Reduction of Fuzzy Linear Systems of Dual Equations

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Abstract

The fuzzy linear system of dual equations is studied. A new algorithm for solving the system is proposed. Some relevant properties of the solutions are also proved.

Keywords: dual equation, fuzzy linear system, Gauss-Jordan method.

1. Introduction

Friedman et al. [14] applied the parametric form of fuzzy numbers [15] to solve the fuzzy linear system $Ax = b$, where A is a real $n \times n$ matrix. They proved that the system can be replaced with a $2n \times 2n$ real linear system. Asady et al. [10] used the same method to solve the more general system in which A is a real $m \times n$ matrix with the condition $m \le n$. Afterwards, Zheng and Wang [20] applied the Moore-Penrose inverse of A to solve that system without the condition $m \le n$. Many numerical methods are also proposed [1-3], [5-9], [11-13], [17], [18]. On the fuzzy linear system of dual equations (FLSDE) $Ax = Bx + b$, where A and B are real square matrices, Ma et al. [16] investigated the existence of the solutions. They claimed that this type system can not be replaced by a fuzzy linear system $Ax = b$. Wang et al. [19] proposed an iteration algorithm (numerical method) for solving the special system $x = Bx + b$, where B is a real square matrix. In this paper, we propose a new algorithm for solving the FLSDE

$$Ax + a = Bx + b \tag{1.1}$$

where A and B are real $m \times n$ matrices.

This paper is organized as follows. In Section 2, the CR-representation of fuzzy numbers is introduced. In section 3, we use the CR-representation to replace Eq. (1.1) with two real linear systems. Therefore, we can determine all solutions of Eq. (1.1) by solving the two corresponding systems. In fact, the solutions may be not fuzzy numbers. In Section 4, we provide a sufficient and necessary condition for discriminating whether the solutions are fuzzy numbers. Some relevant properties of the solutions are proved. If the system is uniquely solvable, and a, b are both triangular (trapezoidal, rectangle), then the unique solution is triangular (trapezoidal, rectangle), too. In Section 5, we propose a new algorithm for computing the solutions of FLSDE. Some examples are given to illustrate our algorithm.

2. CR-representation of fuzzy numbers

An arbitrary fuzzy number x can be represented by an ordered pair of left continuous functions

$$[x^L(\alpha), x^U(\alpha)], \ 0 \le \alpha \le 1,$$

which satisfy the following conditions:

(1) x^L is increasing on [0,1],

(2) x^U is decreasing on [0,1],

(3) $x^L(1) \le x^U(1)$.

For arbitrary fuzzy numbers

$$x = [x^L(\alpha), x^U(\alpha)], \ \ y = [y^L(\alpha), y^U(\alpha)],$$

and a real number $\lambda \in \mathbb{R}$, the addition and scalar multiplication of fuzzy numbers can be described as follows:

$$x + y = [x^L(\alpha) + y^L(\alpha), x^U(\alpha) + y^U(\alpha)] \tag{2.1}$$

and

$$\lambda x = \begin{cases} [\lambda x^L(\alpha), \lambda x^U(\alpha)], \text{ if } \lambda \ge 0 \\ [\lambda x^U(\alpha), \lambda x^L(\alpha)], \text{ if } \lambda < 0 \end{cases}, \tag{2.2}$$

respectively.

Obviously, every closed interval can be equivalently described by its center and radius (half-width). For example, the center of $[x^L(\alpha), x^U(\alpha)]$ is

$$x^C(\alpha) := \frac{x^L(\alpha) + x^U(\alpha)}{2}, \tag{2.3}$$

and the radius of $[x^L(\alpha), x^U(\alpha)]$ is

$$x^R(\alpha) := \frac{x^U(\alpha) - x^L(\alpha)}{2}. \tag{2.4}$$

This means that we may represent x as the ordered pair $(x^C(\alpha), x^R(\alpha))$, which is called the CR-*representation* of x.

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Proposition 2.1. Let

$$x=(x^C(\alpha),x^R(\alpha)),\quad y=(y^C(\alpha),y^R(\alpha))$$

be fuzzy numbers with CR-representations, and λ be a real number. Then,

$$x+y=(x^C(\alpha)+y^C(\alpha),x^R(\alpha)+y^R(\alpha)) \tag{2.5}$$

$$\lambda x=(\lambda x^C(\alpha),|\lambda|x^R(\alpha)) \tag{2.6}$$

Proof. Eq. (2.1) gives

$$x+y=[x^L(\alpha)+y^L(\alpha),x^U(\alpha)+y^U(\alpha)].$$

Consequently, Eqs. (2.3) and (2.4) imply its center and radius are

$$\frac{1}{2}(x^L(\alpha)+y^L(\alpha)+x^U(\alpha)+y^U(\alpha))=x^C(\alpha)+y^C(\alpha)$$

and

$$\frac{1}{2}(x^U(\alpha)+y^U(\alpha)-x^L(\alpha)-y^L(\alpha))=x^R(\alpha)+y^R(\alpha),$$

respectively. This shows Eq. (2.5). If $\lambda\ge 0$, it easily verifies Eq. (2.6). Let $\lambda<0$. Eq. (2.2) implies

$$\lambda x=[\lambda x^U(\alpha),\lambda x^L(\alpha)].$$

Obviously, its center and radius are

$$\frac{1}{2}(\lambda x^U(\alpha)+\lambda x^L(\alpha))=\lambda x^C(\alpha)$$

and

$$\frac{1}{2}(\lambda x^L(\alpha)-\lambda x^U(\alpha))=|\lambda|x^R(\alpha),$$

respectively. This completes the proof.

3. Fuzzy linear system of dual equations

Let $\mathbb{F}$ be the set of all fuzzy numbers, $\mathbb{F}^n$ be the set of all n-column fuzzy vectors, and $\mathbb{R}^{m\times n}$ denote the set of all real $m\times n$ matrices, where m and n are positive integers.

Definition 3.1. Let $A,B\in\mathbb{R}^{m\times n}$, and $a,b\in\mathbb{F}^m$. A fuzzy vector $x=(x_1,x_2,\dots,x_n)^{\mathrm{T}}\in\mathbb{F}^n$ given by α-cuts $x_i=[x_i^L,x_i^U]$ or CR-representations $x_i=(x_i^C,x_i^R)$, $1\le i\le n$, is called a solution of FLSDE if it satisfies Eq. (1.1).

For $A=(A_{ij})\in\mathbb{R}^{m\times n}$, denote by $|A|$ the real matrix with entries $|A_{ij}|$, where $|A_{ij}|$ is the absolute value of A_{ij}. Let $x=(x_1,x_2,\dots,x_n)^{\mathrm{T}}$ be a fuzzy vector, where $x_i=(x_i^C,x_i^R)$. The symbols x^C and x^R will stand the following real functional vectors

$$x^C:=(x_1^C,x_2^C,\dots,x_n^C)^{\mathrm{T}}$$

and

$$x^R:=(x_1^R,x_2^R,\dots,x_n^R)^{\mathrm{T}},$$

respectively.

Theorem 3.2. Let $A,B\in\mathbb{R}^{m\times n}$, and $a,b\in\mathbb{F}^m$. Then the FLSDE $Ax+a=Bx+b$ can be replaced with the following real linear systems:

$$(A-B)x^C=b^C-a^C, \tag{3.1}$$

$$(|A|-|B|)x^R=b^R-a^R. \tag{3.2}$$

Proof. Proposition 2.1 implies that

$$[Ax]_i=\sum_{j=1}^{n}A_{ij}x_j=\sum_{j=1}^{n}(A_{ij}x_j^C,|A_{ij}|x_j^R)$$
$$=(\sum_{j=1}^{n}A_{ij}x_j^C,\sum_{j=1}^{n}|A_{ij}|x_j^R).$$

Hence, the i-th components of $Ax+a$ and $Bx+b$ are

$$(\sum_{j=1}^{n}A_{ij}x_j^C+a_i^C,\sum_{j=1}^{n}|A_{ij}|x_j^R+a_i^R)$$

and

$$(\sum_{j=1}^{n}B_{ij}x_j^C+b_i^C,\sum_{j=1}^{n}|B_{ij}|x_j^R+b_i^R),$$

respectively. This shows that

$$\begin{cases}\displaystyle\sum_{j=1}^{n}A_{ij}x_j^C+a_i^C=\sum_{j=1}^{n}B_{ij}x_j^C+b_i^C,\\ \displaystyle\sum_{j=1}^{n}|A_{ij}|x_j^R+a_i^R=\sum_{j=1}^{n}|B_{ij}|x_j^R+b_i^R,\end{cases}$$

or equivalently

$$\begin{cases}\displaystyle\sum_{j=1}^{n}(A_{ij}-B_{ij})x_j^C=b_i^C-a_i^C,\\ \displaystyle\sum_{j=1}^{n}(|A_{ij}|-|B_{ij}|)x_j^R=b_i^R-a_i^R.\end{cases}$$

That implies Eqs. (3.1) and (3.2).

4. A sufficient and necessary condition

A CR-representation vector $x=(x_1,x_2,\dots,x_n)$ satisfying Eq. (1.1) is called an *extended solution*, in which $x_i=(x_i^C,x_i^R)$ may be not a fuzzy number. In this section, we will provide a sufficient and necessary condition for discriminating whether a given extended solution is fuzzy or not.

Proposition 4.1. Let x^C and x^R be two differentiable functions on [0,1]. Then the CR-representation $x=(x^C,x^R)$ is defined as a fuzzy number if and only if

(1) $x^R(1)\geq 0$, and

(2) $|\frac{d}{d\alpha}x^C(\alpha)|\leq -\frac{d}{d\alpha}x^R(\alpha)$ for all $\alpha\in[0,1]$.

Proof. Eqs. (2.3) and (2.4) together imply

$$x=[x^C-x^R,x^C+x^R].$$

Note that, x^C-x^R and x^C+x^R are increasing and decreasing if and only if

$$\frac{d}{d\alpha}(x^C(\alpha)-x^R(\alpha))=\frac{d}{d\alpha}x^C(\alpha)-\frac{d}{d\alpha}x^R(\alpha)\geq 0$$

and

$$\frac{d}{d\alpha}(x^C(\alpha)+x^R(\alpha))=\frac{d}{d\alpha}x^C(\alpha)+\frac{d}{d\alpha}x^R(\alpha)\leq 0,$$

respectively. Combining the above two inequalities, we get the statement (2). On the other hand,

$$x^C(1)-x^R(1)\leq x^C(1)+x^R(1)$$

implies $x^R(1)\geq 0$. This completes the proof.

Corollary 4.2. The CR-representation

$$x=(a+b\alpha,c+d\alpha),$$

where $a,b,c,d\in\mathbb{R}$, is defined as a trapezoidal fuzzy number if and only if

$$|b|\leq -d\leq c.$$

Moreover, it is triangular if and only if

$$|b|\leq -d=c.$$

Proof. Because that,

$$\frac{d}{d\alpha}(a+b\alpha)=b \text{ and } \frac{d}{d\alpha}(c+d\alpha)=d,$$

Proposition 4.1 implies $x\in\mathbb{F}$ if and only if $c+d\geq 0$ and $|b|\leq -d$. Combining the two inequalities, we get $|b|\leq -d\leq c$. Similarly, we may get the other one.

In [14, Theorem 3], the authors proposed a sufficient and necessary condition for the unique solvability of fuzzy linear system $Ax=b$, in which A is a real square matrix. Later, they generalized the theorem to the case $Ax=Bx+b$. Unfortunately, Allahviranloo [4] claimed that it is only the sufficient condition but not a necessary one. The following Proposition 4.3 gives a modified and generalized one.

For two real vectors $u,v\in\mathbb{R}^n$, denote by $u\leq v$ if each component of u is less than or equal to that of v.

Proposition 4.3. Let $A,B\in\mathbb{R}^{n\times n}$, and $a,b\in\mathbb{F}^n$. Then the FLSDE $Ax+a=Bx+b$ has a unique extended solution if and only if $A-B$ and $|A|-|B|$ are both invertible. Furthermore,

(1) the CR-representation of the unique extended solution is

$$x=((A-B)^{-1}(b^C-a^C),(|A|-|B|)^{-1}(b^R-a^R)), \quad (4.1)$$

(2) if all components of a and b are differentiable, then x is a fuzzy solution if and only if

$$\begin{aligned}&|(A-B)^{-1}\frac{d}{d\alpha}(b^C(\alpha)-a^C(\alpha))|\\&\leq -(|A|-|B|)^{-1}\frac{d}{d\alpha}(b^R(\alpha)-a^R(\alpha))\end{aligned} \quad (4.2)$$

for all $\alpha\in[0,1]$, and

$$(|A|-|B|)^{-1}(b^R(1)-a^R(1))\geq 0. \quad (4.3)$$

Proof. Theorem 3.2 implies that we can replace the FLSDE by Eqs. (3.1) and (3.2). Hence, it has a unique extended solution if and only if Eqs. (3.1) and (3.2) are both uniquely solvable. This is equivalent to $A-B$ and $|A|-|B|$ are both invertible, since A and B are square. Thus, the unique extended solution is

$$\begin{aligned}x^C&=(A-B)^{-1}(b^C-a^C)\\x^R&=(|A|-|B|)^{-1}(b^R-a^R)\end{aligned}.$$

That proves Eq. (4.1). Because that $(A-B)^{-1}$ and $(|A|-|B|)^{-1}$ are both real matrices, we have

$$\frac{d}{d\alpha}x^C(\alpha)=(A-B)^{-1}\frac{d}{d\alpha}(b^C(\alpha)-a^C(\alpha))$$

and

$$\frac{d}{d\alpha}x^R(\alpha)=(|A|-|B|)^{-1}\frac{d}{d\alpha}(b^R(\alpha)-a^R(\alpha)).$$

Proposition 4.1 implies that the extended solution $x=(x^C,x^R)$ is fuzzy if and only if Eqs. (4.2) and (4.3) hold.

Proposition 4.4. Let $A,B\in\mathbb{R}^{m\times n}$, and $a,b\in\mathbb{F}^m$ be triangular (rectangle, trapezoidal). Suppose that $A-B$ and $|A|-|B|$ both have row rank n. Then the unique solution of FLSDE $Ax+a=Bx+b$ is triangular (rectangle, trapezoidal), if it exists.

Proof. Suppose that a and b are triangular fuzzy vectors. Obviously, a_i^C, a_i^R, b_i^C, b_i^R are all polynomials of degree less than or equal to 1. Because that $A-B$ and $|A|-|B|$ both have row rank n, there are two real $n\times m$ matrices P,Q such that

$$P(A-B)=I_n=Q(|A|-|B|).$$

Theorem 3.2 (Eqs. (3.1) and (3.2)) implies

$$x^C=P(b^C-a^C)$$

and

$$x^R=Q(b^R-a^R).$$

This shows that each component of x^C and x^R is a linear combination of a_i^C, a_i^R, b_i^C, b_i^R, so that it is a polynomial of degree less than or equal to 1. Now, it suffices to prove $x^R(1)=0$. Since a and b are triangular fuzzy vectors, we have

$$a_i^R(1)=b_i^R(1)=0\text{, for all }1\le i\le n.$$

That implies $b^R(1)-a^R(1)=0$. Hence,

$$x^R(1)=Q(b^R(1)-a^R(1))=Q\cdot 0=0.$$

This shows that the unique solution is triangular. By the same way, we may prove the other cases.

5. Algorithms and examples

In this section, we present an algorithm for solving FLSDE and some examples to illustrate the algorithm.

Algorithm 5.1. Let $A,B\in\mathbb{R}^{m\times n}$, and a, $b\in\mathbb{F}^m$.

Step 1. Represent fuzzy numbers a_i and $b_i, 1\le i\le m$, to CR-representations.

Step 2. Calculate $A-B$, $|A|-|B|$, b^C-a^C, and b^R-a^R.

Step 3. Apply Gauss-Jordan method to solve the following real linear systems:

$$(A-B)x^C=b^C-a^C,$$
$$(|A|-|B|)x^R=b^R-a^R.$$

Step 4. Determine whether the extended solutions are fuzzy or not.

Example 5.2. Consider the following FLSDE

$$\begin{pmatrix}2 & -1\\ 2 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}[1+2\alpha,5-2\alpha]\\ [-6+3\alpha,2-5\alpha]\end{pmatrix}=\begin{pmatrix}1 & -2\\ -1 & -3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}[-5+3\alpha,3-\alpha]\\ [1+2\alpha,5-2\alpha]\end{pmatrix}.$$

Step 1. By Eqs. (2.3) and (2.4), represent a and b to CR-representation vectors, see as follows:

$$a=\begin{pmatrix}[1+2\alpha,5-2\alpha]\\ [-6+3\alpha,2-5\alpha]\end{pmatrix}=\begin{pmatrix}(3,2-2\alpha)\\ (-2-\alpha,4-4\alpha)\end{pmatrix},$$

$$b=\begin{pmatrix}[-5+3\alpha,3-\alpha]\\ [1+2\alpha,5-2\alpha]\end{pmatrix}=\begin{pmatrix}(-1+\alpha,4-2\alpha)\\ (3,2-2\alpha)\end{pmatrix}.$$

Step 2. Let's Compute the following objectives:

$$A-B=\begin{pmatrix}2 & -1\\ 2 & 1\end{pmatrix}-\begin{pmatrix}1 & -2\\ -1 & -3\end{pmatrix}=\begin{pmatrix}1 & 1\\ 3 & 4\end{pmatrix},$$

$$|A|-|B|=\begin{pmatrix}2 & 1\\ 2 & 1\end{pmatrix}-\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}=\begin{pmatrix}1 & -1\\ 1 & -2\end{pmatrix},$$

$$b^C-a^C=\begin{pmatrix}-1+\alpha\\ 3\end{pmatrix}-\begin{pmatrix}3\\ -2-\alpha\end{pmatrix}=\begin{pmatrix}-4+\alpha\\ 5+\alpha\end{pmatrix},$$

$$b^R-a^R=\begin{pmatrix}4-2\alpha\\ 2-2\alpha\end{pmatrix}-\begin{pmatrix}2-2\alpha\\ 4-4\alpha\end{pmatrix}=\begin{pmatrix}2\\ -2+2\alpha\end{pmatrix}.$$

Step 3. It is easily seen that, $A-B$ and $|A|-|B|$ are both invertible. Moreover,

$$(A-B)^{-1}=\begin{pmatrix}4 & -1\\ -3 & 1\end{pmatrix}$$

and

$$(|A|-|B|)^{-1}=\begin{pmatrix}2 & -1\\ 1 & -1\end{pmatrix}.$$

Thus,

$$\begin{pmatrix}x_1^C\\ x_2^C\end{pmatrix}=(A-B)^{-1}(b^C-a^C)=\begin{pmatrix}4 & -1\\ -3 & 1\end{pmatrix}\begin{pmatrix}-4+\alpha\\ 5+\alpha\end{pmatrix}=\begin{pmatrix}-21+3\alpha\\ 17-2\alpha\end{pmatrix}$$

and

$$\begin{pmatrix}x_1^R\\ x_2^R\end{pmatrix}=(|A|-|B|)^{-1}(b^R-a^R)=\begin{pmatrix}2 & -1\\ 1 & -1\end{pmatrix}\begin{pmatrix}2\\ -2+2\alpha\end{pmatrix}=\begin{pmatrix}6-2\alpha\\ 4-2\alpha\end{pmatrix}.$$

Step 4. By Step 3, we obtain the CR-representation solutions

$$x_1=(-21+3\alpha,6-2\alpha)\text{ and }x_2=(17-2\alpha,4-2\alpha).$$

Consequently, Eqs. (2.3) and (2.4) together imply

$$x_1=[-27+5\alpha,-15+\alpha]\text{ and }x_2=[13,21-4\alpha].$$

Notice that, x_1 is not a fuzzy number, since

$$x_1^U(\alpha)=-15+\alpha$$

is not decreasing. This shows that this FLSDE has no fuzzy solution.

Example 5.3. Let's determine all trapezoidal solutions of the following fuzzy linear system

$$\begin{cases}x_1+x_2=[-5+3\alpha,7-\alpha]\\ x_1-x_2=[-5+\alpha,7-3\alpha]\end{cases}.$$

Step 1. By applying Eqs. (2.3) and (2.4), we will obtain the following equalities:

$$[-5+3\alpha, 7-\alpha] = (1+\alpha, 6-2\alpha)$$

and

$$[-5+\alpha, 7-3\alpha] = (1-\alpha, 6-2\alpha) .$$

Step 2. Observe that

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, |A| = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, b^C = \begin{pmatrix} 1+\alpha \\ 1-\alpha \end{pmatrix}, b^R = \begin{pmatrix} 6-2\alpha \\ 6-2\alpha \end{pmatrix}.$$

Step 3. By Theorem 3.2, it suffices to solve the following system

$$\begin{cases} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1^C \\ x_2^C \end{pmatrix} = \begin{pmatrix} 1+\alpha \\ 1-\alpha \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1^R \\ x_2^R \end{pmatrix} = \begin{pmatrix} 6-2\alpha \\ 6-2\alpha \end{pmatrix} \end{cases} .$$

It easily verifies that

$$\begin{pmatrix} x_1^C \\ x_2^C \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1+\alpha \\ 1-\alpha \end{pmatrix} = \begin{pmatrix} 1 \\ \alpha \end{pmatrix}$$

and

$$x_1^R + x_2^R = 6-2\alpha .$$

Step 4. Since x_1 and x_2 are assumed to be trapezoidal, x_1^R and x_2^R are both polynomials of degree less than or equal to 1. Let

$$x_1^R = s + t\alpha , \text{ where } s, t \in \mathbb{R} .$$

By Step 3, we obtain

$$\begin{cases} x_1 = (x_1^C, x_1^R) = (1, s+t\alpha) \\ x_2 = (x_2^C, x_2^R) = (\alpha, 6-s-(t+2)\alpha) \end{cases} .$$

Corollary 4.2 implies that x_1 and x_2 are both fuzzy numbers if and only if

$$0 \le -t \le s \text{ and } 1 \le t+2 \le 6-s .$$

They are equivalent to

$$-1 \le t \le 0 \text{ and } 0 \le s+t \le 4 . \tag{5.1}$$

Hence, we obtain all trapezoidal solutions:

$$\begin{cases} x_1 = [1-s-t\alpha, 1+s+t\alpha] \\ x_2 = [s-6+(t+3)\alpha, 6-s-(t+1)\alpha] \end{cases},$$

where s and t satisfy Eq. (5.1).

6. Conclusions

In this paper, we consider the fuzzy linear system of dual equations $Ax + a = Bx + b$ with real coefficients matrices A and B. By using the CR-representation of fuzzy numbers, the original system can be replaced with two real linear systems. Then, apply Gauss-Jordan method to solve the two systems. The solutions may be inexistent, extended but not fuzzy, or fuzzy. We also provide a sufficient and necessary condition for discriminating whether the extended solutions are fuzzy or not. Furthermore, if the system is uniquely solvable and a, b are both triangular (trapezoidal, rectangle), then the unique solution is triangular (trapezoidal, rectangle), too.

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8. References

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