s_1 > s_2$  assumed. This set of adjusted probabilities is only one of the six ( $3!=6$ ) possible sets since the DM preference on the states change depending on the placed bets. For example, if the DM bets \$60 on state  $s_1$  (home win), \$30 on state  $s_2$  (draw) and \$10 on state  $s_3$  (away win), the resulting profits are  $c_1 = (60 \times 4/3 - 60 - 30 - 10) = \$-20$ ,  $c_2 = (30 \times 7/2 - 60 - 30 - 10) = \$5$ , and  $c_3 = (10 \times 6 - 60 - 30 - 10) = \$-40$ , and  $s_2 > s_1 > s_3$ .

Let us consider the Wald criterion together with the same intervals analyzed in Section 5.B., that is, the ones defined by the  $\alpha$ -cuts for  $\alpha=0$ :  $P_{1,0}=[0.18, 0.22]$ ,  $P_{2,0}=[0.45, 0.55]$ ,  $P_{3,0}=[0.21, 0.29]$ . There are ( $3!=6$ ) six combination of probability values that represents the pessimistic view of the DM, as shown in Table 1. The amount of money bet on the states  $s_1$ ,  $s_2$ , and  $s_3$ , modifies the DM's preference over those states. In Table 1, the most favorable state takes the lowest possible probability value within its boundary whereas the least favorable state takes the highest one. The third state takes a value within its boundaries in order to complete the addition to one, but if this is not possible, then it becomes necessary to apply the adjustment by expansion technique to the interval. For example, the technique is unnecessary for the cases shown in columns 2 and 3 of Table 1, whereas it is required in the cases shown in columns 4 to 7. Since the probability values in Table 1 are the reduced probabilities of the states, the optimal bets can be calculated with the same formulation of (11). The optimal bets and the corresponding profits are shown in rows 5 to 8 of Table 1. Therefore, the optimal bet is  $x_1=0$ ,  $x_2=\$63.1579$ , and  $x_3=\$36.8421$ , which gives an expected profit of \$68.

Finally, the adjustment by expansion technique together with the Maximax criterion produces the six combinations of probability values shown in Table 2. The optimal bets are shown in rows 5 to 8 being the optimal bet  $x_1=0$ ,  $x_2=100$ , and  $x_3=0$ , which gives an expected profit of \$99.5.

It is important to notice that the following relationship holds:

$$\begin{aligned} \text{Wald}_{\text{pessimistic}} & \leq \text{Wald} \leq \text{Laplace} \leq \\ \$68 & \leq \$72.4211 \leq \$84.625 \leq \\ & \leq \text{Maximax} \leq \text{Maximax}_{\text{optimistic}} \\ & \leq \$92.5 \leq \$99.5 \end{aligned} \tag{14}$$

Table 1. Reduced probabilities ( $p_i$ ), optimal bets ( $x_i$ ) and profit of the six “pessimistic Maximin” cases for the betting problem.

	$s_1 \succ s_2 \succ s_3$	$s_1 \succ s_3 \succ s_2$	$s_2 \succ s_1 \succ s_3$	$s_2 \succ s_3 \succ s_1$	$s_3 \succ s_1 \succ s_2$	$s_3 \succ s_2 \succ s_1$
$p_1$	0.18	0.18	0.22	0.26	0.22	0.24
$p_2$	0.53	0.55	0.45	0.45	0.57	0.55
$p_3$	0.29	0.27	0.33	0.29	0.21	0.21
$x_1$ (\$)	0	0	0	0	0	0
$x_2$ (\$)	0	0	100	63.1579	0	63.1579
$x_3$ (\$)	0	0	0	36.8421	100	36.8421
profit(\$)	0	0	57.5	63.5789	26	68

Table 2. Reduced probabilities ( $p_i$ ), optimal bets ( $x_i$ ) and profit of the six “optimistic Maximax” cases for the betting problem.

	$s_1 \succ s_2 \succ s_3$	$s_1 \succ s_3 \succ s_2$	$s_2 \succ s_1 \succ s_3$	$s_2 \succ s_3 \succ s_1$	$s_3 \succ s_1 \succ s_2$	$s_3 \succ s_2 \succ s_1$
$p_1$	0.24	0.26	0.22	0.18	0.22	0.18
$p_2$	0.55	0.45	0.57	0.55	0.45	0.53
$p_3$	0.21	0.29	0.21	0.27	0.33	0.29
$x_1$ (\$)	0	0	0	0	0	0
$x_2$ (\$)	0	0	100	100	0	63.1579
$x_3$ (\$)	0	0	0	0	100	36.8421
profit(\$)	0	0	99.5	92.5	98	81.2632

## 5. Conclusions

Two techniques to adjust conflicting information provided by NIDMs are proposed and their application together with four criteria of individual decision making under strict uncertainty presented. Both techniques are applied to a problem of betting in sport games in order to illustrate the way of reducing the effects of conflicting values.

The first technique is fundamental in the application of the Laplace criterion and it should be applied as a preliminary step in order to eliminate portions of information that violate the probability constraint of addition to one.

The second technique is proposed to analyze extreme cases implied by the DM’s uncertainty. These extreme cases go beyond the conventional Wald and Maximax criteria and therefore are labeled as “pessimistic maximin” and “optimistic maximax” solutions.

In addition, general expressions to calculate the optimal solutions are given. Although in general the resulting optimization problem is non-linear, the presented analysis also shows that it is possible to formulate them as a linear program and thus, it becomes possible the utilization of the corresponding standard tools such as the function linprog of MATLAB. This is possible since maximization of money instead of utility is sought, although a linear programming formulation is still possible if the amount of money is within the linear

range of the DM’s utility function.

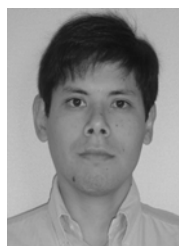
The proposed techniques are expected to help the analysis of individual decision making processes where the complexity of the situation leads the DM to provide conflicting information on his beliefs, which in turn can lead to incorrect conclusions.

## References

- [1] J. Ma, Z. Fan, Y. Jiang, J. Mao, “An optimization approach to multiperson decision making based on different formats of preference information,” *IEEE Trans. Syst., Man, Cybern. A, Syst. Humans*, vol. 36, no. 5, pp. 876-889, Sep. 2006.
- [2] V. Huynh, Y. Nakamori, T. Ho, T. Murai, “Multiple-attribute decision making under uncertainty: the evidential reasoning approach revisited,” *IEEE Trans. Syst., Man, Cybern. A, Syst. Humans*, vol. 36, no. 4, pp. 804-822, July 2006.
- [3] C. Yeh, H. Deng, “A practical approach to fuzzy utilities comparison in fuzzy multicriteria analysis,” *International Journal of Approximate Reasoning*, vol. 35, no. 2, pp. 179-194, Feb. 2004.
- [4] M. A. Gil, M. López-Díaz, “Fundamentals and bayesian analyses of decision problems with fuzzy-valued utilities,” *International Journal of Approximate Reasoning*, vol. 15, no. 3, pp. 203-224, Oct. 1996.

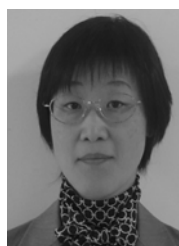
- [5] D. Ben-Arieh, Z. Chen, "Linguistic-labels aggregation and consensus measure for autocratic decision making using group recommendations," *IEEE Trans. Syst., Man, Cybern. A, Syst. Humans*, vol. 36, no. 3, pp. 558-568, May 2006.
- [6] S. R. Watson, J. J. Weiss, and M. L. Donnell, "Fuzzy decision analysis," *IEEE Trans. Syst., Man Cybern.*, vol. 9, no. 1, pp. 1-9, Jan. 1979.
- [7] T. P. Hong, M. J. Chiang, and S. L. Wang, "Fuzzy weighted data mining from quantitative transactions with linguistic minimum supports and confidences," *Intl. Journal of Fuzzy Systems.*, vol. 8, no. 4, pp. 173-182, Dec. 2006.
- [8] N. Nikolova, A. Shulus, D. Toneva, and K. Tenekedjiev, "Fuzzy rationality in quantitative decision analysis," *Journal of Advanced Computational Intelligence and Intelligent Informatics*, vol. 9, no. 1, pp. 65-69, Jan. 2005.
- [9] K. Tenekedjiev, N. Nikolova, D. Dimitrakiev, "Application of the triple bisection method for extraction of subjective utility information," *Proc. Second Intl. Conf. Management and Engineering*, Sofia, Bulgaria, vol. 2(70), pp. 115-117, 2004.
- [10] N. Nikolova, D. Dimitrakiev, K. Tenekedjiev, "Fuzzy rationality in the elicitation of subjective probabilities," *Proc. 2nd Intl. IEEE Conf. Intelligent Systems*, vol.3, pp. 27-31, 2004.
- [11] C. White, H. Holloway, "Resolvability for imprecise multiattribute alternative selection," *IEEE Trans. Syst., Man, Cybern. A, Syst. Humans*, vol. 38, no. 1, pp. 162-169, Jan. 2008.
- [12] K. Katsikopoulos, B. Fasolo, "New tools for decision analysis," *IEEE Trans. Syst., Man, Cybern. A, Syst. Humans*, vol. 36, no. 5, pp. 960-967, Sep. 2006.
- [13] K. Tenekedjiev, N. Nikolova, C. Kobashikawa, K. Hirota, "Fuzzy-rational betting on sport games with interval probabilities," *Intelligent Techniques and Tools for Novel System Architectures*, pp. 431-454, *Studies in Computational Intelligence*, vol. 109, Springer Verlag, 2008.
- [14] K. Tenekedjiev, N. Nikolova, C. Kobashikawa, K. Hirota, "Conservative betting on sport games with intuitionistic fuzzy described uncertainty," *3rd Intl. Conf. Intelligent Systems*, Westminster, UK, pp. 747-754, Sep. 2006.
- [15] S. French, *Decision Theory: an introduction to the mathematics of rationality*, Ellis Horwood, 1993.
- [16] R. R. Yager, V. Kreinovich, "Decision making under interval probabilities", *International Journal of Approximate Reasoning*, vol. 22, no. 3, pp. 195-215, Dec. 1999.
- [17] R. R. Yager, "Decision making with fuzzy probability assessments", *IEEE Transactions on*

*Fuzzy Systems*, vol. 7, no. 4, pp. 462-467, Aug. 1999.



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