

Interval valued $(\in, \in \vee q)$ -fuzzy ideals of pseudo-MV algebras

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Abstract

In this paper, we first introduce the concept of quasi-coincidence of a fuzzy interval value with an interval valued fuzzy set. By using this new idea, we consider the interval valued $(\in, \in \vee q)$ -fuzzy (implicative) ideals of pseudo-MV algebras and investigate some of their related properties. Some characterization theorems of these generalized fuzzy (implicative) ideals are derived.

Keywords: *Pseudo-MV algebra, Ideal, Fuzzy (implicative) ideal, Interval valued $(\in, \in \vee q)$ -fuzzy (implicative) ideal.*

1. Introduction

The study of pseudo-MV algebras was initiated by Georgescu and Iorgulescu [1,2], and independently by Rachunek [3,4] (there they are called generalized MV algebras) as a non-commutative generalization of MV algebras which were introduced by Chang [5]. A non-commutative generalization of reasoning can be found e.g. in psychological processes: in clinical medicine on behalf of transplantation of human organs, an experiment was performed in which the same two questions have been posed to two groups of interviewed people: (1) do you agree to dedicate your organs for medical transplantation after you are dead? (2) do you agree to accept organs of a donor in the case of your need? The order of questions was changed in the second group, and the number of positive answers was higher than in the first group. Therefore, the non-commutative generalization of MV algebras can be useful. For more details, the reader is referred to [3,4,6,7].

The theory of fuzzy sets was first developed by Zadeh [8] and has been applied to many branches in mathematics. Later on, Zadeh [9] also introduced the

concept of interval valued fuzzy set by considering the values of the membership functions as the intervals of numbers instead of the numbers only. The interval valued fuzzy subgroups were first defined and studied by Biswas [10] which are the subgroups of the same nature of the fuzzy subgroups defined by Rosenfeld. In [11], Zeng et al. gave a kind of method to describe the entropy of interval valued fuzzy set based on its similarity measure and discussed their relationship between the similarity measure and the entropy of the interval valued fuzzy sets in detail. However, the obtained results can still be applied in many fields such as pattern recognition, image processing and fuzzy reasoning etc. A new type of fuzzy subgroup (that is, the $(\in, \in \vee q)$ -fuzzy subgroup) was introduced in an earlier paper of Bhakat and Das [12] by using the combined notions of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [13]. In fact, the $(\in, \in \vee q)$ -fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems with other algebraic structures. With this objective in view, Davvaz [14] applied this theory to near-rings and obtained some useful results. Further, Davvaz and Corsini [15] redefined fuzzy H_v -submodule and many valued implications.

Recently, Jun and Walendziak [16] applied the concept of fuzzy set to pseudo-MV algebras. They introduced the notions of fuzzy ideals and fuzzy implicative ideals of a pseudo-MV algebra, gave characterizations of them and provided conditions for a fuzzy set to be a fuzzy (implicative) ideal. Further, Dymek [17] introduced the fuzzy prime ideals of pseudo-MV algebras and obtained some related properties. The other important results can be found in [18-20].

The paper is organized as follows. In section 3, we introduce the notion of interval valued $(\in, \in \vee q)$ -fuzzy ideals of pseudo-MV algebras and investigate some of their related properties. Further, the notion of interval valued $(\in, \in \vee q)$ -fuzzy implicative ideals of pseudo-MV algebras is considered and some characterization theorems of these generalized fuzzy implicative ideals are derived in section 4. We follow in

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Manuscript received 2 Feb. 2008; accepted 16 May 2008.

this paper essentially the results from paper [21].

2. Preliminaries

Let $M = (M; \oplus, -, \sim, 0, 1)$ be an algebra of type $(2, 1, 1, 0, 0)$. We put by definition $y \odot x = (x^- \oplus y^-)^-$ and we consider that the operation \odot has priority to the operation \oplus , i.e., we will write $x \oplus y \odot z$ instead of $x \oplus (y \odot z)$.

Definition 2.1[1,2]: M is called a pseudo-MV algebra if the following axioms are satisfied, for all $x, y, z \in M$:

- (a1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- (a2) $x \oplus 0 = 0 \oplus x = x$;
- (a3) $x \oplus 1 = 1 \oplus x = 1$;
- (a4) $1^- = 0, 1^- = 0$;
- (a5) $((x^- \oplus y^-)^- = (x^- \oplus y^-)^-)$;
- (a6) $x \oplus x^- \odot y = y \oplus y^- \odot x$
 $= x \odot y^- \oplus y = y \odot x^- \oplus x$;
- (a7) $x \odot (x^- \oplus y) = (x \oplus y^-) \odot y$;
- (a8) $(x^-)^- = x$.

If we define $x \leq y$ if and only if $x^- \oplus y = 1$, then \leq is a partial order such that M is a bounded distributive lattice with the join and meet

$$x \vee y = x \oplus x^- \odot y = x \odot y^- \oplus y,$$

$$x \wedge y = x \odot (x^- \oplus y) = (x \oplus y^-) \odot y.$$

Example 2.2[6,7]: Let $G = (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}; +, (0, 0, 0), \leq)$ be the Scrimger 2-group, i.e.,

$$(k_1, m_1, n_1) + (k_2, m_2, n_2) = \begin{cases} (m_1 + k_2, m_2 + k_1, n_1 + n_2) & \text{if } n_2 \text{ is odd,} \\ (k_1 + k_2, m_1 + m_2, n_1 + n_2) & \text{if } n_2 \text{ is even.} \end{cases}$$

Then $0=(0,0,0)$ is the neutral element, and

$$-(k, m, n) = \begin{cases} (-m, -k, -n) & \text{if } n \text{ is odd,} \\ (-k, -m, -n) & \text{if } n \text{ is even.} \end{cases}$$

and G is a non-Abelian l -group with the positive cone $G^+ = (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_{>0}^+ \cup \mathbb{Z}^+ \times \mathbb{Z}^+ \times \{0\})$, or equivalently, $(k_1, m_1, n_1) \leq (k_2, m_2, n_2)$ iff (i) $n_1 < n_2$, or (ii) $n_1 = n_2$, $k_1 \leq k_2, m_1 < m_2$.

Then

$$(k_1, m_1, n_1) + (k_2, m_2, n_2) = \begin{cases} (m_1 + k_2, m_2 + k_1, n_1 + n_2) & \text{if } n_2 \text{ is odd,} \\ (k_1 + k_2, m_1 + m_2, n_1 + n_2) & \text{if } n_2 \text{ is even.} \end{cases}$$

and $u=(1,1,1)$ is a strong unit for G . Consequently, the corresponding pseudo-MV algebra has the form

$$\Gamma(G, u) = \mathbb{Z}^+ \times \mathbb{Z}^+ \times \{0\} \cup \mathbb{Z}_{\leq 1} \times \mathbb{Z}_{\leq 1} \times \{1\},$$

with

$$(k, m, 0)^- = (1 - k, 1 - m, 1),$$

$$(k, m, 0)^- = (1 - m, 1 - k, 1),$$

$$(k, m, 1)^- = (1 - m, 1 - k, 0),$$

$$(k, m, 1)^- = (1 - k, 1 - m, 0),$$

and

$$(k_1, m_1, 0) \oplus (k_2, m_2, 0) = (k_1 + k_2, m_1 + m_2, 0),$$

$$(k_1, m_1, 0) \oplus (k_2, m_2, 1) = ((m_1 + k_2) \wedge 1, (m_2 + k_1) \wedge 1, 1),$$

$$(k_1, m_1, 1) \oplus (k_2, m_2, 0) = ((k_1 + k_2) \wedge 1, (m_2 + m_1) \wedge 1, 1),$$

$$(k_1, m_1, 1) \oplus (k_2, m_2, 1) = (1, 1, 1).$$

A non-empty subset A of a pseudo-MV algebra M is called an ideal of M if it satisfies the following conditions: (i) if $x, y \in A$, then $x \oplus y \in A$; (ii) $\forall x \in A, y \in M, y \leq x \Rightarrow y \in A$. An ideal A of a pseudo-MV algebra M is called implicative if it satisfies: $x \odot y \odot z^- \in A, z \odot y \in A \Rightarrow x \odot y \in A$ (see [4,7,22, 23]).

Recall that a fuzzy set F of a pseudo-MV algebra M is a function $F : M \rightarrow [0, 1]$. We know that $([0, 1], \leq)$ is a complete lattice with 0 as its least element and 1 as its greatest element, where $[0, 1]$ is the real unit interval([16,17,20]).

Definition 2.3[16,17]: A fuzzy set F of a pseudo-MV algebra M is called a fuzzy ideal of M if it satisfies:

- (i) $F(x \oplus y) \geq \min\{F(x), F(y)\}, \forall x, y \in M$,
- (ii) $y \leq x \Rightarrow F(x) \leq F(y), \forall x, y \in M$.

It is easily seen that (ii) implies

- (iii) $F(0) \geq F(x), \forall x \in M$.

Definition 2.4[16,17]: A fuzzy ideal F of a pseudo-MV algebra M is called a fuzzy implicative ideal of M if $\forall x, y, z \in M$, it satisfies:

- (iv) $F(x \odot y) \geq \min\{F(x \odot y \odot z^-), F(z \odot y)\}$.

For any fuzzy set F of M and $t \in (0, 1]$, the set is called a level subset of F .

Theorem 2.5[16]: A fuzzy set F of a pseudo-MV algebra M is a fuzzy ideal of M if and only if each non-empty level subset F_t is an ideal of M .

By an interval number \hat{a} we mean an interval $[a^+, a^T] \subseteq [0, 1]$, where $0 \leq a^+ \leq a^T \leq 1$. The set of

all interval numbers is denoted by $D[0,1]$. The interval $[a,a]$ can be simply identified with the number $a \in [0,1]$.

For the interval numbers

$$\hat{a}_i = [a_i^\perp, a_i^\top], \hat{b}_i = [b_i^\perp, b_i^\top] \in D[0,1], i \in I,$$

we define

$$r \max \{ \hat{a}_i, \hat{b}_i \} = [\max \{ a_i^\perp, b_i^\perp \}, \max \{ a_i^\top, b_i^\top \}],$$

$$r \min \{ \hat{a}_i, \hat{b}_i \} = [\min \{ a_i^\perp, b_i^\perp \}, \min \{ a_i^\top, b_i^\top \}].$$

Then, it is clear that $(D[0,1], \leq)$ is a complete lattice with $0=[0,0]$ as its least element and $1=[1,1]$ as its greatest element.

The interval valued fuzzy sets provide a more adequate description of uncertainty than the traditional fuzzy sets; it is therefore important to use interval valued fuzzy sets in applications. One of the main applications of fuzzy sets is fuzzy control, and one of the most computationally intensive part of fuzzy control is the “defuzzification”. Since a transition to interval valued fuzzy sets usually increases the amount of computations, it is vitally important to design faster algorithms for the corresponding defuzzification. For more details, the reader can find some good examples in [11, 24].

Recall that an interval valued fuzzy set F on any set X ([10]) is the set

$$F = \{ (x, [\mu_F^\perp(x), \mu_F^\top(x)]) \mid x \in X \},$$

where $\mu_F^\perp(x)$ and $\mu_F^\top(x)$ are two fuzzy sets of X such that for all $x \in X$.

3. Interval valued $(\in, \in \vee q)$ – fuzzy ideals

Based on the results in [12], we can extend the concept of quasi-coincidence of fuzzy point within a fuzzy set to the concept of quasi-coincidence of a fuzzy interval value with an interval valued fuzzy set.

An interval valued fuzzy set $F = \{ (x, \mu_F(x)) \}$ of a pseudo-MV algebra M of the form

$$\widehat{\mu}_F(y) = \begin{cases} \hat{t} (\neq [0,0]) & \text{if } y = x, \\ [0,0] & \text{if } y \neq x, \end{cases}$$

is said to be a fuzzy interval valued with support x and interval value \hat{t} and is denoted by $U(x; \hat{t})$. We now say that a fuzzy interval value $U(x; \hat{t})$ belongs to (resp., is quasi-coincident with) an interval valued fuzzy set F , written by $U(x; \hat{t}) \in F$ (resp. $U(x; \hat{t}) q F$) if $\widehat{\mu}_F(x) \geq \hat{t}$ (resp. $\widehat{\mu}_F(x) + \hat{t} > [1,1]$). If $U(x; \hat{t}) \in F$ or (resp. and) $U(x; \hat{t}) q F$, then we write $U(x; \hat{t}) \in \vee q$ (resp.

$\in \wedge q$) F . The symbol $\overline{\in \vee q}$ means that $\in \vee q$ does not hold.

In what follows, M is a pseudo-MV algebra unless otherwise specified. Also, we emphasize that $\widehat{\mu}_F(x) = [\mu_F^\perp(x), \mu_F^\top(x)]$ must satisfy the following properties:

(1) Any two elements of $D[0,1]$ are comparable;

(2) $[\mu_F^\perp(x), \mu_F^\top(x)] < [0.5, 0.5]$ or $[0.5, 0.5] \leq [\mu_F^\perp(x), \mu_F^\top(x)]$, for all $x \in M$.

We point out that if we delete the above conditions, then Theorem 3.6 may not be true.

First, we can extend the concept of fuzzy ideals to the concept of interval valued fuzzy ideals of M as follows:

Definition 3.1: An interval valued fuzzy set F of M is said to be an interval valued fuzzy ideal of M if $\forall x, y \in M$,

$$(F1) \widehat{\mu}_F(x \oplus y) \geq r \min \{ \widehat{\mu}_F(x), \widehat{\mu}_F(y) \}.$$

$$(F2) y \leq x \Rightarrow \widehat{\mu}_F(x) \leq \widehat{\mu}_F(y), \forall x, y \in M.$$

Example 3.2: Let I be an ideal of a pseudo-MV algebra M and F an interval valued fuzzy set of M define by

$$\widehat{\mu}_F(x) = \begin{cases} [0.6, 0.7] & \text{if } x \in I, \\ [0.2, 0.3] & \text{otherwise.} \end{cases}$$

One can easily check that F is an interval valued fuzzy ideal of M .

Let F be an interval valued fuzzy set, Then, for every $\hat{t} \in D[0,1]$, the set $F_{\hat{t}} = \{ x \in M \mid \widehat{\mu}_F(x) \geq \hat{t} \}$ is called an interval valued level subset of F .

Now, we characterize the interval valued fuzzy ideals by using their level ideals.

Theorem 3.3: An interval valued fuzzy set F of M is an interval valued fuzzy ideal of M if and only if for any $[0,0] < \hat{t} \leq [1,1]$ such that $F_{\hat{t}} \neq \emptyset$, $F_{\hat{t}}$ is an ideal of M .

Proof. The proof is similar to Theorem 2.5.

In [17], we introduced the concept of interval valued $(\in \varepsilon \vee q)$ -fuzzy filters of R_0 -algebras. As a dual of the concept, we introduce the following concept in pseudo-MV algebras. Certainly, we could work with the properties of $(\in \varepsilon \vee q)$ -fuzzy filters of pseudo-MV algebras, since pseudo-MV algebras, as R_0 -algebras $\cong NM$ -algebras, MV algebras are involutive structures.

Definition 3.4: An interval valued fuzzy set F of M is said to be an interval valued $(\in \in \vee q)$ -fuzzy ideal of M if for all $\hat{t}, \hat{r} \in D[0,1]$ and $x, y \in M$,

$$(F3) \quad U(x; \hat{t}) \in F \quad \text{and} \quad U(y; \hat{r}) \in F \quad \text{imply} \\ U(x \oplus y; r \min\{\hat{t}, \hat{r}\}) \in \vee qF,$$

$$(F4) \quad U(y; \hat{r}) \in F \quad \text{implies} \quad U(x; \hat{r}) \in \vee qF \quad \text{if} \quad x \leq y.$$

Example 3.5: Let

$M = \{(1, y) \in \mathbb{R}^2 \mid y \geq 0\} \cup \{(2, y) \in \mathbb{R}^2 \mid y \leq 0\}$,
 $\mathbf{0} = (1, 0), \mathbf{1} = (2, 0)$. For any $(a, b), (c, d) \in M$. Define three operations $\oplus, -, \sim$ as follows:

$$(a, b) \oplus (c, d) = \begin{cases} (1, b+d) & \text{if } a=c=1, \\ (2, ad+b) & \text{if } ac=2 \text{ and } ad+b \leq 0, \\ (2, 0) & \text{otherwise.} \end{cases}$$

$$(a, b)^- = \left(\frac{2}{a}, -\frac{2b}{a}\right), (a, b)^{\sim} = \left(\frac{2}{a}, -\frac{b}{a}\right).$$

Then $M = (M, \oplus, -, \sim, \mathbf{0}, \mathbf{1})$ is a pseudo-MV algebra (see [7,23]). Let $M_1 = \{(1, y) \in \mathbb{R}^2 \mid y > 0\}$ and $M_2 = \{(2, y) \in \mathbb{R}^2 \mid y < 0\}$.

Define an interval valued fuzzy set F in M by

$$\widehat{\mu}_F(x) = \begin{cases} [0.7, 0.8] & \text{if } x = 0, \\ [0.3, 0.4] & \text{if } x \in M_1 \\ [0.1, 0.2] & \text{if } x \in M_2 \cup \{1\}. \end{cases}$$

It is now routine to verify that F is an interval valued $(\in \in \vee q)$ -fuzzy ideal of M .

Theorem 3.6: The conditions (F3) and (F4) in Definition 3.4 are equivalent to the following conditions, respectively:

$$(F5) \quad \widehat{\mu}_F(x \oplus y) \geq r \min\{\widehat{\mu}_F(x), \widehat{\mu}_F(y), [0.5, 0.5]\}, \text{ for all } x, y \in M,$$

$$(F6) \quad \forall x, y \in M, x \leq y \Rightarrow \widehat{\mu}_F(x) \geq r \min\{\widehat{\mu}_F(y), [0.5, 0.5]\}, \text{ for all } x, y \in M.$$

Proof. (F3) \Rightarrow (F5): Suppose that $x, y \in M$. Now, we consider the following cases:

$$(a) \quad r \min\{\widehat{\mu}_F(x), \widehat{\mu}_F(y)\} < [0.5, 0.5],$$

$$(b) \quad r \min\{\widehat{\mu}_F(x), \widehat{\mu}_F(y)\} \geq [0.5, 0.5].$$

Case (a): Assume that $\widehat{\mu}_F(x \oplus y) < r \min\{\widehat{\mu}_F(x), \widehat{\mu}_F(y), [0.5, 0.5]\}$. Then we have $\widehat{\mu}_F(x \oplus y) < r \min\{\widehat{\mu}_F(x), \widehat{\mu}_F(y)\}$. Now, we

choose \hat{t} such that

$$\widehat{\mu}_F(x \oplus y) < \hat{t} < r \min\{\widehat{\mu}_F(x), \widehat{\mu}_F(y)\}.$$

$$\text{Then } U(x; \hat{t}) \in F \text{ and } U(y; \hat{t}) \in F,$$

but $U(x \oplus y; \hat{t}) \in \vee qF$, which contradicts (F3).

$$\text{Case (b): Assume that } \widehat{\mu}_F(x \oplus y) < [0.5, 0.5].$$

$$\text{Then } U(x; [0.5, 0.5]) \in F \text{ and } U(y; [0.5, 0.5]) \in F,$$

but $U(x \oplus y; [0.5, 0.5]) \in \vee qF$, a contradiction. Hence (F5) holds.

(F4) \Rightarrow (F6): Suppose that $x, y \in M$. Now, we consider the following two cases:

$$(a) \quad \widehat{\mu}_F(y) < [0.5, 0.5],$$

$$(b) \quad \widehat{\mu}_F(y) \geq [0.5, 0.5].$$

Case (a): Let $x \leq y$. Assume that

$$\widehat{\mu}_F(y) < \hat{t} < [0.5, 0.5] \text{ and } \widehat{\mu}_F(x) < \hat{r} < [0.5, 0.5].$$

Choose \hat{s} such that $\hat{r} < \hat{s} < \hat{t}$ and $\hat{r} + \hat{s} < [1, 1]$.

Then we have $\widehat{\mu}_F(x \oplus y) < r \min\{\widehat{\mu}_F(x), \widehat{\mu}_F(y)\}$. Now, we choose \hat{t} such that

$$\widehat{\mu}_F(x \oplus y) < \hat{t} < r \min\{\widehat{\mu}_F(x), \widehat{\mu}_F(y)\}.$$

Then $U(y; \hat{s}) \in F$, but $U(x; \hat{s}) \in \vee qF$, which contradicts (F4).

Case (b): Let $x \leq y$. Assume that $\widehat{\mu}_F(y) \geq [0.5, 0.5]$. Then $U(y; [0.5, 0.5]) \in F$, but $U(x; [0.5, 0.5]) \in \vee qF$, a contradiction. Hence (F6) holds.

(F5) \Rightarrow (F3): Let $U(x; \hat{t}) \in F$ and $U(y; \hat{r}) \in F$. Then

$$\widehat{\mu}_F(x) \geq \hat{t} \text{ and } \widehat{\mu}_F(y) \geq \hat{r}. \text{ Now, we have} \\ \widehat{\mu}_F(x \oplus y) \geq r \min\{\widehat{\mu}_F(x), \widehat{\mu}_F(y), [0.5, 0.5]\} \\ \geq r \min\{\hat{t}, \hat{r}, [0.5, 0.5]\}.$$

$$\text{If } r \min\{\hat{t}, \hat{r}\} > [0.5, 0.5], \text{ then}$$

$$\widehat{\mu}_F(x \oplus y) \geq [0.5, 0.5], \text{ which implies,}$$

$$\widehat{\mu}_F(x \oplus y) + r \min\{\hat{t}, \hat{r}\} > [1, 1].$$

$$\text{If } r \min\{\hat{t}, \hat{r}\} \leq [0.5, 0.5], \text{ then}$$

$$\widehat{\mu}_F(x \oplus y) \geq r \min\{\hat{t}, \hat{r}\}, \text{ which implies,}$$

$$U(x \oplus y; r \min\{\hat{t}, \hat{r}\}) \in \vee qF.$$

Similarly, we can prove (F6) \Rightarrow (F4).

By Definition 3.4 and Theorem 3.6, we immediately obtain the following corollary:

Corollary 3.7: An interval valued fuzzy set F of M is an

interval valued $(\in \varepsilon \vee q)$ -fuzzy ideal of M if and only if the conditions (F5) and (F6) in Theorem 3.6 hold.

By the above discussion, we can obtain the following two propositions:

Proposition 3.8: Every interval valued $(\in \varepsilon \vee q)$ -fuzzy ideal F of M satisfies the following two inequalities:

(F7) $\widehat{\mu}_F(y) \geq \text{rmin}\{\widehat{\mu}_F(x), \widehat{\mu}_F(x \sim \odot y), [0.5, 0.5]\}$, for all $x, y \in M$,

(F8) $\widehat{\mu}_F(y) \geq \text{rmin}\{\widehat{\mu}_F(x), \widehat{\mu}_F(x \sim \odot y), [0.5, 0.5]\}$, for all $x, y \in M$.

Proof. By [16, Proposition 3.5 and 3.6], we can easily prove this proposition and we omit the proof.

Proposition 3.9: For an interval valued fuzzy set F of M , the following are equivalent:

(i) F is an interval valued $(\in \varepsilon \vee q)$ -fuzzy ideal of M ;

(ii) F satisfies the conditions (F7) and

(F9) $\widehat{\mu}_F(0) \geq \text{rmin}\{\widehat{\mu}_F(x), [0.5, 0.5]\}$, for all $x \in M$;

(iii) F satisfies the conditions (F8) and (F9).

Proof. By [16, Theorem 3.8], we can easily prove this proposition and we omit the proof.

Now, we characterize the interval valued $(\in \varepsilon \vee q)$ -fuzzy ideals by using their level ideals.

Theorem 3.10: Let F be an interval valued $(\in \varepsilon \vee q)$ -fuzzy ideal of M . Then for all $[0, 0] < \hat{t} \leq [0.5, 0.5]$, $F_{\hat{t}}$ is an empty set or an ideal of M . Conversely, if F is an interval valued fuzzy set of M , for every $[0, 0] < \hat{t} \leq [0.5, 0.5]$ such that $F_{\hat{t}} \neq \emptyset$, $F_{\hat{t}}$ is an ideal of M , then F is an interval valued $(\in \varepsilon \vee q)$ -fuzzy ideal of M .

Proof. Let F be an interval valued $(\in \varepsilon \vee q)$ -fuzzy ideal of M and $[0, 0] < \hat{t} \leq [0.5, 0.5]$. If $x, y \in F_{\hat{t}}$, then $\widehat{\mu}_F(x) \geq \hat{t}$ and $\widehat{\mu}_F(y) \geq \hat{t}$. Now we have $\widehat{\mu}_F(x \oplus y) \geq \text{rmin}\{\widehat{\mu}_F(x), \widehat{\mu}_F(y), [0.5, 0.5]\} = \hat{t}$. This implies that $x \oplus y \in F_{\hat{t}}$. Let $x, y \in M$ be such that $x \leq y$. If $y \in F_{\hat{t}}$, then by (F6), we have $\widehat{\mu}_F(x) \geq \text{rmin}\{\widehat{\mu}_F(y), [0.5, 0.5]\} = \hat{t}$, which implies $x \in F_{\hat{t}}$. Hence, $F_{\hat{t}}$ is an ideal of M .

Conversely, let F be an interval valued fuzzy set of M

such that $F_{\hat{t}} (\neq \emptyset)$ is an ideal of M for all $[0, 0] < \hat{t} \leq [0.5, 0.5]$. Then, for every $x, y \in M$, we can write

$$\widehat{\mu}_F(x) \geq \text{rmin}\{\widehat{\mu}_F(x), \widehat{\mu}_F(y), [0.5, 0.5]\} = \hat{t}_0,$$

$$\widehat{\mu}_F(y) \geq \text{rmin}\{\widehat{\mu}_F(x), \widehat{\mu}_F(y), [0.5, 0.5]\} = \hat{t}_0.$$

Thus, $x, y \in F_{\hat{t}_0}$, and so $x \oplus y \in F_{\hat{t}_0}$. This proves that

$\widehat{\mu}_F(x \oplus y) \geq \text{rmin}\{\widehat{\mu}_F(x), \widehat{\mu}_F(y), [0.5, 0.5]\}$. Also, let

$x, y \in M$ be such that $x \leq y$. If we put $\hat{s}_0 = \text{rmin}\{\widehat{\mu}_F(x), [0.5, 0.5]\} \leq \widehat{\mu}_F(y)$, then $y \in F_{\hat{s}_0}$, and so $x \in F_{\hat{s}_0}$. Thus

$\widehat{\mu}_F(y) \geq \hat{s}_0 = \text{rmin}\{\widehat{\mu}_F(y), [0.5, 0.5]\}$. Therefore, F is an interval valued $(\in \varepsilon \vee q)$ -fuzzy ideal of M .

Naturally, we can also establish a corresponding result when $F_{\hat{t}}$ is an ideal of M , for all $[0.5, 0.5] < \hat{t} \leq [1, 1]$.

Theorem 3.11: Let F be an interval valued fuzzy set of M .

Then for any $[0.5, 0.5] < \hat{t} \leq [1, 1]$ such that $F_{\hat{t}} \neq \emptyset$, $F_{\hat{t}}$ is an ideal of M if and only if for all $x, y \in M$,

$$(F10) \quad \text{rmax}\{\widehat{\mu}_F(x \oplus y), [0.5, 0.5]\} \geq \text{rmin}\{\widehat{\mu}_F(x), \widehat{\mu}_F(y)\},$$

$$(F11) \quad \text{rmax}\{\widehat{\mu}_F(x), [0.5, 0.5]\} \geq \widehat{\mu}_F(y) \quad \text{if } x \leq y.$$

Proof. Assume that $F_{\hat{t}} (\neq \emptyset)$ is an ideal of M .

Suppose that for some $x, y \in M$

$$\text{rmax}\{\widehat{\mu}_F(x \oplus y), [0.5, 0.5]\} < \text{rmin}\{\widehat{\mu}_F(x), \widehat{\mu}_F(y)\} = \hat{t}.$$

Then $[0.5, 0.5] < \hat{t} \leq 1$, $\widehat{\mu}_F(x \oplus y) < \hat{t}$, and $x, y \in F_{\hat{t}}$. Since $x, y \in F_{\hat{t}}$ and $F_{\hat{t}}$ is an ideal of M , so

$x \oplus y \in F_{\hat{t}}$, and hence $\widehat{\mu}_F(x \oplus y) \geq \hat{t}$, which

contradicts to $\widehat{\mu}_F(x \oplus y) < \hat{t}$. Hence, (F10) holds.

If there exist $x, y \in M$ with $x \leq y$ such that

$$\text{rmax}\{\widehat{\mu}_F(x), [0.5, 0.5]\} < \widehat{\mu}_F(y) = \hat{t},$$

then $[0.5, 0.5] < \hat{t} \leq [1, 1]$, $\widehat{\mu}_F(x) < \hat{t}$ and $y \in F_{\hat{t}}$. Since $y \in F_{\hat{t}}$, we

have $x \in F_{\hat{t}}$, and hence $\widehat{\mu}_F(x) \geq \hat{t}$, which is again a

contradiction. Hence (F11) holds.

Conversely, suppose that the conditions (F10) and (F11) hold. In order to prove that $F_{\hat{t}}$ is an ideal of M ,

assume that $[0.5, 0.5] < \hat{t} \leq [1, 1]$, $x, y \in F_{\hat{t}}$. Then

$[0.5, 0.5] < \hat{t} \leq \min\{\widehat{\mu}_F(x), \widehat{\mu}_F(y)\} \leq$
 $\max\{\widehat{\mu}_F(x \oplus y), [0.5, 0.5]\} = \widehat{\mu}_F(x \oplus y)$. This
 implies that $x \oplus y \in F_{\hat{t}}$. Let $x \leq y$ and $y \in F_{\hat{t}}$. Then
 $[0.5, 0.5] < \hat{t} \leq \widehat{\mu}_F(y) \leq \max\{\widehat{\mu}_F(x),$
 $[0.5, 0.5]\} = \widehat{\mu}_F(x)$, and so $x \in F_{\hat{t}}$. Therefore, $F_{\hat{t}}$ is an
 ideal of M .

Let F be an interval valued fuzzy set of a pseudo-MV algebra M and $J = \{\hat{t} \mid \hat{t} \in D[0, 1] \text{ and } F_{\hat{t}} \text{ is an empty set or an ideal of } M\}$. In particular, if $J = \{\hat{t} \mid [0, 0] < \hat{t}\}$, then F is an ordinary interval valued fuzzy ideal of M (Theorem 3.3); if $J = \{\hat{t} \mid \hat{t} \leq [0.5, 0.5]\}$, F is an interval valued $(\in \in \vee q)$ -fuzzy ideal of M (Theorem 3.10).

In [25], Yuan, Zhang and Ren gave the definition of a fuzzy subgroup with thresholds which is a generalization of Resenfeld's fuzzy subgroup, and also Bhakat and Das's fuzzy subgroup. Based on the results of [25], we can extend the concept of a fuzzy subgroup with thresholds to the concept of a fuzzy ideal with thresholds in following way:

Definition 3.12. Let $\hat{\alpha}, \hat{\beta} \in D[0, 1]$ and $\hat{\alpha} < \hat{\beta}$. Then an interval valued fuzzy set F of M is called an interval valued fuzzy ideal with thresholds $(\hat{\alpha}, \hat{\beta})$ of M if for all $x, y \in M$,

$$(F12) \max\{\widehat{\mu}_F(x \oplus y), \hat{\alpha}\} \geq \min\{\widehat{\mu}_F(x), \widehat{\mu}_F(y), \hat{\beta}\},$$

$$(F13) \max\{\widehat{\mu}_F(x), \hat{\alpha}\} \geq \min\{\widehat{\mu}_F(y), \hat{\beta}\} \text{ if } x \leq y.$$

We now characterize the interval valued fuzzy ideals with thresholds by using their level ideals.

Theorem 3.13: An interval valued fuzzy set F of M is an interval valued fuzzy ideal with thresholds $(\hat{\alpha}, \hat{\beta})$ of M if and only if for every $\hat{\alpha} < \hat{t} \leq \hat{\beta}$ such that $F_{\hat{t}} \neq \emptyset$, $F_{\hat{t}}$ is an ideal of M .

Proof. The proof is similar to the proof of Theorem 3.10 and 3.11.

4. Interval valued $(\in, \in \vee q)$ -fuzzy implicative ideals

In this section, we mainly investigate the properties of interval valued $(\in, \in \vee q)$ -fuzzy implicative ideals in

pseudo-MV algebras.

First, we introduce the concept of interval valued fuzzy implicative ideals of pseudo-MV algebras as follows:

Definition 4.1: An interval valued fuzzy ideal of M is called an interval valued fuzzy implicative ideal of M if for all $x, y, z \in M$, it satisfies:

$$(F14) \widehat{\mu}_F(x \odot y) \geq \min\{\widehat{\mu}_F(x \odot y \odot z^-), \widehat{\mu}_F(z \odot y)\}.$$

Example 4.2: Consider Example 3.2. Let I be an implicative ideal of M , then F is an interval valued fuzzy implicative ideal of M .

Now, we introduce the following concept:

Definition 4.3: An interval valued $(\in \in \vee q)$ -fuzzy ideal of M is called an interval valued $(\in \in \vee q)$ -fuzzy implicative ideal of M if for all $x, y, z \in M$, it satisfies:

$$(F15) \widehat{\mu}_F(x \odot y) \geq \min\{\widehat{\mu}_F(x \odot y \odot z^-), \widehat{\mu}_F(z \odot y), [0.5, 0.5]\}.$$

The following example shows that interval valued $(\in \in \vee q)$ -fuzzy implicative ideals exist.

Example 4.4: Consider the pseudo-MV algebra M in Example 3.5. Define an interval valued fuzzy set F of M by

$$\widehat{\mu}_F(x) = \begin{cases} [0.7, 0.8] & \text{if } x \in M_1 \cup \{0\}, \\ [0.1, 0.2] & \text{if } x \in M_2 \cup \{1\}. \end{cases}$$

Then, one can easily verify that F is an interval valued $(\in \in \vee q)$ -fuzzy implicative ideal of M .

The following proposition is obvious:

Proposition 4.5: An interval valued $(\in \in \vee q)$ -fuzzy ideal of M is an interval valued $(\in \in \vee q)$ -fuzzy implicative ideal of M if and only if for all $x, y, z \in M$,

$$(F16) \widehat{\mu}_F(x \odot y) \geq \min\{\widehat{\mu}_F(x \odot y \odot z), \widehat{\mu}_F(z^- \odot y), [0.5, 0.5]\}.$$

By using the level implicative ideals of pseudo-MV algebras, we can characterize the interval valued $(\in \in \vee q)$ -fuzzy implicative ideals as follows:

Theorem 4.6: Let F be an interval valued $(\in \varepsilon \vee q)$ -fuzzy implicative ideal of M . Then for all $[0,0] < \hat{t} \leq [0.5,0.5]$, $F_{\hat{t}}$ is an empty set or an implicative ideal of M . Conversely, if F is an interval valued fuzzy set of M , for any $[0,0] < \hat{t} \leq [0.5,0.5]$ such that $F_{\hat{t}} \neq \emptyset$, $F_{\hat{t}}$ is an implicative ideal of M , then F is an interval valued $(\in \varepsilon \vee q)$ -fuzzy implicative ideal of M .

Proof. Let F be an interval valued $(\in \varepsilon \vee q)$ -fuzzy implicative ideal of M and $[0,0] < \hat{t} \leq [0.5,0.5]$ such that $F_{\hat{t}} \neq \emptyset$. Then, by Theorem 3.10, we know that $F_{\hat{t}}$ is an ideal of M . If $x \odot y \odot z^- \in F_{\hat{t}}$ and $x \odot y \in F_{\hat{t}}$, then $\widehat{\mu}_F(x \odot y \odot z^-) \geq \hat{t}$ and $\widehat{\mu}_F(x \odot y) \geq \hat{t}$. By using (F15), it follows that $\widehat{\mu}_F(x \odot y) \geq \text{rmin}\{\widehat{\mu}_F(x \odot y \odot z^-), \widehat{\mu}_F(z \odot y), [0.5, 0.5]\} \geq \text{rmin}\{\hat{t}, [0.5, 0.5]\} = \hat{t}$, that is, $x \odot y \in F_{\hat{t}}$. Thus, $F_{\hat{t}}$ is an implicative ideal of M .

Conversely, let F be an interval valued fuzzy set of M such that $F_{\hat{t}} (\neq \emptyset)$ is an implicative ideal of M for all $[0,0] < \hat{t} \leq [0.5,0.5]$. Then by Theorem 3.10, F is an interval valued $(\in \varepsilon \vee q)$ -fuzzy ideal of M . For every $x, y, z \in M$, we can write $\widehat{\mu}_F(x \odot y \odot z^-) \geq \text{rmin}\{\widehat{\mu}_F(x \odot y \odot z^-), \widehat{\mu}_F(z \odot y), [0.5, 0.5]\} = \hat{t}_0$, $\widehat{\mu}_F(z \odot y) \geq \text{rmin}\{\widehat{\mu}_F(x \odot y \odot z^-), \widehat{\mu}_F(z \odot y), [0.5, 0.5]\} = \hat{t}_0$.

Thus, $x \odot y \odot z^- \in F_{\hat{t}_0}$ and $z \odot y \in F_{\hat{t}_0}$. Since $F_{\hat{t}_0}$ is an implicative ideal of M , we have $x \odot y \in F_{\hat{t}_0}$. Hence, $\widehat{\mu}_F(x \odot y) \geq \hat{t}_0 = \text{rmin}\{\widehat{\mu}_F(x \odot y \odot z^-), \widehat{\mu}_F(z \odot y), [0.5, 0.5]\}$.

Therefore, F is an interval valued $(\in \varepsilon \vee q)$ -fuzzy implicative ideal of M .

Naturally, we can establish a corresponding result when $F_{\hat{t}}$ is an implicative ideal of M , for all $[0.5,0.5] < \hat{t} \leq [1,1]$.

Theorem 4.7: Let F be an interval valued fuzzy set of M . Then for any $[0.5,0.5] < \hat{t} \leq [1,1]$ such that $F_{\hat{t}} \neq \emptyset$, $F_{\hat{t}}$ is an implicative ideal of M if and only if it satisfies (F10),

(F11) and (F17)

$$\begin{aligned} & \text{rmax}\{\widehat{\mu}_F(x \odot y), [0.5, 0.5]\} \\ & \geq \text{rmin}\{\widehat{\mu}_F(x \odot y \odot z^-), \widehat{\mu}_F(z \odot y)\}. \end{aligned}$$

Proof. Assume that $F_{\hat{t}} (\neq \emptyset)$ is an implicative ideal of M . Then, it follows from Theorem 3.11 that (F10) and (F11) hold. If there exist $x, y, z \in M$ such that $\text{rmax}\{\widehat{\mu}_F(x \odot y), [0.5, 0.5]\} < \text{rmin}\{\widehat{\mu}_F(x \odot y \odot z^-), \widehat{\mu}_F(z \odot y), [0.5, 0.5]\} = \hat{t}$, then

$[0.5, 0.5] < \hat{t} \leq [1, 1]$, $\widehat{\mu}_F(x \odot y) < \hat{t}$ and $x \odot y \odot z^- \in F_{\hat{t}}$, $z \odot y \in F_{\hat{t}}$. Since $F_{\hat{t}}$ is an implicative ideal of M , $x \odot y \in F_{\hat{t}}$, and so $\widehat{\mu}_F(x \odot y) \geq \hat{t}$, which is a contradiction. Hence (F17) holds.

Conversely, suppose that the conditions (F10), (F11) and (F17) hold. Then, it follows from Theorem 3.11 that $F_{\hat{t}}$ is an ideal of M . Assume that $[0.5, 0.5] < \hat{t} \leq [1, 1]$, $x \odot y \odot z^- \in F_{\hat{t}}$ and $z \odot y \in F_{\hat{t}}$. Then $[0.5, 0.5] < \hat{t} \leq \text{rmin}\{\widehat{\mu}_F(x \odot y \odot z^-), \widehat{\mu}_F(x \odot y)\} \leq \text{rmax}\{\widehat{\mu}_F(x \odot y), [0.5, 0.5]\} < \widehat{\mu}_F(x \odot y)$, which implies that $x \odot y \in F_{\hat{t}}$. Thus, $F_{\hat{t}}$ is indeed an implicative ideal of M .

Based on the method in [25], we can extend the concept of a fuzzy subgroup with thresholds to the concept of an interval valued fuzzy implicative ideal with thresholds in the following way:

Definition 4.8: Let $\hat{\alpha}, \hat{\beta} \in D[0,1]$ with $\hat{\alpha} < \hat{\beta}$. Then an interval valued fuzzy set F of M is called an interval valued fuzzy implicative ideal with thresholds $(\hat{\alpha}, \hat{\beta})$ of M if it satisfies (F12), (F13) and (F18) $\text{rmax}\{\widehat{\mu}_F(x \odot y), \hat{\alpha}\} \geq \text{rmin}\{\widehat{\mu}_F(x \odot y \odot z^-), \widehat{\mu}_F(z \odot y), \hat{\beta}\}$.

Now, we characterize the interval valued fuzzy implicative ideals with thresholds.

Theorem 4.9: An interval valued fuzzy set F of M is an interval valued fuzzy implicative ideal with thresholds $(\hat{\alpha}, \hat{\beta})$ of M if and only if for any $\hat{\alpha} < \hat{t} \leq \hat{\beta}$ such that $F_{\hat{t}} \neq \emptyset$, $F_{\hat{t}}$ is an implicative ideal of M .

Proof. The proof is similar to the proof of Theorem 4.6 and 4.7.

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