

Fuzzy Number Intuitionistic Fuzzy Arithmetic Aggregation Operators

Xinfan Wang

Abstract

A fuzzy number intuitionistic fuzzy set (FNIFS) is a generalization of intuitionistic fuzzy set. The fundamental characteristic of FNIFS is that the values of its membership function and non-membership function are trigonometric fuzzy numbers rather than exact numbers. In this paper, we define some operational laws of fuzzy number intuitionistic fuzzy numbers, and, based on these operational laws, develop some new arithmetic aggregation operators, such as the fuzzy number intuitionistic fuzzy weighted averaging (FIFWA) operator, the fuzzy number intuitionistic fuzzy ordered weighted averaging (FIFOWA) operator and the fuzzy number intuitionistic fuzzy hybrid aggregation (FIFHA) operator for aggregating fuzzy number intuitionistic fuzzy information. Furthermore, we give an application of the FIFHA operator to multiple attribute decision making based on fuzzy number intuitionistic fuzzy information. Finally, an illustrative example is given to verify the developed approach.

Keywords: *Intuitionistic fuzzy set, fuzzy number intuitionistic fuzzy set, operational laws, FIFWA operator, FIFOWA operator, FIFHA operator.*

1. Introduction

Atanassov [1, 2] introduced the concept of intuitionistic fuzzy set (IFS), which is a generalization of the concept of fuzzy set [3]. Gau and Buehrer [4] introduced the concept of vague set, but Bustine and Burillo [5] showed that vague sets are intuitionistic fuzzy sets. In [6], Xu and Yager defined some multiplicative operational laws of intuitionistic fuzzy values, and, based on these operational laws, proposed some intuitionistic fuzzy geometric aggregation operators, such as the intuitionistic fuzzy weighted

geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator and the intuitionistic fuzzy hybrid geometric (IFHG) operator. In [7], Xu defined some additive operational laws of intuitionistic fuzzy values and developed some intuitionistic fuzzy arithmetic aggregation operators, such as the intuitionistic fuzzy weighted averaging (IFWA) operator, the intuitionistic fuzzy ordered weighted averaging (IFOWA) operator and the intuitionistic fuzzy hybrid aggregation (IFHA) operator. Later, Atanassov and Gargov [8-9] further introduced the interval-valued intuitionistic fuzzy set (IVIFS), which is a generalization of the IFS. The fundamental characteristic of the IVIFS is that the values of its membership function and non-membership function are intervals rather than exact numbers. In [10], based on some operational laws of interval-valued intuitionistic fuzzy numbers [11], Xu and Chen proposed some interval-valued intuitionistic fuzzy arithmetic aggregation operators, such as the interval-valued intuitionistic fuzzy weighted averaging (IIFWA) operator, the interval-valued intuitionistic fuzzy ordered weighted averaging (IIFOWA) operator and the interval-valued intuitionistic fuzzy hybrid aggregation (IIFHA) operator. On the basis of the interval-valued intuitionistic fuzzy weighted geometric (IIFWG) operator [11], Xu and Chen [12] developed some interval-valued intuitionistic fuzzy geometric aggregation operators, such as the interval-valued intuitionistic fuzzy ordered weighted geometric (IIFOWG) operator and the interval-valued intuitionistic fuzzy hybrid geometric (IIFHG) operator. Recently, Liu [13] also extended the IFS, introduced the fuzzy number intuitionistic fuzzy set (FNIFS) whose fundamental characteristic is that the values of its membership function and non-membership function are trigonometric fuzzy numbers rather than exact numbers. However, it seems that in the literature there is little investigation on aggregation operators for aggregating a collection of FNIFSs [14]. In this paper, we try to define some operational laws of fuzzy number intuitionistic fuzzy numbers, and, based on these operational laws, develop some new arithmetic aggregation operators, such as the fuzzy number intuitionistic fuzzy weighted averaging (FIFWA) operator, the fuzzy number intuitionistic fuzzy ordered weighted averaging (FIFOWA) operator and the fuzzy number intuitionistic fuzzy hybrid aggregation

Corresponding Author: Xinfan Wang. Author is with the School of Science, Hunan University of Technology, Zhuzhou, Hunan, China, 412008. This work was supported by Scientific Research Fund of Hunan Provincial Education Department under grant 07C232.
E-mail: zzwxfydm@126.com
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(FIFHA) operator.

2. Basic Concepts and Relations

The concept of intuitionistic fuzzy set was introduced by Atanassov [1] to deal with vagueness, which can be defined as follows.

Definition 2.1 [1]. Let X be a universe of discourse, then an IFS A in X is given by:

$$A = \{(x, \langle t_A(x), f_A(x) \rangle) \mid x \in X\} \quad (1)$$

where the functions $t_A : X \rightarrow [0,1]$ and $f_A : X \rightarrow [0,1]$ determine the degree of membership and the degree of non-membership of the element $x \in X$, respectively, and for every $x \in X$:

$$0 \leq t_A(x) + f_A(x) \leq 1 \quad (2)$$

Sometime it is not approximate to assume that the membership degrees for certain elements of A are exactly defined, but a value range can be given. In such cases, Atanassov and Gargov [8] defined the notion of IVIFS as below.

Definition 2.2 [8]. An IVIFS \tilde{A} in X is an object having the following form:

$$\tilde{A} = \{(x, \langle \tilde{t}_A(x), \tilde{f}_A(x) \rangle) \mid x \in X\} \quad (3)$$

Where $\tilde{t}_A(x) = (\tilde{t}_A^1(x), \tilde{t}_A^2(x)) \subset [0,1]$ and $\tilde{f}_A(x) = (\tilde{f}_A^1(x), \tilde{f}_A^2(x)) \subset [0,1]$ are intervals, and for every $x \in X$:

$$0 \leq \tilde{t}_A^2(x) + \tilde{f}_A^2(x) \leq 1 \quad (4)$$

Especially, if each of the intervals $\tilde{t}_A(x)$ and $\tilde{f}_A(x)$ contains exactly one element, i.e., if for every $x \in X$: $\tilde{t}_A(x) = \tilde{t}_A^1(x) = \tilde{t}_A^2(x)$, $\tilde{f}_A(x) = \tilde{f}_A^1(x) = \tilde{f}_A^2(x)$, then, the given IVIFS \tilde{A} is transformed to an ordinary intuitionistic fuzzy set.

However, interval lacks gravity center, cannot emphasis center point which is most impossible to be given value, so it is more suitable that the degree of membership and the degree of non-membership are expressed by trigonometric fuzzy number [15]. In such cases, Liu [13] defined the notion of FNIFS as follow.

Definition 2.3 [13, 15]. If $\beta = (l, p, q) \in F(I)$, $I = [0,1]$, we call β a trigonometric fuzzy number in I , its membership function $\mu_\beta(x) : I \rightarrow [0,1]$ is defined by:

$$\mu_\beta(x) = \begin{cases} (x-l)/(p-l) & l \leq x \leq p \\ (x-q)/(p-q) & p \leq x \leq q \\ 0 & \text{other} \end{cases}$$

where $x \in I, 0 \leq l \leq p \leq q \leq 1$, l is called lower limit of β , q is called upper limit of β , and p is called gravity center of β . On the basis of document [15], the mean of β is defined as below:

$$E(\beta)^{(\theta)} = \frac{(1-\theta)l + p + \theta q}{2}, \quad \theta \in [0,1] \quad (5)$$

where θ is an index that reflects the decision maker's risk-bearing attitude. If $\theta > 0.5$, then the decision maker is a risk lover. If $\theta < 0.5$, then the decision maker is a risk averter. In general, let $\theta = 0.5$, then the attitude of the decision maker is neutral to the risk, and

$$E(\beta)^{(\theta)} = \frac{l + 2p + q}{4} \quad (6)$$

Definition 2.4 [13]. A FNIFS \bar{A} in X is an object having the following form:

$$\bar{A} = \{(x, \langle \bar{t}_A(x), \bar{f}_A(x) \rangle) \mid x \in X\} \quad (7)$$

where $\bar{t}_A(x) = (\bar{t}_A^1(x), \bar{t}_A^2(x), \bar{t}_A^3(x)) \in F(I)$ and $\bar{f}_A(x) = (\bar{f}_A^1(x), \bar{f}_A^2(x), \bar{f}_A^3(x)) \in F(I)$ are two trigonometric fuzzy numbers, and for every $x \in X$:

$$0 \leq \bar{t}_A^3(x) + \bar{f}_A^3(x) \leq 1 \quad (8)$$

Especially, if each of the trigonometric fuzzy numbers $\bar{t}_A(x)$ and $\bar{f}_A(x)$ contains exactly one element, i.e., if for every $x \in X$: $\bar{t}_A(x) = \bar{t}_A^1(x) = \bar{t}_A^2(x) = \bar{t}_A^3(x)$, $\bar{f}_A(x) = \bar{f}_A^1(x) = \bar{f}_A^2(x) = \bar{f}_A^3(x)$, then, the given FNIFS \bar{A} is transformed to an ordinary intuitionistic fuzzy set.

In [11], Xu defined the notion of interval-valued intuitionistic fuzzy number (IVIFN) and introduced some operational laws of IVIFNs. Based on these, we define the concept of fuzzy number intuitionistic fuzzy number (FNIFN) and introduce some operational laws of FNIFNs.

Definition 2.5. Let $\bar{A} = \{(x, \langle \bar{t}_A(x), \bar{f}_A(x) \rangle) \mid x \in X\}$ be a FNIFS, then we call the pair $\langle \bar{t}_A(x), \bar{f}_A(x) \rangle$ a fuzzy number intuitionistic fuzzy number (FNIFN).

For convenience, we denote a FNIFN by $\langle (a, b, c), (l, p, q) \rangle$, where

$$(a, b, c) \in F(I), (l, p, q) \in F(I), 0 \leq c + q \leq 1 \quad (9)$$

and let Ω be the set of all FNIFNs.

Definition 2.6. Let $\bar{\beta}_1 = \langle (a_1, b_1, c_1), (l_1, p_1, q_1) \rangle$, and $\bar{\beta}_2 = \langle (a_2, b_2, c_2), (l_2, p_2, q_2) \rangle$ be any two FNIFNs, then some operational laws of $\bar{\beta}_1$ and $\bar{\beta}_2$ can be defined as:

$$(1) \bar{\beta}_1 \oplus \bar{\beta}_2 = \langle (a_1 + a_2 - a_1 a_2, b_1 + b_2 - b_1 b_2, c_1 + c_2 - c_1 c_2), (l_1 l_2, p_1 p_2, q_1 q_2) \rangle ;$$

$$(2) \lambda \bar{\beta}_1 = \langle (1 - (1 - a_1)^\lambda, 1 - (1 - b_1)^\lambda, 1 - (1 - c_1)^\lambda), (l_1^\lambda, p_1^\lambda, q_1^\lambda) \rangle, \lambda > 0.$$

It can be easily proven that all the above results are also FNIFNs. Based on Definition 2.6, we can further obtain the following relation:

- (1) $\bar{\beta}_1 \oplus \bar{\beta}_2 = \bar{\beta}_2 \oplus \bar{\beta}_1$;
- (2) $\lambda(\bar{\beta}_1 \oplus \bar{\beta}_2) = \lambda \bar{\beta}_1 \oplus \lambda \bar{\beta}_2, \lambda > 0$;
- (3) $(\lambda_1 + \lambda_2)\bar{\beta}_1 = \lambda_1 \bar{\beta}_1 \oplus \lambda_2 \bar{\beta}_1, \lambda_1, \lambda_2 > 0$.

Chen and Tan [16] introduced a score function to measure an intuitionistic fuzzy value. Later, Hong and Chio [17] defined an accuracy function to evaluate the accuracy degree of an intuitionistic fuzzy value. Furthermore, Xu [11] developed a score function and an accuracy function to measure an IVIFN, respectively. In the following, we develop two score function to measure a FNIFN.

Definition 3.1. Let $\bar{\beta} = \langle (a, b, c), (l, p, q) \rangle$ be a FNIFN, then we call

$$\bar{S}(\bar{\beta}) = \frac{a + 2b + c}{4} - \frac{l + 2p + q}{4} \quad (10)$$

a score function of $\bar{\beta}$, where $\bar{S}(\bar{\beta}) \in [-1, 1]$. The greater the value of $\bar{S}(\bar{\beta})$, the greater the FNIFN $\bar{\beta}$. Especially, if $\bar{S}(\bar{\beta}) = 1$, then $\bar{\beta} = \langle (1, 1, 1), (0, 0, 0) \rangle$, which is the largest FNIFN; if $\bar{S}(\bar{\beta}) = -1$, then $\bar{\beta} = \langle (0, 0, 0), (1, 1, 1) \rangle$, which is the smallest FNIFN.

Based on the score function $L(E(A_i)) = t_{A_i} \cdot (2 - t_{A_i} - f_{A_i})$ of intuitionistic fuzzy value developed in [18], we define a new score function.

Definition 3.2. Let $\bar{\beta} = \langle (a, b, c), (l, p, q) \rangle$ be a FNIFN, then we call

$$\bar{L}(\bar{\beta}) = \frac{a + 2b + c}{4} \cdot (2 - \frac{a + 2b + c}{4} - \frac{l + 2p + q}{4}) \quad (11)$$

a extended score function of $\bar{\beta}$, where $\bar{L}(\bar{\beta}) \in [0, 1]$. The greater the value of $\bar{L}(\bar{\beta})$, the better the FNIFN $\bar{\beta}$.

According to Definition 3.1 and Definition 3.2, we develop a method for comparison between two FNIFNs.

Definition 3.3. Let $\bar{\beta}_1$ and $\bar{\beta}_2$ be any two FNIFNs, then

- (1) If $\bar{S}(\bar{\beta}_1) < \bar{S}(\bar{\beta}_2)$, then $\bar{\beta}_1$ is smaller than $\bar{\beta}_2$, denoted by $\bar{\beta}_1 < \bar{\beta}_2$;
- (2) If $\bar{S}(\bar{\beta}_1) = \bar{S}(\bar{\beta}_2)$, then 1) If $\bar{L}(\bar{\beta}_1) = \bar{L}(\bar{\beta}_2)$, then $\bar{\beta}_1$ and $\bar{\beta}_2$ represent the same information, denoted by $\bar{\beta}_1 = \bar{\beta}_2$; 2) If $\bar{L}(\bar{\beta}_1) < \bar{L}(\bar{\beta}_2)$, then $\bar{\beta}_1$ is

smaller than $\bar{\beta}_2$, denoted by $\bar{\beta}_1 < \bar{\beta}_2$.

3. Fuzzy Number Intuitionistic Fuzzy Aggregation Operators

Up to now, many operators have been proposed for aggregating information [19]-[26]. The weighted averaging (WA) operator [19] and the ordered weighted averaging (OWA) operator [20] are the most common operators for aggregating arguments.

Let $a_j (j = 1, 2, \dots, n)$ be a collection of real numbers, $w = (w_1, w_2, \dots, w_n)^T$ be the weight vector of $a_j (j = 1, 2, \dots, n)$, with $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$, then a WA operator is defined as [19]:

$$WA_w(a_1, a_2, \dots, a_n) = \sum_{j=1}^n w_j a_j$$

An OWA operator [20] of dimension n is a mapping OWA: $R^n \rightarrow R$, that has an associated vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, such that $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$.

Furthermore,

$$OWA_\omega(a_1, a_2, \dots, a_n) = \sum_{j=1}^n \omega_j b_j$$

where b_j is the j th largest of $a_j (j = 1, 2, \dots, n)$.

From above, we know that the WA operator first weights all the given arguments and then aggregates all these weighted arguments into a collective one. The fundamental aspect of the OWA operator is the reordering step, it first reorders all the given arguments in descending order and then weights these ordered arguments, and finally aggregates all these ordered weighted arguments into a collective one. In [7], Xu extended the WA operator and the OWA operator to accommodate the situations where the input arguments are intuitionistic fuzzy values. In [10], Xu and Chen also extended the WA operator and the OWA operator to accommodate the situations where the input arguments are interval-valued intuitionistic fuzzy numbers. In the following we shall extend the WA operator and the OWA operator to accommodate the situations where the input arguments are FNIFNs, and develop some new arithmetic aggregation operators for aggregating fuzzy number intuitionistic fuzzy information.

A. Fuzzy Number Intuitionistic Fuzzy Weighted Averaging (FIFWA) Operator

Definition 4.1. Let $\bar{\beta}_j = \langle (a_j, b_j, c_j), (l_j, p_j, q_j) \rangle$

($j = 1, 2, \dots, n$) be a collection of FNIFNs, and let FIFWA: $\Omega^n \rightarrow \Omega$, if

$$\text{FIFWA}_w(\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_n) = w_1 \bar{\beta}_1 \oplus w_2 \bar{\beta}_2 \oplus \dots \oplus w_n \bar{\beta}_n \quad (12)$$

then FIFWA is called a fuzzy number intuitionistic fuzzy weighted averaging (FIFWA) operator of dimension n , where $w = (w_1, w_2, \dots, w_n)^T$ be the weight vector of

$$\bar{\beta}_j (j = 1, 2, \dots, n), \text{ with } w_j \in [0, 1] \text{ and } \sum_{j=1}^n w_j = 1.$$

Especially, if $w = (1/n, 1/n, \dots, 1/n)^T$, then the FIFWA operator is reduced to a fuzzy number intuitionistic fuzzy averaging (FIFA) operator of dimension n , which is defined as follows:

$$\text{FIFA}_w(\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_n) = \frac{1}{n}(\bar{\beta}_1 \oplus \bar{\beta}_2 \oplus \dots \oplus \bar{\beta}_n) \quad (13)$$

By Definition 2.6 and Definition 4.1, we can get the following result by using mathematical induction on n .

Theorem 4.1. Let $\bar{\beta}_j = \langle (a_j, b_j, c_j), (l_j, p_j, q_j) \rangle$ ($j = 1, 2, \dots, n$) be a collection of FNIFNs, then their aggregated value by using the FIFWA operator is also a FNIFN and

$$\begin{aligned} & \text{FIFWA}_w(\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_n) = \\ & \langle (1 - \prod_{j=1}^n (1 - a_j)^{w_j}, 1 - \prod_{j=1}^n (1 - b_j)^{w_j}, 1 - \\ & \prod_{j=1}^n (1 - c_j)^{w_j}), (\prod_{j=1}^n l_j^{w_j}, \prod_{j=1}^n p_j^{w_j}, \prod_{j=1}^n q_j^{w_j}) \rangle \quad (14) \end{aligned}$$

where $w = (w_1, w_2, \dots, w_n)^T$ be the weight vector of

$$\bar{\beta}_j (j = 1, 2, \dots, n), \text{ with } w_j \in [0, 1] \text{ and } \sum_{j=1}^n w_j = 1.$$

Proof: The first result follows quickly from Definition 2.6. Below we prove (14) by using mathematical induction on n .

We first prove that (14) holds for $n = 2$.

Since

$$\begin{aligned} w_1 \bar{\beta}_1 &= \langle (1 - (1 - a_1)^{w_1}, 1 - (1 - b_1)^{w_1}, \\ & 1 - (1 - c_1)^{w_1}), (l_1^{w_1}, p_1^{w_1}, q_1^{w_1}) \rangle \\ w_2 \bar{\beta}_2 &= \langle (1 - (1 - a_2)^{w_2}, 1 - (1 - b_2)^{w_2}, \\ & 1 - (1 - c_2)^{w_2}), (l_2^{w_2}, p_2^{w_2}, q_2^{w_2}) \rangle \end{aligned}$$

then

$$\begin{aligned} \text{FIFWA}_w(\bar{\beta}_1, \bar{\beta}_2) &= w_1 \bar{\beta}_1 \oplus w_2 \bar{\beta}_2 \\ &= \langle (1 - (1 - a_1)^{w_1} + 1 - (1 - a_2)^{w_2} - (1 - (1 - a_1)^{w_1} \\ & (1 - (1 - a_2)^{w_2}), (1 - (1 - b_1)^{w_1} + 1 - (1 - b_2)^{w_2} - \\ & (1 - (1 - b_1)^{w_1})(1 - (1 - b_2)^{w_2}), (1 - (1 - c_1)^{w_1} + \\ & 1 - (1 - c_2)^{w_2} - (1 - (1 - c_1)^{w_1})(1 - (1 - c_2)^{w_2})), \\ & (l_1^{w_1} l_2^{w_2}, p_1^{w_1} p_2^{w_2}, q_1^{w_1} q_2^{w_2}) \rangle \end{aligned}$$

$$\begin{aligned} &= \langle (1 - (1 - a_1)^{w_1} (1 - a_2)^{w_2}, 1 - (1 - b_1)^{w_1} (1 - b_2)^{w_2}, \\ & 1 - (1 - c_1)^{w_1} (1 - c_2)^{w_2}), (l_1^{w_1} l_2^{w_2}, p_1^{w_1} p_2^{w_2}, q_1^{w_1} q_2^{w_2}) \rangle \end{aligned}$$

If (14) holds for $n = k$, that is

$$\begin{aligned} & \text{FIFWA}_w(\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_k) = \\ & \langle (1 - \prod_{j=1}^k (1 - a_j)^{w_j}, 1 - \prod_{j=1}^k (1 - b_j)^{w_j}, 1 - \\ & \prod_{j=1}^k (1 - c_j)^{w_j}), (\prod_{j=1}^k l_j^{w_j}, \prod_{j=1}^k p_j^{w_j}, \prod_{j=1}^k q_j^{w_j}) \rangle \end{aligned}$$

then, when $n = k + 1$, by the operational laws in Definition 2.6, we have

$$\begin{aligned} & \text{FIFWA}_w(\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_k, \bar{\beta}_{k+1}) \\ &= \langle (1 - \prod_{j=1}^k (1 - a_j)^{w_j} + 1 - (1 - a_{k+1})^{w_{k+1}} - \\ & (1 - \prod_{j=1}^k (1 - a_j)^{w_j})(1 - (1 - a_{k+1})^{w_{k+1}}), \\ & 1 - \prod_{j=1}^k (1 - b_j)^{w_j} + 1 - (1 - b_{k+1})^{w_{k+1}} - \\ & (1 - \prod_{j=1}^k (1 - b_j)^{w_j})(1 - (1 - b_{k+1})^{w_{k+1}}), \\ & 1 - \prod_{j=1}^k (1 - c_j)^{w_j} + 1 - (1 - c_{k+1})^{w_{k+1}} - \\ & (1 - \prod_{j=1}^k (1 - c_j)^{w_j})(1 - (1 - c_{k+1})^{w_{k+1}})), \\ & (\prod_{j=1}^k l_j^{w_j} l_{k+1}^{w_{k+1}}, \prod_{j=1}^k p_j^{w_j} p_{k+1}^{w_{k+1}}, \prod_{j=1}^k q_j^{w_j} q_{k+1}^{w_{k+1}}) \rangle \\ &= \langle (1 - \prod_{j=1}^{k+1} (1 - a_j)^{w_j}, 1 - \prod_{j=1}^{k+1} (1 - b_j)^{w_j}, 1 - \\ & \prod_{j=1}^{k+1} (1 - c_j)^{w_j}), (\prod_{j=1}^{k+1} l_j^{w_j}, \prod_{j=1}^{k+1} p_j^{w_j}, \prod_{j=1}^{k+1} q_j^{w_j}) \rangle. \end{aligned}$$

i.e., (14) holds for $n = k + 1$.

Therefore, (14) holds for all n , which completes the proof of Theorem 4.1.

B. Fuzzy Number Intuitionistic Fuzzy Ordered Weighted Averaging (FIFOWA) Operator

Definition 4.2. Let $\bar{\beta}_j = \langle (a_j, b_j, c_j), (l_j, p_j, q_j) \rangle$

($j = 1, 2, \dots, n$) be a collection of FNIFNs. A fuzzy number intuitionistic fuzzy ordered weighted averaging (FIFOWA) operator of dimension n is a mapping FIFOWA: $\Omega^n \rightarrow \Omega$, that has an associated vector $\omega =$

$(\omega_1, \omega_2, \dots, \omega_n)^T$, such that $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$.

Furthermore,

$$\text{FIFOWA}_\omega(\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_n)$$

$$= \omega_1 \bar{\beta}_{\sigma(1)} \oplus \omega_2 \bar{\beta}_{\sigma(2)} \oplus \dots \oplus \omega_n \bar{\beta}_{\sigma(n)} \quad (15)$$

where $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1, 2, \dots, n)$ such that $\bar{\beta}_{\sigma(j-1)} \geq \bar{\beta}_{\sigma(j)}$ for all j . Especially, if $\omega = (1/n, 1/n, \dots, 1/n)^T$, then the FIFOWA operator is reduced to a FIFA operator of dimension n .

Similar Theorem 4.1, we have the following.

Theorem 4.2. Let $\bar{\beta}_j = \langle (a_j, b_j, c_j), (l_j, p_j, q_j) \rangle$ ($j = 1, 2, \dots, n$) be a collection of FNIFNs, then their aggregated value by using the FIFOWA operator is also a FNIFN and

$$\begin{aligned} \text{FIFOWA}_{\omega}(\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_n) &= \langle (1 - \prod_{j=1}^n (1 - a_{\sigma(j)})^{\omega_j}, \\ &(1 - \prod_{j=1}^n (1 - b_{\sigma(j)})^{\omega_j}, 1 - \prod_{j=1}^n (1 - c_{\sigma(j)})^{\omega_j}), \\ &(\prod_{j=1}^n l_{\sigma(j)}^{\omega_j}, \prod_{j=1}^n p_{\sigma(j)}^{\omega_j}, \prod_{j=1}^n q_{\sigma(j)}^{\omega_j})) \rangle \end{aligned} \quad (16)$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weighting vector of the FIFOWA operator, with $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$,

which can be determined by using the normal distribution based method [24].

The FIFOWA weights can be determined similar to the OWA weights (for example, we can use the normal distribution based method to determine the FIFOWA weights [24]).

C. Fuzzy Number Intuitionistic Fuzzy Hybrid Aggregation (FIFHA) Operator

Consider that the FIFWA operator weights only the FNIFNs, while the FIFOWA operator weights only the ordered positions of the FNIFNs instead of weighting the FNIFNs themselves. To overcome this limitation, motivated by the idea of combining the WA and the OWA operator [25, 26], in what follows, we develop a fuzzy number intuitionistic fuzzy hybrid aggregation (FIFHA) operator.

Definition 4.3. Let $\bar{\beta}_j = \langle (a_j, b_j, c_j), (l_j, p_j, q_j) \rangle$ ($j = 1, 2, \dots, n$) be a collection of FNIFNs. A fuzzy number intuitionistic fuzzy hybrid aggregation (FIFHA) Operator of dimension n is a mapping FIFHA: $\Omega^n \rightarrow \Omega$, which has an associated vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$,

such that

$$\begin{aligned} \text{FIFHA}_{w, \omega}(\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_n) \\ = \omega_1 \bar{\beta}'_{\sigma(1)} \oplus \omega_2 \bar{\beta}'_{\sigma(2)} \oplus \dots \oplus \omega_n \bar{\beta}'_{\sigma(n)} \end{aligned} \quad (17)$$

where $\bar{\beta}'_{\sigma(j)}$ is the j th largest of the weighted FNIFNs $\bar{\beta}'_j$ ($\bar{\beta}'_j = n w_j \bar{\beta}_j$, $j = 1, 2, \dots, n$, $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of $\bar{\beta}_j$ ($j = 1, 2, \dots, n$), with $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$, and n is the balancing

coefficient, which plays a role of balance (in this case, if the vector $w = (w_1, w_2, \dots, w_n)^T$ approaches $(1/n, 1/n, \dots, 1/n)^T$, then the vector $(n w_1 \bar{\beta}_1, n w_2 \bar{\beta}_2, \dots, n w_n \bar{\beta}_n)$ approaches $(\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_n)$).

Theorem 4.3. Let $\bar{\beta}_j = \langle (a_j, b_j, c_j), (l_j, p_j, q_j) \rangle$ ($j = 1, 2, \dots, n$) be a collection of FNIFNs, then their aggregated value by using the FIFHA operator is also a FNIFN and satisfies:

$$\begin{aligned} \text{FIFHA}_{w, \omega}(\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_n) &= \langle (1 - \prod_{j=1}^n (1 - a'_{\sigma(j)})^{\omega_j}, \\ &(1 - \prod_{j=1}^n (1 - b'_{\sigma(j)})^{\omega_j}, 1 - \prod_{j=1}^n (1 - c'_{\sigma(j)})^{\omega_j}), \\ &(\prod_{j=1}^n l'^{\omega_j}_{\sigma(j)}, \prod_{j=1}^n p^{\omega_j}_{\sigma(j)}, \prod_{j=1}^n q'^{\omega_j}_{\sigma(j)}) \rangle \end{aligned} \quad (18)$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weighting vector of the FIFHA operator, with $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$,

$\bar{\beta}'_{\sigma(j)} = \langle (a'_{\sigma(j)}, b'_{\sigma(j)}, c'_{\sigma(j)}), (l'_{\sigma(j)}, p'_{\sigma(j)}, q'_{\sigma(j)}) \rangle$ ($j = 1, 2, \dots, n$) is the j th largest of the weighted FNIFNs $\bar{\beta}'_j$ ($\bar{\beta}'_j = n w_j \bar{\beta}_j$, $j = 1, 2, \dots, n$), $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of $\bar{\beta}_j$ ($j = 1, 2, \dots, n$), with $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$, and n is the balancing coefficient.

Theorem 4.4. The FIFWA operator is a special case of the FIFHA operator.

Proof: Let $\omega = (1/n, 1/n, \dots, 1/n)^T$, then

$$\begin{aligned} \text{FIFHA}_{w, \omega}(\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_n) \\ = \omega_1 \bar{\beta}'_{\sigma(1)} \oplus \omega_2 \bar{\beta}'_{\sigma(2)} \oplus \dots \oplus \omega_n \bar{\beta}'_{\sigma(n)} \\ = \frac{1}{n} (\bar{\beta}'_{\sigma(1)} \oplus \bar{\beta}'_{\sigma(2)} \oplus \dots \oplus \bar{\beta}'_{\sigma(n)}) \\ = w_1 \bar{\beta}_1 \oplus w_2 \bar{\beta}_2 \oplus \dots \oplus w_n \bar{\beta}_n \\ = \text{FIFWA}_w(\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_n) \end{aligned}$$

This completes the proof of Theorem 4.4.

Theorem 4.5. The FIFOWA operator is a special case of the FIFHA operator.

Proof: Let $w = (1/n, 1/n, \dots, 1/n)^T$, then $\bar{\beta}'_j = n w_j \bar{\beta}_j$

$$\begin{aligned}
 &= \bar{\beta}_j \ (j = 1, 2, \dots, n), \text{ thus} \\
 &\text{FIFHA}_{w,\omega}(\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_n) \\
 &= \omega_1 \bar{\beta}'_{\sigma(1)} \oplus \omega_2 \bar{\beta}'_{\sigma(2)} \oplus \dots \oplus \omega_n \bar{\beta}'_{\sigma(n)} \\
 &= \omega_1 \bar{\beta}_{\sigma(1)} \oplus \omega_2 \bar{\beta}_{\sigma(2)} \oplus \dots \oplus \omega_n \bar{\beta}_{\sigma(n)} \\
 &= \text{FIFOWA}_{\omega}(\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_n)
 \end{aligned}$$

which completes the proof of Theorem 4.5.

Clearly, from Theorem 4.4 and Theorem 4.5, we know that the FIFHA operator generalizes both the FIFOWA and FIFWA operators and reflects the importance degrees of both the given fuzzy number intuitionistic fuzzy argument and the ordered position of the argument.

4. An application of the FIFHA operator to multiple attribute decision making

In the following, we apply the FIFHA operator to multiple attribute decision making based on fuzzy number intuitionistic fuzzy information.

Let $A = \{A_1, A_2, \dots, A_m\}$ be a set of alternatives, and let $B = \{B_1, B_2, \dots, B_n\}$ be a set of attributes. $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of B_j ($j = 1, 2, \dots, n$), with $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$. Assume that the characteristics of the alternatives A_i ($i = 1, 2, \dots, m$), is represented by the fuzzy number intuitionistic fuzzy set:

$$\begin{aligned}
 A_i = \{ &(B_1, \langle (a_{i1}, b_{i1}, c_{i1}), (l_{i1}, p_{i1}, q_{i1}) \rangle), \\
 &(B_2, \langle (a_{i2}, b_{i2}, c_{i2}), (l_{i2}, p_{i2}, q_{i2}) \rangle), \\
 &\dots, (B_n, \langle (a_{in}, b_{in}, c_{in}), (l_{in}, p_{in}, q_{in}) \rangle) \} \quad (19)
 \end{aligned}$$

where trigonometric fuzzy number (a_{ij}, b_{ij}, c_{ij}) indicates the degree that the alternative A_i satisfies the attribute B_j , (l_{ij}, p_{ij}, q_{ij}) indicates the degree that the alternative A_i does not satisfy the attribute B_j , $(a_{ij}, b_{ij}, c_{ij}) \in F(I)$, $(l_{ij}, p_{ij}, q_{ij}) \in F(I)$, $0 \leq c_{ij} + q_{ij} \leq 1$, $1 \leq i \leq m, 1 \leq j \leq n$.

Let $\bar{\beta}_{ij} = \langle (a_{ij}, b_{ij}, c_{ij}), (l_{ij}, p_{ij}, q_{ij}) \rangle$ for all i, j , then equation (19) can written as

$$A_i = \{(B_j, \bar{\beta}_{ij}) \mid B_j \in B\} \quad (20)$$

To get the best alternative(s), we can utilize the FIFHA operator:

$$\bar{\beta}_i = \text{FIFHA}_{w,\omega}(\bar{\beta}_{i1}, \bar{\beta}_{i2}, \dots, \bar{\beta}_{in}), \ i = 1, 2, \dots, m \quad (21)$$

to derive the overall values $\bar{\beta}_i = \langle (a_i, b_i, c_i), (l_i, p_i, q_i) \rangle$,

($i = 1, 2, \dots, m$) of the alternatives A_i ($i = 1, 2, \dots, m$) where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weighting vector of the FIFHA operator, with $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$, which can be determined by the normal distribution based method (Xu 2005 [24]).

Then by equation (10), we calculate the scores $\bar{S}(\bar{\beta}_i)$ ($i = 1, 2, \dots, m$) of the overall values $\bar{\beta}_i$ ($i = 1, 2, \dots, m$), and utilize the scores $\bar{S}(\bar{\beta}_i)$ ($i = 1, 2, \dots, m$) to rank the alternatives A_i ($i = 1, 2, \dots, m$), and then select the best one(s) (if there is no difference between two scores $\bar{S}(\bar{\beta}_i)$ and $\bar{S}(\bar{\beta}_j)$, then we need to calculate the extended scores $\bar{L}(\bar{\beta}_i)$ and $\bar{L}(\bar{\beta}_j)$ of the overall values $\bar{\beta}_i$ and $\bar{\beta}_j$, respectively, and then rank the alternatives A_i and A_j , in accordance with the extended scores $\bar{L}(\bar{\beta}_i)$ and $\bar{L}(\bar{\beta}_j)$).

5. Illustrative example

In this section, a multiple attribute decision-making problem involves the prioritization of a set of information technology improvement projects (adapted from [12, 27]) is used to illustrate the developed procedures. The information management steering committee of Midwest American Manufacturing Corp. must prioritize for development and implementation a set of four information technology improvement projects A_i ($i = 1, 2, 3, 4$), which have been proposed by area managers. The committee is concerned that the projects are prioritized from highest to lowest potential contribution to the firm's strategic goal of gaining competitive advantage in the industry. In assessing the potential of each project, three factors are considered, B_1 --productivity, B_2 -- differentiation, and B_3 --management. The productivity factor assesses the potential of a proposed project to increase effectiveness and efficiency of the firm's manufacturing and service operations. The differentiation factor assesses the potential of a proposed project to fundamentally differentiate the firm's products and services from its competitors, and to make them more desirable to its customers. The management factor assesses the potential of a proposed project to assist management in improving their planning, controlling and decision-making activities. The following is the list of proposed

information systems projects: 1) A_1 --Quality Management Information, 2) A_2 --Inventory Control, 3) A_3 --Customer Order Tracking, 4) A_4 --Materials Purchasing Management. Suppose that the weight vector of B_1, B_2 and B_3 is $w = (0.3442, 0.2500, 0.4058)^T$, and the committee represents the characteristics of the projects A_i ($i = 1, 2, 3, 4$) by the FNIFNs $\bar{\beta}_{ij}$ ($i = 1, 2, 3, 4, j = 1, 2, 3$) with respect to the factors B_j ($j = 1, 2, 3$), listed in the following.

$$A_1 = \{(B_1, \langle(0.3, 0.4, 0.5), (0.3, 0.3, 0.4)\rangle), (B_2, \langle(0.4, 0.5, 0.6), (0.2, 0.3, 0.4)\rangle), (B_3, \langle(0.1, 0.2, 0.3), (0.5, 0.6, 0.7)\rangle)\}$$

$$A_2 = \{(B_1, \langle(0.5, 0.6, 0.7), (0.2, 0.2, 0.3)\rangle), (B_2, \langle(0.6, 0.6, 0.6), (0.1, 0.2, 0.3)\rangle), (B_3, \langle(0.4, 0.6, 0.7), (0.1, 0.2, 0.2)\rangle)\}$$

$$A_3 = \{(B_1, \langle(0.3, 0.4, 0.6), (0.2, 0.3, 0.4)\rangle), (B_2, \langle(0.4, 0.5, 0.6), (0.3, 0.3, 0.3)\rangle), (B_3, \langle(0.7, 0.7, 0.7), (0.1, 0.2, 0.3)\rangle)\}$$

$$A_4 = \{(B_1, \langle(0.7, 0.7, 0.8), (0.1, 0.2, 0.2)\rangle), (B_2, \langle(0.6, 0.6, 0.7), (0.1, 0.2, 0.3)\rangle), (B_3, \langle(0.2, 0.3, 0.4), (0.1, 0.2, 0.3)\rangle)\}$$

To rank the given four projects, we first weight all the FNIFVs $\bar{\beta}_{ij}$ ($i = 1, 2, 3, 4, j = 1, 2, 3$) by the vector $w = (0.3442, 0.2500, 0.4058)^T$ of attributes B_j ($j = 1, 2, 3$) and multiplies these values by the balancing coefficient $n = 3$, and gets the weighted FNIFVs $\bar{\beta}'_{ij}$, where $\bar{\beta}'_{ij} = n w_j \bar{\beta}_{ij}$ ($i = 1, 2, 3, 4, j = 1, 2, 3$):

$$\bar{\beta}'_{11} = \langle(0.3081, 0.4099, 0.5112), (0.2885, 0.2885, 0.3882)\rangle$$

$$\bar{\beta}'_{12} = \langle(0.3183, 0.4054, 0.4970), (0.2991, 0.4054, 0.5030)\rangle$$

$$\bar{\beta}'_{13} = \langle(0.1204, 0.2379, 0.3522), (0.4301, 0.5369, 0.6478)\rangle$$

$$\bar{\beta}'_{21} = \langle(0.5112, 0.6118, 0.7115), (0.1898, 0.1898, 0.2885)\rangle$$

$$\bar{\beta}'_{22} = \langle(0.4970, 0.4970, 0.4970), (0.1778, 0.2991, 0.4054)\rangle$$

$$\bar{\beta}'_{23} = \langle(0.4631, 0.6722, 0.7691), (0.0606, 0.1410, 0.1410)\rangle$$

$$\bar{\beta}'_{31} = \langle(0.3081, 0.4099, 0.6118), (0.1898, 0.2885, 0.3882)\rangle$$

$$\bar{\beta}'_{32} = \langle(0.3183, 0.4054, 0.4970), (0.4054, 0.4054, 0.4054)\rangle$$

$$\bar{\beta}'_{33} = \langle(0.7691, 0.7691, 0.7691), (0.0606, 0.1410, 0.2309)\rangle$$

$$\bar{\beta}'_{41} = \langle(0.7115, 0.7115, 0.8102), (0.0928, 0.1898, 0.1898)\rangle$$

$$\bar{\beta}'_{42} = \langle(0.4970, 0.4970, 0.5946), (0.1778, 0.2991, 0.4054)\rangle$$

$$\bar{\beta}'_{43} = \langle(0.2379, 0.3522, 0.4631), (0.0606, 0.1410, 0.2309)\rangle$$

By equation (10), we calculate the scores of $\bar{\beta}'_{ij}$

$$(i = 1, 2, 3, 4, j = 1, 2, 3):$$

$$\bar{S}(\bar{\beta}'_{11}) = 0.0964, \bar{S}(\bar{\beta}'_{12}) = 0.0033, \bar{S}(\bar{\beta}'_{13}) = -0.3008$$

$$\bar{S}(\bar{\beta}'_{21}) = 0.3971, \bar{S}(\bar{\beta}'_{22}) = 0.2017, \bar{S}(\bar{\beta}'_{23}) = 0.5232$$

$$\bar{S}(\bar{\beta}'_{31}) = 0.1462, \bar{S}(\bar{\beta}'_{32}) = 0.0011, \bar{S}(\bar{\beta}'_{33}) = 0.6257$$

$$\bar{S}(\bar{\beta}'_{41}) = 0.5706, \bar{S}(\bar{\beta}'_{42}) = 0.2261, \bar{S}(\bar{\beta}'_{43}) = 0.2080$$

Thus

$$\bar{\beta}'_{1\sigma(1)} = \bar{\beta}'_{11}, \bar{\beta}'_{1\sigma(2)} = \bar{\beta}'_{12}, \bar{\beta}'_{1\sigma(3)} = \bar{\beta}'_{13}$$

$$\bar{\beta}'_{2\sigma(1)} = \bar{\beta}'_{23}, \bar{\beta}'_{2\sigma(2)} = \bar{\beta}'_{21}, \bar{\beta}'_{2\sigma(3)} = \bar{\beta}'_{22}$$

$$\bar{\beta}'_{3\sigma(1)} = \bar{\beta}'_{33}, \bar{\beta}'_{3\sigma(2)} = \bar{\beta}'_{31}, \bar{\beta}'_{3\sigma(3)} = \bar{\beta}'_{32}$$

$$\bar{\beta}'_{4\sigma(1)} = \bar{\beta}'_{41}, \bar{\beta}'_{4\sigma(2)} = \bar{\beta}'_{42}, \bar{\beta}'_{4\sigma(3)} = \bar{\beta}'_{43}$$

Suppose that $\omega = (0.2429, 0.5142, 0.2429)^T$ (derived by the normal distribution based method (Xu 2005 [24])) is the weighting vector of the FIFHA operator, then, by equation (18), we get the overall values $\bar{\beta}_i$ ($i = 1, 2, 3, 4$) of the alternatives A_i ($i = 1, 2, 3, 4$):

$$\bar{\beta}_1 = \langle(0.2721, 0.3696, 0.4688), (0.3238, 0.3996, 0.5023)\rangle$$

$$\bar{\beta}_2 = \langle(0.4964, 0.6032, 0.6872), (0.1416, 0.1972, 0.2633)\rangle$$

$$\bar{\beta}_3 = \langle(0.4719, 0.5293, 0.6356), (0.1729, 0.2633, 0.3458)\rangle$$

$$\bar{\beta}_4 = \langle(0.5139, 0.5327, 0.6390), (0.1169, 0.2231, 0.2941)\rangle$$

By equation (10), we calculate the scores of the overall values $\bar{\beta}_i$ ($i = 1, 2, 3, 4$):

$$\bar{S}(\bar{\beta}_1) = -0.0363, \bar{S}(\bar{\beta}_2) = 0.3977$$

$$\bar{S}(\bar{\beta}_3) = 0.2802, \bar{S}(\bar{\beta}_4) = 0.3403$$

Since

$$\bar{S}(\bar{\beta}_2) > \bar{S}(\bar{\beta}_4) > \bar{S}(\bar{\beta}_3) > \bar{S}(\bar{\beta}_1)$$

Thus

$$A_2 \succ A_4 \succ A_3 \succ A_1$$

then the most desirable alternative is A_2 .

6. Conclusion

In this paper, we have introduced some operational laws of FNIFNs, and developed some new arithmetic aggregation operators, including the FIFWA operator, the FIFOWA operator and the FIFHA operator, which extend the WA operator and the OWA operator to accommodate the situations where the given arguments are fuzzy number intuitionistic fuzzy numbers. The FIFHA operator generalizes both the FIFOWA and FIFWA operators and reflects the importance degrees of both the given fuzzy number intuitionistic fuzzy number and the ordered position of the argument. This paper develops the theories both the aggregation operators and

the fuzzy number intuitionistic fuzzy sets. The applications of the developed operators in many fields such as group decision making, pattern recognition, and medical diagnosis, etc., will be the direction for future research.

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References

- [1] K. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol.20, no.1, pp.87-96, 1986.
- [2] K. Atanassov, "More on intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol.33, no.1, pp.37-46, 1989.
- [3] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol.8, no.4, pp.338-356, 1965.
- [4] W. L. Gau and D. J. Buehrer, "Vague sets," *IEEE Transactions on Systems, Man and Cybernetics*, vol.23, no.2, pp.610-614, 1993.
- [5] H. Bustine and P. Burillo, "Vague sets are intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol.79, no.3, pp.403-405, 1996.
- [6] Z. S. Xu and R. R. Yager, "Some geometric aggregation operators based on intuitionistic fuzzy sets," *International Journal of General Systems*, vol.35, no.4, pp.417-433, 2006.
- [7] Z. S. Xu, "Intuitionistic fuzzy aggregation operators," *IEEE Transactions on Fuzzy Systems*, vol.15, no.6, pp.1-10, 2007.
- [8] K. Atanassov and G. Gargov, "Interval-valued intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol.31, no.3, pp.343-349, 1989.
- [9] K. Atanassov, "Operators over Interval-valued intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol.64, no.2, pp.159-174, 1994.
- [10] Z. S. Xu and J. Chen, "An approach to group decision making based on interval-valued intuitionistic judgment matrices," *Systems Engineering-Theory and Practice*, vol.27, no.4, pp.126-133, 2007.
- [11] Z. S. Xu, "Methods for aggregating interval-valued intuitionistic fuzzy information and application to decision making," *Control and Decision*, vol.22, no.2, pp.215-219, 2007.
- [12] Z. S. Xu and J. Chen, "On geometric aggregation over interval-valued intuitionistic fuzzy information," In *Fourth International Conference on Fuzzy Systems and Knowledge Discovery (FSKD)*, Haikou, China, pp. 466-471, Aug. 2007.
- [13] F. Liu, X. H. Yuan, "Fuzzy number intuitionistic fuzzy set," *Fuzzy Systems and Mathematics*, vol.21, no.1, pp.88-91, 2007.
- [14] X. F. Wang, "Fuzzy Number Intuitionistic Fuzzy geometric Aggregation Operators and Their Application to Decision Making," *Control and Decision*, in press.
- [15] T. S. Liou, M. J. Wang, "Ranking fuzzy numbers with integral value," *Fuzzy Sets and Systems*, vol.50, no.2, pp.247-255, 1992.
- [16] S. M. Chen, J. M. Tan, "Handling multi-criteria fuzzy decision-making problems based on vague set theory," *Fuzzy Sets and Systems*, vol.67, no.2, pp.163-172, 1994.
- [17] D. H. Hong, C. H. Choi, "Multi-criteria fuzzy decision-making problems based on vague set theory," *Fuzzy Sets and Systems*, vol.114, no.1, pp.103-113, 2000.
- [18] H. W. Liu, "Vague set methods of multicriteria fuzzy decision making," *Systems Engineering-Theory and Practice*, vol.24, no.5, pp.103-109, 2004.
- [19] J. C. Harsanyi, "Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility," *J. Political Econ.*, vol.63, no.4, pp.309-321, 1955.
- [20] R. R. Yager, "On ordered weighted averaging aggregation operators in multicriteria decision making," *IEEE Transactions on Systems, Man, and Cybernetics*, vol.18, no.1, pp.183-190, 1988.
- [21] R. R. Yager, J. Kacprzyk, *The Ordered Weighted Averaging Operators: Theory and Applications*, Norwell, MA: Kluwer, 1997.
- [22] B. Buchon-Meunier, R. R. Yager, *Information, Uncertainty, Fusion*, Norwell, MA: Kluwer, 2000.
- [23] Z. S. Xu, *Uncertain Multiple Decision Making: Methods and Applications*, Beijing: Tsinghua University Press, 2004.
- [24] Z. S. Xu, "An overview of methods for determining OWA weights," *International Journal of Intelligent Systems*, vol.20, no.8, pp.843-865, 2005.
- [25] Z. S. Xu, Q. L. Da, "An overview of operators for aggregating information," *International Journal of Intelligent Systems*, vol.18, no.9, pp.953-969, 2003.
- [26] V. Torra, "The weighted OWA operator," *International Journal of Intelligent Systems*, vol.12, no.2, pp.153-166, 1997.
- [27] O. Ngwenyama, and N. Bryson, "Eliciting and mapping qualitative preferences to numeric rankings in group decision making," *European Journal of Operational Research*, vol.116, no.3, pp.487-497, 1999.