Computational Effective Stability Conditions for Time-delay Fuzzy Systems

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Abstract

This paper investigates the system stability of the time-delay fuzzy-model-based control systems based on delay-independent and -dependent approaches. LMI-based delay-independent and -dependent stability conditions are derived to guarantee the system stability using Lyapunov-based technique. It can be seen that some existing stability analysis approaches require high computational demand to solve the solution due to the large number of stability conditions. Consequently, feasible solution may not be obtained with the use of numerical methods due to the limitation of the computer system especially for complicated fuzzy systems with a large number of rules. In this paper, under a particular system formulation, the number of stability conditions can be reduced to alleviate the computational demand on searching for the solution. Furthermore, the stability conditions offer a larger upper bound of time delay compared with some existing approaches. LMI-based performance conditions are also derived to guarantee the system performance. Simulation examples are given to illustrate the merits of the proposed approaches.

1. Introduction

Fuzzy controllers are good at handling complex nonlinear systems owing to its superior generalization abilities. System stability and performance are important concerns in the design of fuzzy control systems. To investigate the system stability and performance of the fuzzy control systems, fuzzy-model-based control approach is the most common approach. Based on a TS-fuzzy model [1]-[2] to represent a nonlinear system, a fuzzy controller can be designed accordingly. Fruitful stability and performance conditions were obtained to guarantee the system stability and performance [3]-[10] of the fuzzy control systems. The system stability and performance conditions can be expressed in the form of linear matrix inequalities (LMIs) [11] which can be solved numerically and efficiently using some convex programming techniques.

Recently, the attention of the researchers on fuzzy control discipline is shifting to time-delay nonlinear systems [12]. As the time-delay nonlinear systems can be found in many real-life engineering processes and the time delay is one of the sources to cause system instability, it is thus important to extend the fuzzy control techniques to this class of nonlinear system in order to put the fuzzy controllers into practice. To deal with the time-delay nonlinear systems using fuzzy control technique, two approaches can be found in the literature, namely delay-independent and -dependent approaches. Delay-independent stability conditions for time-delay fuzzy control systems were derived in [13]-[15] based on Lyapunov-Krasovksii or -Razumikhin approaches. For the delay-independent approach, the stability conditions are not related to the time-delay information. Once the time-delay fuzzy control system is guaranteed to be stable, it is stable for any value of time delay. Hence, delay-independent stability conditions are particular useful for nonlinear systems subject to unknown or inestimable value of time delay. In [16]-[24], delay-dependent stability conditions were derived based on the Lyapunov-Krasovksii approach. During the stability analysis, the time-delay information is considered. To deal with the time-delay information, various inequalities have been proposed. In [13], the Leibniz-Newton formula was employed to approximate the time-delay system states with current system states. To relax the conservativeness of the stability analysis, other forms of inequalities were proposed in [27]-[26] to serve the same purpose. These inequalities have been employed in [16]-[24] to investigate the system stability of time-delay fuzzy-model-based control system. It has been shown in [20]-[24] that relaxed inequalities have led to relaxed stability analysis results. Furthermore, by introducing some free matrices, the stability conditions can be further relaxed. Compared with the delay-independent approach, the analysis procedure of the delay-dependent approach is more complicated. For the delay-dependent stability conditions, the time-delay information is one of the elements to determine the system

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stability. As a result, less conservative stability conditions may be produced. The delay-dependent stability conditions are good for time-delay fuzzy control systems with known or estimable values of time delays. Consequently, both delay-independent and -dependent stability analysis results have their own advantages for different kinds of the time-delay nonlinear systems.

In this paper, the system stability and performance of time-delay fuzzy control systems are investigated based on delay-independent and -dependent approaches. LMI-based stability conditions are derived based on an augmented system representation using Lyapunov-based approach. It can be seen from [13]-[24] that the number of stability conditions is large due to the inherent property of multiplication terms of the fuzzy models, which increases the computational demand on solving the solution. Moreover, during the stability analysis, some free matrices and approximation inequalities are employed to relax the conservativeness and handle the time-delay information, which lead to high matrix dimension of the stability conditions further increasing the computational demand. Some convex programming techniques are usually employed to solve the solution numerically. However, it can be shown that when the number of stability conditions is large enough, feasible solution can be obtained due to the limitation of the computer systems. In fact, conservativeness of the stability conditions can be further relaxed by considering some advanced stability analysis techniques using in delay-free fuzzy-model-based control systems [6]-[10]. However, it can be seen in [6]-[10] that the system stability is governed by a huge matrix which is formed by all sub-system stability conditions. Under such a case, the computational demand is increased drastically. The situation is much worse for time-delay fuzzy-model-based control systems. To put the stability design for time-delay fuzzy-model-based control systems into practice, it is important to alleviate the computational demand to solve the solution of the stability conditions. In this paper, a fuzzy descriptor representation [27]-[29] is employed to represent the fuzzy-model-based control systems. In [29], we have shown that computational demand can be alleviated under a particular representation of the system which effectively handles the multiplication terms for delay-free fuzzy systems. Compared with the stability analysis results in [13]-[24], the derived stability conditions for time-delay fuzzy-model-based control systems offers less number of LMIs. As a result, computational demand on searching for the solution of the stability conditions can be alleviated. The computational advantage can be revealed especially when the time-delay fuzzy control system consists of large number of rules. Furthermore, LMI-based performance conditions are derived to guarantee the system performance. The derived LMI-based stability and performance can be employed to aid the design of stable and well-performed fuzzy controllers for time-delay nonlinear systems.

This paper is organized as follows. In section II, the time-delay TS-fuzzy model and the fuzzy controller are presented. In section III, the LMI-based delay-independent and -dependent stability and performance conditions are derived. In section IV, simulations examples are presented to illustrate the merits of the proposed approach. A conclusion is drawn in section V.

2. FUZZY MODEL AND FUZZY CONTROLLER

A time-delay fuzzy-model-based control system formed by a time-delay nonlinear plant represented by a fuzzy model and a fuzzy controller is introduced.

A. Fuzzy Model

Let p be the number of fuzzy rules describing the nonlinear plant. The i-th rule is of the following format,

Rule i: IF $f_1(\mathbf{x}(t))$ is M_1^i AND … AND $f_\Psi(\mathbf{x}(t))$ is M_Ψ^i THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{A}_{di}\mathbf{x}(t-\tau_d) + \mathbf{B}_i\mathbf{u}(t)$ (1)

where M_α^i is a fuzzy term of rule i corresponding to the function $f_\alpha(\mathbf{x}(t))$, $\alpha = 1, 2, \ldots, \Psi$, $i = 1, 2, \ldots, p$, Ψ is a positive integer; $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{A}_{di} \in \Re^{n\times n}$ are the known constant system matrices; $\mathbf{B}_i \in \Re^{n\times m}$ is the constant input matrix; $\mathbf{x}(t) \in \Re^{n\times 1}$ is the system state vector and $\mathbf{u}(t) \in \Re^{m\times 1}$ is the input vector, $\tau_d \geq 0$ denotes a constant time delay. The system dynamics are described as,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\begin{bmatrix}\mathbf{A}_i & \mathbf{A}_{di} & \mathbf{B}_i\end{bmatrix}\begin{bmatrix}\mathbf{x}(t) \\ \mathbf{x}(t-\tau_d) \\ \mathbf{u}(t)\end{bmatrix} \tag{2}$$

where,

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1, \quad w_i(\mathbf{x}(t)) \in \begin{bmatrix}0 & 1\end{bmatrix} \text{ for all } i \tag{3}$$

$$w_i(\mathbf{x}(t)) = \frac{\mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{M}_\Psi^i}(f_\Psi(\mathbf{x}(t)))}{\sum_{k=1}^{p}\left(\mu_{\mathrm{M}_1^k}(f_1(\mathbf{x}(t))) \times \mu_{\mathrm{M}_2^k}(f_2(\mathbf{x}(t))) \times \cdots \times \mu_{\mathrm{M}_\Psi^k}(f_\Psi(\mathbf{x}(t)))\right)} \tag{4}$$

is a nonlinear function of $\mathbf{x}(t)$ and $\mu_{\mathrm{M}_\alpha^i}(f_\alpha(\mathbf{x}(t)))$, $\alpha = 1, 2, \ldots, \Psi$, is the grade of membership corresponding to the fuzzy term of M_α^i. It is assumed that $\mathbf{x}(t) = \boldsymbol{\varphi}(t)$ for $t \in [-\tau_d \quad 0]$ where $\boldsymbol{\varphi}(t)$ denotes the initial condition of $\mathbf{x}(t)$.

B. Fuzzy Controller

A fuzzy controller with p fuzzy rules is employed to handle the time-delay nonlinear plant. The j-th rule of the fuzzy controller is of the following format.

Rule j: IF $f_1(\mathbf{x}(t))$ is M_1^j AND … AND $f_\Psi(\mathbf{x}(t))$ is M_Ψ^j THEN $\mathbf{u}(t)=\mathbf{G}_j\mathbf{x}(t)$ (5)

where $\mathbf{G}_j \in \Re^{m\times n}$ is the feedback gains of rule j. The inferred output of the fuzzy controller is given by,

$$\mathbf{u}(t)=\sum_{j=1}^{p} w_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t) \tag{6}$$

3. STABILITY ANALYSIS AND PERFORMANCE DESIGN

In this section, the system stability of the time-delay fuzzy control system formed by (2) and (6) is investigated based on delay-independent and -dependent approaches. LMI-based stability conditions are derived to guarantee the system stability. Furthermore, LMI-based performance conditions are derived to serve as constraints for the feedback gains to achieve the system performance measured by a performance index.

A. *Time-Delay Independent Approach*

The system stability of the time-delay fuzzy control system based on delay-independent approach is investigated. In the following analysis, $w_i(\mathbf{x}(t))$ is denoted by w_i for simplicity. Furthermore, the property of $\sum_{i=1}^{p} w_i = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j = 1$ will be used. To investigate the system stability of the time-delay fuzzy control system, the following Lyapunov function candidate is considered,

$$V_1(t)=\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}_1\mathbf{x}(t)+\int_{t-\tau_d}^{t}\mathbf{x}(\varphi)^{\mathrm{T}}\mathbf{S}\mathbf{x}(\varphi)d\varphi \tag{7}$$

where $\mathbf{P}_1=\mathbf{P}_1^{\mathrm{T}}\in\Re^{n\times n}>0$ and $\mathbf{S}=\mathbf{S}^{\mathrm{T}}\in\Re^{n\times n}>0$. It will be shown that $\dot{V}(t)\le 0$ (equality holds when x(t) = x(t − τ_d) = 0) which implies the asymptotic stability of the time-delay fuzzy control system. From (2) and (7),

$$\begin{aligned}\dot{V}_1(t)&=\mathbf{x}(t)^{\mathrm{T}}\mathbf{P}_1\dot{\mathbf{x}}(t)+\dot{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{P}_1\mathbf{x}(t)+\mathbf{x}(t)^{\mathrm{T}}\mathbf{S}\mathbf{x}(t)\\&\quad-\mathbf{x}(t-\tau_d)^{\mathrm{T}}\mathbf{S}\mathbf{x}(t-\tau_d)\\&=\sum_{i=1}^{p}w_i\begin{bmatrix}\mathbf{x}(t)\\\mathbf{x}(t-\tau_d)\\\mathbf{u}(t)\end{bmatrix}^{\mathrm{T}}\left(\mathbf{P}^{\mathrm{T}}\begin{bmatrix}\mathbf{A}_i&\mathbf{A}_{di}&\mathbf{B}_i\\\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\end{bmatrix}+\begin{bmatrix}\mathbf{A}_i&\mathbf{A}_{di}&\mathbf{B}_i\\\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\end{bmatrix}^{\mathrm{T}}\mathbf{P}+\begin{bmatrix}\mathbf{S}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&-\mathbf{S}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\end{bmatrix}\right)\begin{bmatrix}\mathbf{x}(t)\\\mathbf{x}(t-\tau_d)\\\mathbf{u}(t)\end{bmatrix}\end{aligned} \tag{8}$$

where $\mathbf{P}=\begin{bmatrix}\mathbf{P}_1&\mathbf{0}&\mathbf{0}\\\mathbf{P}_2&\mathbf{P}_3&\mathbf{0}\\\mathbf{P}_4&\mathbf{P}_5&\mathbf{P}_6\end{bmatrix}\in\Re^{(2n+m)\times(2n+m)}$, $\mathbf{P}_2\in\Re^{n\times n}$, $\mathbf{P}_3\in\Re^{n\times n}$, $\mathbf{P}_4\in\Re^{m\times n}$, $\mathbf{P}_5\in\Re^{m\times n}$ and $\mathbf{P}_6\in\Re^{m\times m}$. From (6), we have the following property which is applied in the following analysis,

$$\sum_{i=1}^{p}w_i\begin{bmatrix}\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{G}_i&\mathbf{0}&-\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{x}(t-\tau_d)\\\mathbf{u}(t)\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{0}\end{bmatrix}$$

$$\sum_{i=1}^{p}w_i\begin{bmatrix}\mathbf{x}(t)\\\mathbf{x}(t-\tau_d)\\\mathbf{u}(t)\end{bmatrix}^{\mathrm{T}}\left(\mathbf{P}^{\mathrm{T}}\begin{bmatrix}\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{G}_i&\mathbf{0}&-\mathbf{I}\end{bmatrix}+\begin{bmatrix}\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{G}_i&\mathbf{0}&-\mathbf{I}\end{bmatrix}^{\mathrm{T}}\mathbf{P}\right)\begin{bmatrix}\mathbf{x}(t)\\\mathbf{x}(t-\tau_d)\\\mathbf{u}(t)\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{0}\end{bmatrix} \tag{9}$$

From (8) and (9), we have,

$$\dot{V}_1(t)=\sum_{i=1}^{p}w_i\begin{bmatrix}\mathbf{x}(t)\\\mathbf{x}(t-\tau_d)\\\mathbf{u}(t)\end{bmatrix}^{\mathrm{T}}\mathbf{X}^{-1^{\mathrm{T}}}\mathbf{X}^{\mathrm{T}}\left(\mathbf{P}^{\mathrm{T}}\begin{bmatrix}\mathbf{A}_i&\mathbf{A}_{di}&\mathbf{B}_i\\\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{G}_i&\mathbf{0}&-\mathbf{I}\end{bmatrix}+\begin{bmatrix}\mathbf{A}_i&\mathbf{A}_{di}&\mathbf{B}_i\\\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{G}_i&\mathbf{0}&-\mathbf{I}\end{bmatrix}^{\mathrm{T}}\mathbf{P}+\begin{bmatrix}\mathbf{S}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&-\mathbf{S}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\end{bmatrix}\right)\mathbf{X}\mathbf{X}^{-1}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{x}(t-\tau_d)\\\mathbf{u}(t)\end{bmatrix} \tag{10}$$

where $\mathbf{X}=\begin{bmatrix}\mathbf{X}_1&\mathbf{0}&\mathbf{0}\\\mathbf{X}_1&\mathbf{X}_1&\mathbf{0}\\\mathbf{X}_2&\mathbf{X}_3&\mathbf{X}_4\end{bmatrix}=\mathbf{P}^{-1}\in\Re^{(2n+m)\times(2n+m)}$, $\mathbf{X}_1=\mathbf{X}_1^{\mathrm{T}}=\mathbf{P}_1\in\Re^{n\times n}>0$, $\mathbf{X}_2\in\Re^{m\times n}$, $\mathbf{X}_3\in\Re^{m\times n}$ and $\mathbf{X}_4\in\Re^{m\times m}$. Let the feedback gains be designed as $\mathbf{G}_i=\mathbf{N}_i\mathbf{X}_1^{-1}$ where $\mathbf{N}_i\in\Re^{m\times n}$, i = 1, 2, ..., p, and $\mathbf{Y}=\mathbf{X}_1\mathbf{S}\mathbf{X}_1$. From (10), we have,

$$\dot{V}_1(t)=\sum_{i=1}^{p}w_i\mathbf{z}(t)^{\mathrm{T}}\mathbf{Q}_i\mathbf{z}(t) \tag{11}$$

where $\mathbf{z}(t)=\mathbf{X}^{-1}\begin{bmatrix}\mathbf{x}(t)\\\mathbf{x}(t-\tau_d)\\\mathbf{u}(t)\end{bmatrix}$ and

$$\mathbf{Q}_i=\begin{bmatrix}(\mathbf{A}_i+\mathbf{A}_{di})\mathbf{X}_1+\mathbf{X}_1(\mathbf{A}_i+\mathbf{A}_{di})^{\mathrm{T}}+\mathbf{B}_i\mathbf{X}_2+\mathbf{X}_2^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}&*&*\\\mathbf{X}_1\mathbf{A}_{di}^{\mathrm{T}}+\mathbf{X}_3^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}-\mathbf{Y}&-\mathbf{Y}&*\\\mathbf{N}_i-\mathbf{X}_2+\mathbf{X}_4^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}&-\mathbf{X}_3&-\mathbf{X}_4-\mathbf{X}_4^{\mathrm{T}}\end{bmatrix} \tag{12}$$

The symbol "*" denotes the transposed element in the corresponding position of the matrix. Referring to (11), it can be seen that $\dot{V}_1(t)<0$ if $\mathbf{Q}_i$ < 0 for i = 1, 2, ..., p, which implies the asymptotically stability of the time-delay fuzzy control system.

Remark 1: The number of LMI stability conditions are reduced to $p + 2$ compared with those in the stability conditions of [13]-[24]. Consequently, the computational demand on searching for the solution to the stability conditions can be alleviated. The computational advantage is obvious when the time-delay fuzzy-model-based control system has a large number of rules.

B. Performance Design

The LMI-based performance conditions are derived for the time-delay fuzzy-model-based control system. The system performance is quantitatively measured by the following performance index which is commonly used in optimal control techniques [30].

$$J = \int_0^{\infty} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} dt \tag{13}$$

where $\mathbf{J}_1 = \mathbf{J}_1^{\mathrm{T}} \in \Re^{n \times n} > 0$, $\mathbf{J}_2 \in \Re^{n \times m}$, $\mathbf{J}_3 = \mathbf{J}_3^{\mathrm{T}} \in \Re^{m \times m} > 0$ and $\begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix} \in \Re^{(n+m)\times(n+m)} > 0$.

From (6) and (13), we have,

$$J = \int_0^{\infty} \begin{bmatrix} \mathbf{x}(t)^{\mathrm{T}} & \left(\sum_{i=1}^{p} w_i \mathbf{G}_i \mathbf{x}(t) \right)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \sum_{j=1}^{p} w_j \mathbf{G}_j \mathbf{x}(t) \end{bmatrix} dt$$

$$= \int_0^{\infty} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sum_{i=1}^{p} w_i \mathbf{G}_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sum_{j=1}^{p} w_j \mathbf{G}_j \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t) \end{bmatrix} dt \tag{14}$$

Let

$$J < \eta \int_0^{\infty} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{X}_1^{-1}\mathbf{X}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_1^{-1}\mathbf{X}_1^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t) \end{bmatrix} dt \tag{15}$$

where η is a non-zero positive scalar. The scalar performance index of J can be attenuated to a prescribed level governed by the scalar value of η. From (14) and (15), we have,

$$\int_0^{\infty} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t) \end{bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sum_{i=1}^{p} w_i \mathbf{G}_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sum_{j=1}^{p} w_j \mathbf{G}_j \end{bmatrix} - \eta \begin{bmatrix} \mathbf{X}_1^{-1}\mathbf{X}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_1^{-1}\mathbf{X}_1^{-1} \end{bmatrix} \right) \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t) \end{bmatrix} dt < 0 \tag{16}$$

From (16) and recalling $\mathbf{G}_i = \mathbf{N}_i \mathbf{X}_1^{-1}$, $i = 1, 2, ..., p$, we have,

$$\int_0^{\infty} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{X}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_1^{-1} \end{bmatrix} \mathbf{W} \begin{bmatrix} \mathbf{X}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_1^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t) \end{bmatrix} dt < 0 \tag{17}$$

where

$$\mathbf{W} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{0} & \sum_{i=1}^{p} w_i \mathbf{N}_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{0} & \sum_{j=1}^{p} w_j \mathbf{N}_j \end{bmatrix} - \eta \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{18}$$

It can be seen that the inequality of (17) holds when $\mathbf{W} < 0$. From (18) and by Schur complement, $\mathbf{W} < 0$ is equivalent to the following inequality.

$$\sum_{i=1}^{p} w_i \mathbf{W}_i < 0 \tag{19}$$

where $\begin{bmatrix} \mathbf{K}_1 & \mathbf{K}_2 \\ \mathbf{K}_2^{\mathrm{T}} & \mathbf{K}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix}^{-1} > 0$,

$$\mathbf{W}_i = \begin{bmatrix} -\eta\mathbf{I} & \mathbf{0} & \mathbf{X}_1 & \mathbf{0} \\ \mathbf{0} & -\eta\mathbf{I} & \mathbf{0} & \mathbf{N}_i^{\mathrm{T}} \\ \mathbf{X}_1 & \mathbf{0} & -\mathbf{K}_1 & -\mathbf{K}_2 \\ \mathbf{0} & \mathbf{N}_i & -\mathbf{K}_2^{\mathrm{T}} & -\mathbf{K}_3 \end{bmatrix}, \; i = 1, 2, ..., p.$$

It can be seen that the inequality of (19) holds when $\mathbf{W}_i < 0$, $i = 1, 2, ..., p$. The LMI conditions of $\begin{bmatrix} \mathbf{K}_1 & \mathbf{K}_2 \\ \mathbf{K}_2^{\mathrm{T}} & \mathbf{K}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix}^{-1} > 0$ and $\mathbf{W}_i < 0$, $i = 1, 2, ..., p$ are the performance conditions. The stability and performance conditions are summarized in the following theorem.

Theorem 1 (Delay-Independent Approach): The time-delay fuzzy-model-based control system formed by the time-delay nonlinear system in the form of (2) *and the fuzzy controller of* (6) *is asymptotically stable if there exists a non-zero positive scalar* η *and there exist matrices* $\mathbf{X}_1 = \mathbf{X}_1^{\mathrm{T}} \in \Re^{n \times n}$, $\mathbf{X}_2 \in \Re^{m \times n}$, $\mathbf{X}_3 \in \Re^{m \times n}$, $\mathbf{X}_4 \in \Re^{m \times m}$ *and* $\mathbf{N}_i \in \Re^{m \times n}$, $i = 1, 2, ..., p$, $\mathbf{Y} = \mathbf{Y}^{\mathrm{T}} \in \Re^{n \times n}$, $\mathbf{J}_1 = \mathbf{J}_1^{\mathrm{T}} \in \Re^{n \times n}$, $\mathbf{J}_2 \in \Re^{n \times m}$ *and* $\mathbf{J}_3 = \mathbf{J}_3^{\mathrm{T}} \in \Re^{m \times m}$ *such that the following LMI-based stability and performance conditions are satisfied.*

Stability Conditions:

$\mathbf{X}_1 > 0$*;* $\mathbf{Y} > 0$*;* (12) *where the feedback gains are designed as* $\mathbf{G}_i = \mathbf{N}_i \mathbf{X}_1^{-1}$*,* $i = 1, 2, ..., p$.

Performance Conditions:

$$\begin{bmatrix} \mathbf{K}_1 & \mathbf{K}_2 \\ \mathbf{K}_2^{\mathrm{T}} & \mathbf{K}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix}^{-1} > 0;$$

$$\mathbf{W}_i = \begin{bmatrix} -\eta\mathbf{I} & \mathbf{0} & \mathbf{X}_1 & \mathbf{0} \\ \mathbf{0} & -\eta\mathbf{I} & \mathbf{0} & \mathbf{N}_i^{\mathrm{T}} \\ \mathbf{X}_1 & \mathbf{0} & -\mathbf{K}_1 & -\mathbf{K}_2 \\ \mathbf{0} & \mathbf{N}_i & -\mathbf{K}_2^{\mathrm{T}} & -\mathbf{K}_3 \end{bmatrix} < 0, \; i = 1, 2, ..., p.$$

Remark 2: It should be noted that the stability and performance conditions in Theorem 1 do not relate to the time-delay information. Hence, they are the time-delay independent stability and performance conditions.

Remark 3: The performance conditions only affect the system performance and can be taken away from Theorem 1 when the system performance is not under concerned during design. Prior to applying the performance conditions, the constant weighting matrices $\mathbf{J}_1$, $\mathbf{J}_2$ and $\mathbf{J}_3$ have to be determined.

Remark 4: If there exists a solution to the stability conditions in Theorem 1, it implies that $\mathbf{X}_1 > 0$ and $-\mathbf{X}_4 - \mathbf{X}_4^{\mathrm{T}} < 0$. These are the sufficient conditions to ensure that $\mathbf{X}$ is a non-singular matrix. Hence, the existence of $\mathbf{X} = \mathbf{P}^{-1}$ is ensured.

C. Time-Delay Dependent Approach

The system stability of the time-delay fuzzy-model-based control system based on delay-independent approach is investigated. The following Lyapunov function candidate is considered to investigate the system stability.

$$V(t) = V_1(t) + V_2(t) \tag{20}$$

where $V_1(t)$ is defined in (7) and

$$V_2(t) = \int_{-h_d}^{0}\int_{t+\sigma}^{t} \dot{\mathbf{x}}(\varphi)^{\mathrm{T}}\mathbf{R}\dot{\mathbf{x}}(\varphi)d\varphi d\sigma \tag{21}$$

From (20), the time derivative of $V(t)$ is as follows,

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) \tag{22}$$

Similar to the derivation of the delay-independent approach, from (10), we have,

$$\dot{V}_1(t) = \sum_{i=1}^{p} w_i \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t-\tau_d) \\ \mathbf{u}(t) \end{bmatrix}^{\mathrm{T}} \left(\begin{matrix} \mathbf{P}^{\mathrm{T}} \begin{bmatrix} \mathbf{A}_i & \mathbf{A}_{di} & \mathbf{B}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{G}_i & \mathbf{0} & -\mathbf{I} \end{bmatrix} + \begin{bmatrix} \mathbf{A}_i & \mathbf{A}_{di} & \mathbf{B}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{G}_i & \mathbf{0} & -\mathbf{I} \end{bmatrix}^{\mathrm{T}} \mathbf{P} \\ + \begin{bmatrix} \mathbf{S} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \end{matrix} \right) \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t-\tau_d) \\ \mathbf{u}(t) \end{bmatrix} \tag{23}$$

Denote the upper bound of τ_d as h_d. From (2), (21), and using the property that R > 0 and $(\mathbf{a}_i - \mathbf{a}_j)^{\mathrm{T}}\mathbf{R}(\mathbf{a}_i - \mathbf{a}_j) \geq 0 \Rightarrow$ $\mathbf{a}_i^{\mathrm{T}}\mathbf{R}\mathbf{a}_i + \mathbf{a}_j^{\mathrm{T}}\mathbf{R}\mathbf{a}_j \geq \mathbf{a}_i^{\mathrm{T}}\mathbf{R}\mathbf{a}_j + \mathbf{a}_j^{\mathrm{T}}\mathbf{R}\mathbf{a}_i$ where $\mathbf{a}_i \in \Re^{n\times 1}$, i, j = 1, 2, ..., p, are arbitrary vectors, the time derivative of $V_2(t)$ is as follows.

$$\begin{aligned} \dot{V}_2(t) &= h_d\dot{\mathbf{x}}(t)^{\mathrm{T}}\mathbf{R}\dot{\mathbf{x}}(t) - \int_{t-h_d}^{t} \dot{\mathbf{x}}(\varphi)^{\mathrm{T}}\mathbf{R}\dot{\mathbf{x}}(\varphi)d\varphi \\ &\leq h_d \sum_{i=1}^{p} w_i \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t-\tau_d) \\ \mathbf{u}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{A}_i^{\mathrm{T}} \\ \mathbf{A}_{di}^{\mathrm{T}} \\ \mathbf{B}_i^{\mathrm{T}} \end{bmatrix} \mathbf{R} \begin{bmatrix} \mathbf{A}_i^{\mathrm{T}} \\ \mathbf{A}_{di}^{\mathrm{T}} \\ \mathbf{B}_i^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t-\tau_d) \\ \mathbf{u}(t) \end{bmatrix} \\ &- \int_{t-h_d}^{t} \dot{\mathbf{x}}(\varphi)^{\mathrm{T}}\mathbf{R}\dot{\mathbf{x}}(\varphi)d\varphi \end{aligned} \tag{24}$$

Before proceeding further, the following Lemma is introduced.

Lemma 1 [26]: *The following integral inequality holds for any arbitrary matrices of* $\mathbf{T}_1 \in \Re^{n\times n}$, $\mathbf{T}_2 \in \Re^{n\times n}$ *and* $\mathbf{R} = \mathbf{R}^{\mathrm{T}} \in \Re^{n\times n} > 0$, *and a scalar* $h_d \geq 0$.

$$-\int_{t-h_d}^{t} \dot{\mathbf{x}}(\varphi)\mathbf{R}\dot{\mathbf{x}}(\varphi)d\varphi \leq \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t-h_d) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{T}_1 + \mathbf{T}_1^{\mathrm{T}} & -\mathbf{T}_1^{\mathrm{T}} + \mathbf{T}_2 \\ -\mathbf{T}_1 + \mathbf{T}_2^{\mathrm{T}} & -\mathbf{T}_2 - \mathbf{T}_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t-h_d) \end{bmatrix} + h_d \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t-h_d) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{T}_1^{\mathrm{T}} \\ \mathbf{T}_2^{\mathrm{T}} \end{bmatrix} \mathbf{R}^{-1} \begin{bmatrix} \mathbf{T}_1^{\mathrm{T}} \\ \mathbf{T}_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t-h_d) \end{bmatrix}$$

where $\mathbf{x}(t) \in \Re^{n\times 1}$ *with first-order continuous-derivative entries.*

Based on Lemma 1 and from (24), we have

$$\begin{aligned} \dot{V}_2(t) \leq{}& h_d \sum_{i=1}^{p} w_i \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t-\tau_d) \\ \mathbf{u}(t) \end{bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \mathbf{A}_i^{\mathrm{T}} \\ \mathbf{A}_{di}^{\mathrm{T}} \\ \mathbf{B}_i^{\mathrm{T}} \end{bmatrix} \mathbf{R} \begin{bmatrix} \mathbf{A}_i^{\mathrm{T}} \\ \mathbf{A}_{di}^{\mathrm{T}} \\ \mathbf{B}_i^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} + \begin{bmatrix} \mathbf{T}_{1i}^{\mathrm{T}} \\ \mathbf{T}_{2i}^{\mathrm{T}} \\ \mathbf{0} \end{bmatrix} \mathbf{R}^{-1} \begin{bmatrix} \mathbf{T}_{1i}^{\mathrm{T}} \\ \mathbf{T}_{2i}^{\mathrm{T}} \\ \mathbf{0} \end{bmatrix}^{\mathrm{T}} \right) \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t-\tau_d) \\ \mathbf{u}(t) \end{bmatrix} \\ &+ \sum_{i=1}^{p} w_i \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t-\tau_d) \\ \mathbf{u}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{T}_{1i} + \mathbf{T}_{1i}^{\mathrm{T}} & -\mathbf{T}_{1i}^{\mathrm{T}} + \mathbf{T}_{2i} & \mathbf{0} \\ -\mathbf{T}_{1i} + \mathbf{T}_{2i}^{\mathrm{T}} & -\mathbf{T}_{2i} - \mathbf{T}_{2i}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t-\tau_d) \\ \mathbf{u}(t) \end{bmatrix} \end{aligned} \tag{25}$$

where $\mathbf{T}_{1i} \in \Re^{n\times n}$ and $\mathbf{T}_{2i} \in \Re^{n\times n}$, i = 1, 2, ..., p, are arbitrary matrices. From (23) and (25), we have,

$$\dot{V}(t) \leq= \sum_{i=1}^{p} w_i \mathbf{z}(t)^{\mathrm{T}}\mathbf{X}^{\mathrm{T}}\overline{\mathbf{Q}}_i\mathbf{X}\mathbf{z}(t) \tag{26}$$

where

$$\begin{aligned} \overline{\mathbf{Q}}_i ={}& \mathbf{P}^{\mathrm{T}} \begin{bmatrix} \mathbf{A}_i & \mathbf{A}_{di} & \mathbf{B}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{G}_i & \mathbf{0} & -\mathbf{I} \end{bmatrix} + \begin{bmatrix} \mathbf{A}_i & \mathbf{A}_{di} & \mathbf{B}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{G}_i & \mathbf{0} & -\mathbf{I} \end{bmatrix}^{\mathrm{T}} \mathbf{P} + \begin{bmatrix} \mathbf{S} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{T}_{1i} + \mathbf{T}_{1i}^{\mathrm{T}} & -\mathbf{T}_{1i}^{\mathrm{T}} + \mathbf{T}_{2i} & \mathbf{0} \\ -\mathbf{T}_{1i} + \mathbf{T}_{2i}^{\mathrm{T}} & -\mathbf{T}_{2i} - \mathbf{T}_{2i}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + h_d \begin{bmatrix} \mathbf{A}_i^{\mathrm{T}} \\ \mathbf{A}_{di}^{\mathrm{T}} \\ \mathbf{B}_i^{\mathrm{T}} \end{bmatrix} \mathbf{R} \begin{bmatrix} \mathbf{A}_i^{\mathrm{T}} \\ \mathbf{A}_{di}^{\mathrm{T}} \\ \mathbf{B}_i^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} + h_d \begin{bmatrix} \mathbf{T}_{1i}^{\mathrm{T}} \\ \mathbf{T}_{2i}^{\mathrm{T}} \\ \mathbf{0} \end{bmatrix} \mathbf{R}^{-1} \begin{bmatrix} \mathbf{T}_{1i}^{\mathrm{T}} \\ \mathbf{T}_{2i}^{\mathrm{T}} \\ \mathbf{0} \end{bmatrix}^{\mathrm{T}} \end{aligned} \tag{27}$$

It can be seen that $\dot{V}(t) < 0$ when $\mathbf{X}^{\mathrm{T}}\overline{\mathbf{Q}}_i\mathbf{X} < 0$ for all i = 1, 2, ..., p, which imply the asymptotically stability of the time-delay fuzzy control system. Let $\mathbf{M} = \mathbf{R}^{-1} \in \Re^{n\times n}$, $\mathbf{Y} = \mathbf{X}_1\mathbf{S}\mathbf{X}_1$, $\mathbf{G}_i = \mathbf{N}_i\mathbf{X}_1^{-1}$, $\mathbf{N}_i \in \Re^{m\times n}$, $\mathbf{U}_{1i} = \mathbf{X}_1\mathbf{T}_{1i}\mathbf{X}_1 \in \Re^{n\times n}$, $\mathbf{U}_{2i} = \mathbf{X}_1^{\mathrm{T}}\mathbf{T}_{2i}\mathbf{X}_1 \in \Re^{n\times n}$, i = 1, 2, ..., p. Considering $\mathbf{X}^{\mathrm{T}}\overline{\mathbf{Q}}_i\mathbf{X}$, we have,

$$\begin{aligned} \mathbf{X}^{\mathrm{T}}\overline{\mathbf{Q}}_i\mathbf{X} ={}& \begin{bmatrix} \mathbf{A}_i & \mathbf{A}_{di} & \mathbf{B}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{G}_i & \mathbf{0} & -\mathbf{I} \end{bmatrix} \mathbf{X} + \mathbf{X}^{\mathrm{T}} \begin{bmatrix} \mathbf{A}_i & \mathbf{A}_{di} & \mathbf{B}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{G}_i & \mathbf{0} & -\mathbf{I} \end{bmatrix}^{\mathrm{T}} + \mathbf{X}^{\mathrm{T}} \begin{bmatrix} \mathbf{S} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{X} \\ &+ \mathbf{X}^{\mathrm{T}} \begin{bmatrix} \mathbf{T}_{1i} + \mathbf{T}_{1i}^{\mathrm{T}} & -\mathbf{T}_{1i}^{\mathrm{T}} + \mathbf{T}_{2i} & \mathbf{0} \\ -\mathbf{T}_{1i} + \mathbf{T}_{2i}^{\mathrm{T}} & -\mathbf{T}_{2i} - \mathbf{T}_{2i}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{X} + h_d\mathbf{X}^{\mathrm{T}} \begin{bmatrix} \mathbf{A}_i^{\mathrm{T}} \\ \mathbf{A}_{di}^{\mathrm{T}} \\ \mathbf{B}_i^{\mathrm{T}} \end{bmatrix} \mathbf{R} \begin{bmatrix} \mathbf{A}_i^{\mathrm{T}} \\ \mathbf{A}_{di}^{\mathrm{T}} \\ \mathbf{B}_i^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \mathbf{X} \\ &+ h_d\mathbf{X}^{\mathrm{T}} \begin{bmatrix} \mathbf{T}_{1i}^{\mathrm{T}} \\ \mathbf{T}_{2i}^{\mathrm{T}} \\ \mathbf{0} \end{bmatrix} \mathbf{X}_1\left(\mathbf{X}_1^{-1}\mathbf{R}^{-1}\mathbf{X}_1^{-1}\right)\mathbf{X}_1 \begin{bmatrix} \mathbf{T}_{1i}^{\mathrm{T}} \\ \mathbf{T}_{2i}^{\mathrm{T}} \\ \mathbf{0} \end{bmatrix}^{\mathrm{T}} \mathbf{X} \end{aligned} \tag{28}$$

With the property that $\mathbf{R} = \mathbf{R}^{\mathrm{T}} > 0$, we consider the following inequality,

$$\begin{aligned} &\left(\mathbf{X}_1 - \zeta\mathbf{R}^{-1}\right)^{\mathrm{T}}\mathbf{R}\left(\mathbf{X}_1 - \zeta\mathbf{R}^{-1}\right) = \mathbf{X}_1^{\mathrm{T}}\mathbf{R}\mathbf{X}_1 - \zeta\mathbf{X}_1^{\mathrm{T}} - \zeta\mathbf{X}_1 + \zeta^2\mathbf{R}^{-1} \geq 0 \\ &\Rightarrow \mathbf{X}_1\mathbf{R}\mathbf{X}_1 \geq 2\zeta\mathbf{X}_1 - \zeta^2\mathbf{M} \end{aligned} \tag{29}$$

By Schur complement, from (28) and (29), $\mathbf{X}^{\mathrm{T}}\overline{\mathbf{Q}}_i\mathbf{X} < 0$ for all i = 1, 2, ..., p, are implied by the following LMIs.

$$\boldsymbol{\Theta}_i = \begin{bmatrix} (\mathbf{A}_i+\mathbf{A}_{di})\mathbf{X}_1+\mathbf{X}_1(\mathbf{A}_i+\mathbf{A}_{di})^{\mathrm{T}}+\mathbf{B}_i\mathbf{X}_2+\mathbf{X}_2^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} & * & * & * & * \\ \mathbf{X}_1\mathbf{A}_{di}^{\mathrm{T}}+\mathbf{X}_3^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}-\mathbf{Y}-\mathbf{U}_{1i}-\mathbf{U}_{2i} & -\mathbf{Y}-\mathbf{U}_{2i}-\mathbf{U}_{2i}^{\mathrm{T}} & * & * & * \\ \mathbf{N}_i-\mathbf{X}_2+\mathbf{X}_4^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} & -\mathbf{X}_3 & -\mathbf{X}_4-\mathbf{X}_4^{\mathrm{T}} & * & * \\ h_d((\mathbf{A}_i+\mathbf{A}_{di})\mathbf{X}_1+\mathbf{B}_i\mathbf{X}_2) & h_d(\mathbf{A}_{di}\mathbf{X}_1+\mathbf{B}_i\mathbf{X}_3) & h_d\mathbf{B}_i\mathbf{X}_4 & -h_d\mathbf{M} & * \\ h_d(\mathbf{U}_{1i}+\mathbf{U}_{2i}) & h_d\mathbf{U}_{2i} & \mathbf{0} & \mathbf{0} & -h_d(2\zeta\mathbf{X}_1-\zeta^2\mathbf{M}) \end{bmatrix} < 0,$$

i = 1, 2, ..., p (30)

It can be seen that the fuzzy control system is asymptotically stable if the stability conditions of (30) are satisfied. The performance design is the same as that in the delay-independent approach. The stability and performance stability conditions are summarized in the following theorem.

Theorem 2 (Delay-Dependent Approach): The time-delay fuzzy-model-based control system formed by the time-delay nonlinear system in form of (2) *and the fuzzy controller of* (6) *is asymptotically stable if there exist non-zero positive scalars* h_d, ζ *and* η, *and matrices* $\mathbf{X}_1 = \mathbf{X}_1^{\mathrm{T}} \in \Re^{n\times n}$, $\mathbf{X}_2 \in \Re^{m\times n}$, $\mathbf{X}_3 \in \Re^{m\times n}$, $\mathbf{X}_4 \in \Re^{m\times m}$, $\mathbf{M} = \mathbf{M}^{\mathrm{T}} \in \Re^{n\times n}$, $\mathbf{Y} = \mathbf{Y}^{\mathrm{T}} \in \Re^{n\times n}$, $\mathbf{J}_1 = \mathbf{J}_1^{\mathrm{T}} \in \Re^{n\times n}$, $\mathbf{J}_2 \in \Re^{n\times m}$, $\mathbf{J}_3 = \mathbf{J}_3^{\mathrm{T}} \in \Re^{m\times m}$, $\mathbf{U}_{1i} = \mathbf{X}_1\mathbf{T}_{1i}\mathbf{X}_1 \in \Re^{n\times n}$ *and* $\mathbf{U}_{2i} = \mathbf{X}_1^{\mathrm{T}}\mathbf{T}_{2i}\mathbf{X}_1 \in \Re^{n\times n}$, *i = 1, 2, ..., p, such that the following LMI-based stability and performance conditions are satisfied.*

Stability Conditions:

$\mathbf{X}_1 > 0$; $\mathbf{M} > 0$; $\mathbf{Y} > 0$; (30) *where the feedback gains are designed as* $\mathbf{G}_i = \mathbf{N}_i\mathbf{X}_1^{-1}$, *i = 1, 2, ..., p.*

Performance Conditions in Theorem 1

Remark 5: It can be seen that the stability and performance conditions in Theorem 2 relate to the time-delay information. Hence, they are the delay-dependent stability and performance conditions.

4. SIMULATION EXAMPLES

Three simulation examples are employed to illustrate the effectiveness of the proposed approach. The proposed fuzzy controller of (6) is employed to control some fuzzy systems with time delay. With the aid of the derived LMI-based stability and performance conditions, the feedback gains of the fuzzy controller can obtained to guarantee the system stability and performance. In the following examples, comparisons are done based on the versions of stability analysis results with constant time delay in various published papers.

A. Simulation Example 1: Time-Delay Independent Approach

Consider a fuzzy model with the following rules.

Rule i: IF $x_1(t)$ is about M_1^i THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_1\mathbf{x}(t) + \mathbf{A}_{d1}\mathbf{x}(t-t_d) + \mathbf{B}_i u(t)$, i = 1, 2 (31)

where $\mathbf{A}_1 = \begin{bmatrix} -2.1 & 0.1 \\ -0.2 & -0.9 \end{bmatrix}$, $\mathbf{A}_2 = \begin{bmatrix} -1.9 & 0 \\ -0.2 & -1.1 \end{bmatrix}$, $\mathbf{A}_{d1} = \begin{bmatrix} -1.1 & 0.1 \\ -0.8 & -0.9 \end{bmatrix}$, $\mathbf{A}_{d2} = \begin{bmatrix} -0.9 & 0 \\ -1.1 & -1.2 \end{bmatrix}$, $\mathbf{B}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{B}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. The membership functions are chosen as $w_1(x_1(t)) = \mu_{\mathrm{M}_1^1}(x_1(t)) = \dfrac{1}{1+e^{-2x_1(t)}}$ and $w_2(x_1(t)) = \mu_{\mathrm{M}_1^2}(x_1(t)) = 1 - w_1(x_1(t))$ [18]. Considering the open-loop fuzzy-model-based system (i.e., setting $\mathbf{B}_i = \mathbf{0}$) and using the delay-independent stability conditions in Theorem 1, feasible solution can be obtained with the help of MATLAB LMI toolbox but not for the stability conditions in [13] and [23].

To illustrate the effectiveness of the LMI-based performance conditions, we consider the closed-loop fuzzy-model-based control system and design the feedback gains based on Theorem 1. With η = 10^{-10}, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{J}_3$ = 1, the feedback gains under $\mathbf{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{J}_1 = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix}$, are obtained respectively and tabulated in Table I. It can be seen that different $\mathbf{J}_1$ put different weights on different system states. These weights will constrain the parameter space of the feedback gains to satisfy the performance conditions. Take $\mathbf{J}_1 = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$ as an example and refer to the performance index of (13), the weight of 100 is put on $x_1(t)$ which is to suppress the accumulation of energy contributed by $x_1(t)$ 100 times more than those contributed by the rest, i.e., $x_2(t)$.

Fig. 1 shows the system state responses with τ_d = 2s and the control signals of the time-delay fuzzy control systems under the initial state condition of $\mathbf{x}(t)$ = $[1 \quad 0]^{\mathrm{T}}$. The initial system state function is defined as $\boldsymbol{\varphi}(t) = [1 \quad 0]^{\mathrm{T}}$ for $t \in [-\tau_d \quad 0]$. Referring to this figure, it can be seen that the fuzzy controllers with different feedback gains can stabilize the time-delay nonlinear system. The fuzzy controller with $\mathbf{J}_1 = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$ offers better system response on $x_1(t)$ in terms of rising time and overshoot/undershoot magnitude as the heaviest weight is put on x_1(t) in $\mathbf{J}_1$. For the feedback gains corresponding to $\mathbf{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix}$, as the heaviest weight is put on $x_2(t)$ in $\mathbf{J}_1$, this fuzzy controller offers better system response on $x_2(t)$ in terms of rising time and overshoot/undershoot magnitude.

B. Simulation Example 2: Time-Delay Dependent Approach

Consider a fuzzy model with the following rules [18].

Rule i: IF $x_1(t)$ is about M_1^i THEN

$$\dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{A}_{di}\mathbf{x}(t-t_d) + \mathbf{B}_i u(t),\ i = 1, 2 \tag{32}$$

where $\mathbf{A}_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}$, $\mathbf{A}_2 = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix}$, $\mathbf{A}_{d1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$, $\mathbf{A}_{d2} = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix}$, $\mathbf{B}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{B}_2 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$. The membership functions are the same as those in simulation example 1. Considering $\mathbf{B}_i = 0$, the upper bounds of time delay given by some existing approaches and Theorem 2 are listed in Table II. It can be seen that Theorem 2 of this paper offers the largest upper bounds of time delay, i.e., 2.3767s with $\xi = 0.5$.

LMI-based stability and performance conditions in Theorem 2 are employed to obtain a set of feedback gains for the fuzzy system of (32). With the help of MATLAB LMI toolbox, the feedback gains are tabulated in Table III with τ_d = 2s, $\xi = 0.5$, $\eta = 10^{-10}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{J}_3 = 1$, corresponding to $\mathbf{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{J}_1 = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix}$ respectively. Fig. 2 shows the system state responses with τ_d = 2s and the control signals of the time-delay fuzzy control systems under the initial state condition of $\mathbf{x}(t) = [1 \quad 0]^{\mathrm{T}}$. The initial system state function is defined as $\boldsymbol{\varphi}(t) = [1 \quad 0]^{\mathrm{T}}$ for $t \in [-\tau_d \quad 0]$. Referring to this figure, it can be seen that the fuzzy controllers with different feedback gains can stabilize the time-delay nonlinear system. Furthermore, the fuzzy controller with $\mathbf{J}_1 = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$ offers better system response on $x_1(t)$ in terms of rising time and overshoot/undershoot magnitude as the heaviest weight is put on $x_1(t)$ in $\mathbf{J}_1$. For the fuzzy controller with $\mathbf{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix}$, it can be seen that the overshoot/undershoot of $x_2(t)$ is suppressed effectively.

C. Simulation Example 3: Time-Delay Dependent Approach

Consider a fuzzy model with the following rules.

Rule i: IF $x_1(t)$ is about M_1^i THEN

$$\dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{A}_{di}\mathbf{x}(t-t_d) + \mathbf{B}_i u(t),\ i = 1, 2 \tag{33}$$

where $\mathbf{A}_1 = \begin{bmatrix} 0.5091 & 0 & 0 \\ -0.5091 & 0 & 0 \\ 0.5091 & -4 & 0 \end{bmatrix}$, $\mathbf{A}_2 = \begin{bmatrix} 0.5091 & 0 & 0 \\ -0.5091 & 0 & 0 \\ 0.8102 & -6.3662 & 0 \end{bmatrix}$, $\mathbf{A}_{d1} = \begin{bmatrix} 0.2182 & 0 & 0 \\ -0.2182 & 0 & 0 \\ 0.2182 & 0 & 0 \end{bmatrix}$, $\mathbf{A}_{d2} = \begin{bmatrix} 0.2182 & 0 & 0 \\ -0.2182 & 0 & 0 \\ 0.3472 & 0 & 0 \end{bmatrix}$, $\mathbf{B}_1 = \begin{bmatrix} -1.4286 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{B}_2 = \begin{bmatrix} -10 \\ -1 \\ 0 \end{bmatrix}$. The membership functions are the same as those in simulation example 1. Based on Theorem 1, with τ_d = 2s, $\xi = 0.01$, $\eta = 10^{-9}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{J}_3 = 1$, the feedback gains under $\mathbf{J}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{J}_1 = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{J}_1 = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, are obtained respectively and tabulated in Table IV. It can be seen that different $\mathbf{J}_1$ put different weights on different system states to constrain the parameter space of the feedback gains to satisfy the performance conditions.

Fig. 3 shows the system state responses with τ_d = 2s and the control signals of the time-delay fuzzy control systems under the initial state condition of $\mathbf{x}(t) = [3 \quad -2 \quad -15]^{\mathrm{T}}$. The initial system state function is defined as $\boldsymbol{\varphi}(t) = [3 \quad -2 \quad -15]^{\mathrm{T}}$ for $t \in [-\tau_d \quad 0]$. Referring to this figure, it can be seen that the fuzzy controllers with different feedback gains can stabilize the time-delay nonlinear system. The fuzzy controller with $\mathbf{J}_1 = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ offers better system response on $x_1(t)$ in terms of rising time and overshoot/undershoot magnitude as the heaviest weight is put on $x_1(t)$ in $\mathbf{J}_1$. For the feedback gains corresponding to $\mathbf{J}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}$, as the heaviest weight is put on $x_3(t)$ in $\mathbf{J}_1$, this fuzzy controller offers better system response on $x_3(t)$ in terms of rising time and overshoot/undershoot magnitude.

It can be shown that all approaches in Table III are able to offer stable design of fuzzy controller for the fuzzy system of (33) with time delay of 2s. In order to illustrate the computational advantage of the proposed approach, the following experiment is carried out. Additional r rules are added to the fuzzy model in the above example. As a result, there are $r + 2$ rules for the fuzzy model in total. The k-th rule, $k = 1, 2, ..., r$ is characterized by $\mathbf{A}_k = \mathbf{A}_1 a_k + \mathbf{A}_2(1 - a_k)$,

$\mathbf{A}_{dk} = \mathbf{A}_{d1}a_k + \mathbf{A}_{d2}(1-a_k)$ and $\mathbf{B}_k = \mathbf{B}_1$ where $a_k \in [0 \quad 1]$ is a randomly generated scalar. The existing delay-dependent and -independent stability conditions and the proposed approaches are employed to testify the system stability. The solution is solved with the help of MATLAB LMI toolbox running under a personal computer with 3G Hz P4 CPU and 1GB RAM. In this simulation, we choose r = 15 (there are 17 rules for the fuzzy model in total) and h_d = 2s (for delay-dependent approach). The experiment is repeated for 20 times. The average computational time for various algorithms is tabulated in Table V. It can be seen from this table that the proposed approach offer the least computation time for both delay-independent and -dependent stability approaches. The computational advantage of the proposed approaches is more obvious for large value of r = 50. Furthermore, some existing delay-dependent stability conditions offer no solution as the search process terminates due to slow process for r = 50.

5. Conclusions

The system stability of the time-delay fuzzy-model-based control system has been investigated under delay-independent and -independent approaches. Based on the descriptor representation, the multiplication terms of the fuzzy system can be handled effectively. As a result, less number of LMI-based stability conditions, which can alleviate the computational demand on searching for the solution to the stability conditions, has been derived using the Lyapunov-based approach. Furthermore, LMI-based performance conditions have been derived to guarantee the system performance. Simulation examples have been given to show the merits of the proposed approach.

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References

[1] T. Takagi and M. Sugeno, “Fuzzy identification of systems and its applications to modeling and control,” *IEEE Trans. Sys., Man., Cybern.*, vol. smc-15 no. 1, pp. 116-132, Jan., 1985.

[2] M. Sugeno and G.T. Kang, “Structure identification of fuzzy model,” *Fuzzy Sets and Systems*, vol. 28, pp. 15-33, 1988.

[3] C.L. Chen, P.C. Chen, C.K. Chen, “Analysis and design of fuzzy-model-based control,” *Fuzzy Sets and Systems,* vol. 57, no 2, 26, pp. 125-140, Jul. 1993.

[4] H.O. Wang, K. Tanaka, and M.F. Griffin, “An approach to fuzzy control of nonlinear systems: stability and the design issues,” *IEEE Trans. Fuzzy Syst*., vol. 4, no. 1, pp. 14-23, Feb., 1996.

[5] K. Tanaka, T. Ikeda and H.O. Wang, “Fuzzy regulator and fuzzy observer: Relaxed stability conditions and LMI-based designs,” *IEEE Trans. Fuzzy Syst.*, vol. 6, no. 2, pp. 250-265, 1998.

[6] E. Kim and H. Lee, “New approaches to relaxed quadratic stability conditions of fuzzy control systems,” *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 5, pp. 523-534, 2000.

[7] X. Liu and Q. Zhang, “New approaches to H_∞ controller designs based on fuzzy observers for T-S fuzzy systems via LMI,” *Automatica*, vol. 39, no. 9, pp. 1571-1582, Sep. 2003.

[8] X. Liu and Q. Zhang, “Approaches to quadratic stability conditions and H_∞ control designs for T-S fuzzy systems,” *IEEE Trans. Fuzzy Syst.*, vol. 11, no. 6, pp. 830-839, 2003.

[9] M.C.M. Teixeira, E. Assunção and R.G. Avellar, “On relaxed LMI-based designs for fuzzy regulators and fuzzy observers,” *IEEE Trans. on Fuzzy Systems*, vol. 11, no. 5, pp. 613-623, Oct. 2003.

[10] C.H. Fang, Y.S. Liu, S.W. Kau, L. Hong and C.H. Lee, “A new LMI-based approach to relaxed quadratic stabilization of T-S fuzzy control systems,” *IEEE Trans. on Fuzzy Systems*, vol. 14, no. 3, pp. 386-397, Jun. 2006.

[11] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*, ser. SIAM studies in Applied Mathematics, Philadelphia, PA: SIAM, 1994.

[12] G. Feng, “A Survey on Analysis and Design of Model-Based Fuzzy Control Systems,” *IEEE Trans Fuzzy Systems*, vol. 14, no. 5, pp. 676- 697, 2006

[13] Y.Y. Cao and P.M. Frank, “Analysis and synthesis of nonlinear time-delay systems via fuzzy control approach,” *IEEE Trans. Fuzzy Systems*, vol. 8, no. 2, pp. 200-211, Apr. 2000.

[14] Y.Y. Cao and P.M. Frank, “Stability analysis and synthesis of nonlinear time-delay systems via linear Takagi-Sugeno fuzzy models,” *Fuzzy Sets and Systems*, vol. 124, no. 2, pp. 213-219, Apr. 2001.

[15] R.J. Wong, W.W. Lin and W.J. Wang, “Stabilizability of linear quadratic state feedback for uncertain fuzzy time-delay systems,” *IEEE Trans. Syst., Man and Cybern., Part B: Cybernetics*, vol. 34, no. 2, pp.1288-1292 Apr. 2004.

[16] X.P. Guan and C.L. Chen, “Delay-dependent guaranteed cost control for T-S fuzzy systems with time delays,” *IEEE Trans. Fuzzy Systems*, vol. 12, no. 2, pp.236-249, Apr. 2004.

[17] B. Chen and X. Liu, “Delay-dependent robust $H\infty$ control for T–S fuzzy systems with time delay,” *IEEE Trans. Fuzzy Systems*, vol. 13, no. 4, pp.544-556, Aug. 2005.

[18] E. Tian and C. Peng, “Delay-dependently stability analysis and synthesis of uncertain T-S fuzzy systems with time-varying delay,” *Fuzzy Sets and Systems*, vol. 157, pp. 544-559, 2006.

[19] B. Chen, X. Liu and S. Tong, "Delay-dependent stability analysis and control synthesis of fuzzy dynamic systems with time delay," *Fuzzy Sets and Systems*, vol. 157, pp. 2224-2240, 2006.

[20] J. Yoneyama, "New robust stability conditions and design of robust stabilizing controllers for Takagi-Sugeno fuzzy time-delay systems," *IEEE Trans. Fuzzy Systems*, vol. 15, no. 5, pp. 828-839, Oct, 2007.

[21] C. Li, H. Wang and X. Liao, "Delay-dependent robust stability of uncertain fuzzy systems with time-varying delays," *IEE Proc, Control Theory Application*, vol. 151, no. 4, pp.417-421, July, 2004

[22] C.H. Lien, "Stabilization for uncertain Takagi-Sugen fuzzy systems with time-varying delays and bounded uncertainties," *Chaos, Solitons and Fractals*, vol. 32, pp. 645-652, 2007.

[23] Z. Wang, D.W.C. Ho and X. Liu, "A note on the robust stability of uncertain stochastic fuzzy systems with time-delays," *IEEE Trans. Sys., Man., Cybern. – Part A*, vol.34, no. 4, pp. 116-132, July, 2004.

[24] B. Chen, X. Liu and S. Tong, "New delay-dependent stabilization conditions of T-S fuzzy systems with constant delay," *Fuzzy Sets and Systems*, 2007 (Available online).

[25] M. Wu, Y. He, J.-H. She, and G.-P. Liu, "Delay-dependent criteria for robust stability of time-varying delay systems," *Automatica*, vol. 40, pp. 1435-1439, 2004.

[26] X.M. Zhang, M. Wu, J.-H. She, and Y. He, "Delay-dependent stabilization of linear systems with time-varying state and input delays," *Automatica*, vol. 41, pp. 1405-1412, 2005.

[27] E. Fridman and U. Shaked, "Delay-dependent stability and H_∞ control: constant and time-varying delays," *International Journal of Control*, vol. 76, no. 1, pp. 48-60, 2003.

[28] K. Tanaka, H. Ohtake and H.O. Wang, "A descriptor system approach to fuzzy control system design via fuzzy Lyapunov functions," *IEEE Trans. Fuzzy Systems*, vol. 15, no. 3, pp. 333-341, Jun. 2007.

[29] H.K. Lam and F.H.F. Leung, "LMI relaxed stability conditions for fuzzy-model-based control systems," in *Proc. of the International Conference on Fuzzy Systems 2006 (FUZZ-IEEE 2006)*, Vancouver, BC, Canada, July 16-21, 2006, pp. 9064-9070.

[30] B.D.O. Anderson and J.B. Moore, *Optimal Control: Linear Quadratic Methods*. Prentice Hall, 1990.

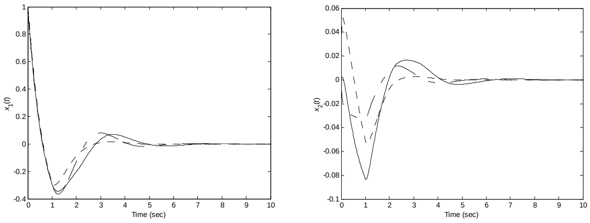

Fig. 1(a). $x_1(t)$. Fig. 1(b). $x_2(t)$.

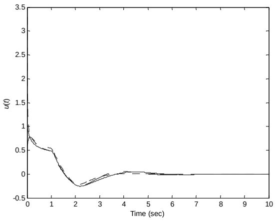

Fig. 1(c). $u(t)$.

Fig. 1. System state responses and control signals of example 1 with $\tau_d = 1$s under the fuzzy controller with $\mathbf{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (solid lines), $\mathbf{J}_1 = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$ (dotted lines) and $\mathbf{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix}$ (dash-dot lines).

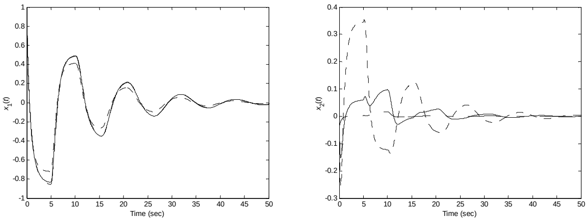

Fig. 2(a). $x_1(t)$. Fig. 2(b). $x_2(t)$.

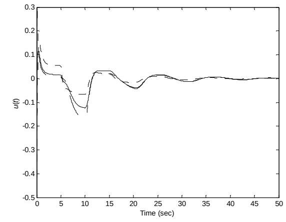

Fig. 2(c). $u(t)$.

Fig. 2. System state responses and control signals of example 2 with $\tau_d = 5$s under the fuzzy controller with $\mathbf{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (solid lines), $\mathbf{J}_1 = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$ (dotted lines) and $\mathbf{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix}$ (dash-dot lines).

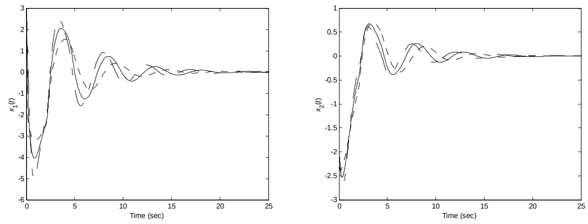

Fig. 3(a). $x_1(t)$. Fig. 3(b). $x_2(t)$.

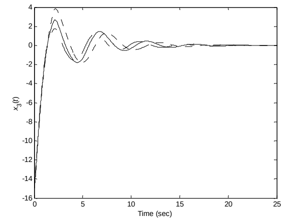

Fig. 3(c). $x_3(t)$.

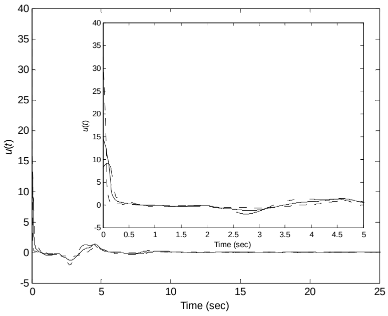

Fig. 3(d). $u(t)$.

Fig. 3. System state responses and control signals of example 3 with τ_d = 2s under the fuzzy controller with $\mathbf{J}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (solid lines), $\mathbf{J}_1 = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (dotted lines) and $\mathbf{J}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}$ (dash-dot lines).

Table I. Feedback gains under delay-dependent stability approach with different $\mathbf{J}_1$ for example 1.

$\mathbf{J}_1$	Feedback Gains
$\mathbf{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\mathbf{G}_1 = [1.0349 \quad -10.0025]$ $\mathbf{G}_2 = [0.9584 \quad -8.9833]$
$\mathbf{J}_1 = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$	$\mathbf{G}_1 = [2.4915 \quad -38.8894]$ $\mathbf{G}_2 = [9.2812 \quad -52.5048]$
$\mathbf{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix}$	$\mathbf{G}_1 = [0.4619 \quad -19.9519]$ $\mathbf{G}_2 = [0.3537 \quad -18.7751]$

Table II. Maximum time delay of h_d for some published and the proposed stability conditions.

Stability Conditions	Maximum time delay of h_d (Sec.)
Theorem 2	2.3767
[18]	1.5970
[22]	1.5416
[21]	1.4785
[17]	1.2246
[20]	1.0449
[16]	1.0124
[19]	0.7171
[24]	0.0343

Table III. Feedback gains under delay-independent stability approach with different $\mathbf{J}_1$ for example 2.

$\mathbf{J}_1$	Feedback Gains
$\mathbf{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\mathbf{G}_1 = [-0.0728 \quad -0.9403]$ $\mathbf{G}_2 = [-0.0779 \quad -0.8023]$
$\mathbf{J}_1 = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$	$\mathbf{G}_1 = [-0.4058 \quad -0.9449]$ $\mathbf{G}_2 = [-0.4883 \quad -0.8045]$
$\mathbf{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix}$	$\mathbf{G}_1 = [-0.0392 \quad -9.9649]$ $\mathbf{G}_2 = [-0.0286 \quad -6.9614]$

Table IV. Feedback gains under delay-dependent stability with different $\mathbf{J}_1$ for example 3.

$\mathbf{J}_1$	Feedback Gains
$\mathbf{J}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\mathbf{G}_1 = [4.9775 \quad -14.5541 \quad 1.9301]$ $\mathbf{G}_2 = [3.2072 \quad -9.0984 \quad 1.2968]$
$\mathbf{J}_1 = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\mathbf{G}_1 = [9.2414 \quad -20.3801 \quad 2.1765]$ $\mathbf{G}_2 = [6.0483 \quad -13.6325 \quad 1.4814]$
$\mathbf{J}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}$	$\mathbf{G}_1 = [6.4084 \quad -23.0313 \quad 3.8262]$ $\mathbf{G}_2 = [4.1202 \quad -14.4536 \quad 2.6412]$

Table V. Average computational time for some published and the proposed delay independent (DI) and delay dependent (DD) stability conditions.

Stability Analysis Approach	Stability Conditions	Average Computational Time (Sec.)	
		r = 15	r = 50
DI	Theorem 1	0.3344	2.1359
DI	[14]	1.0110	13.9453
DD	Theorem 2	3.1158	53.2656
DD	[22]	44.4060	No Solution
DD	[17]	51.5940	No Solution
DD	[16]	59.5630	No Solution
DD	[18]	No Solution	No Solution
DD	[19]	No Solution	No Solution