Remarks on the Subtraction and Division Operations over Intuitionistic Fuzzy Sets and Interval-Valued Fuzzy Sets

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Abstract

Atanassov (Fuzzy Sets and Systems 20 (1986) 87-96) defined basic operations over the intuitionistic fuzzy sets. Here we shall introduce two operations, subtraction and division, derived from the deconvolution for equations using addition and multiplication operations, respectively. In addition, we shall give some conditions to examine the correctness regarding the defined operations. The remarks in this paper can be immediately expressed in terms of interval-valued fuzzy sets.

Keywords: Operation, intuitionistic fuzzy set, deconvolution, inteval-valued fuzzy sets.

1. Introduction

Intuitionistic fuzzy set (IFS) theory [1] is an extension of ordinary fuzzy set theory. IFSs assign to each element of the universe not only a membership degree but also a non-membership degree, and furthermore the sum of these two degrees is less than or equal to 1. Atanassov [2], [3] defined relations, operations, and operators over IFSs and studied their basic properties; however, two basic operations, subtraction and division, have not been discussed in the past literature [5], [7], [10], [14] involving IFS operations.

Let A and B denote two IFSs of the universe of discourse X, where $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle | x\in X\}$ and $B=\{\langle x,\mu_B(x),\nu_B(x)\rangle | x\in X\}$. Burillo and Bustince [6] defined the following expressions:

- $A\le B$ if and only if $\mu_A(x)\le\mu_B(x)$ and $\nu_A(x)\ge\nu_B(x)$ for all $x\in X$.
- $A\preceq B$ if and only if $\mu_A(x)\le\mu_B(x)$ and $\nu_A(x)\le\nu_B(x)$ for all $x\in X$.

In addition, $A\ge B$ if and only if $B\le A$; $A\succeq B$ if and only if $B\preceq A$.

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Atanassov [1-3], [5] defined addition and multiplication operations as follows:

$$A+B=\{\langle x,\mu_A(x)+\mu_B(x)-\mu_A(x)\cdot\mu_B(x),\nu_A(x)\cdot\nu_B(x)\rangle | x\in X\},\tag{1}$$

$$A\cdot B=\{\langle x,\mu_A(x)\cdot\mu_B(x),\nu_A(x)+\nu_B(x)-\nu_A(x)\cdot\nu_B(x)\rangle | x\in X\}.\tag{2}$$

It is easy to demonstrate the correctness of the above-mentioned operations. First, consider four non-negative real numbers $\mu_A(x)$, $\nu_A(x)$, $\mu_B(x)$, and $\nu_B(x)$, for which $0\le\mu_A(x)+\nu_A(x)\le 1$ and $0\le\mu_B(x)+\nu_B(x)\le 1$. Since $1-\mu_A(x)\ge 0$ and $1-\mu_B(x)\ge 0$,

$$\mu_A(x)+\mu_B(x)-\mu_A(x)\cdot\mu_B(x) =1-(1-\mu_A(x))\cdot(1-\mu_B(x))\le 1.$$

In addition,

$$\mu_A(x)+\mu_B(x)-\mu_A(x)\cdot\mu_B(x) =\mu_A(x)(1-\mu_B(x))+\mu_B(x)\ge\mu_B(x)\ge 0.$$

It follows that

$$\begin{aligned}0&\le\nu_A(x)\cdot\nu_B(x)\\ &\le\nu_A(x)\cdot\nu_B(x)+\mu_A(x)(1-\mu_B(x))+\mu_B(x)\\ &\le(1-\mu_A(x))\cdot(1-\mu_B(x))+\mu_A(x)(1-\mu_B(x))+\mu_B(x)\\ &=1-\mu_A(x)-\mu_B(x)+\mu_A(x)\cdot\mu_B(x)\\ &\quad+\mu_A(x)-\mu_A(x)\cdot\mu_B(x)+\mu_B(x)=1.\end{aligned}$$

From these inequalities, we know that $A+B$ is an IFS. Regarding the multiplication operation, it is similar to verify the correctness.

However, the subtraction and division operations cannot be directly derived by addition and multiplication operations. Let $\pi_A(x)$ denote the intuitionistic index and $\pi_A(x)=1-\mu_A(x)-\nu_A(x)$. It is obvious that for every $x\in X$, $0\le\pi_A(x)\le 1$. If we define subtraction and division through addition and multiplication, it is possible that the intuitionistic index $\pi_A(x)$ is negative or an infeasible solution is obtained. For example, let $A=\{\langle x,0.7,0.2\rangle\}$ and $B=\{\langle x,0.4,0.3\rangle\}$. For the sake of satisfying $(A-B)+B=A$, $A-B=\{\langle x,0.5,0.67\rangle\}$, but $\pi_{A-B}(x)=-0.17$. This violates the condition of the intuitionistic index. We show another example by $A=\{\langle x,0.2,0.4\rangle\}$ and $B=\{\langle x,0.5,0.8\rangle\}$. It follows that $A-B=\{\langle x,-0.6,0.5\rangle\}$, and this is not permitted by the definition of an IFS. Thus, we intend to discuss the correctness of two operations, subtraction and division, derived from addition and multiplication, respectively.

Through the deconvolution for equations using addition and multiplication operations, we shall introduce two arithmetic operations over IFSs: subtraction ($A - B$) and division (A / B).

2. Subtraction Operation in IFSs

Let us consider an equation of the type:

$$C + B = A, \tag{3}$$

where the IFSs A and B are given, and the problem is to find the unknown IFS C which satisfies $0 \le \mu_C(x) + \nu_C(x) \le 1$. Using (1), we know that

$\mu_C(x) + \mu_B(x) - \mu_C(x) \cdot \mu_B(x) = \mu_A(x)$,

$\nu_C(x) \cdot \nu_B(x) = \nu_A(x)$.

Then,

$$\mu_C(x) = \frac{\mu_A(x) - \mu_B(x)}{1 - \mu_B(x)}, \tag{4}$$

$$\nu_C(x) = \frac{\nu_A(x)}{\nu_B(x)}. \tag{5}$$

Unfortunately, C with (4) and (5) may not be an IFS.

The membership degree of C must take values in the interval [0,1], i.e.,

$$0 \le \frac{\mu_A(x) - \mu_B(x)}{1 - \mu_B(x)} \le 1. \tag{6}$$

The right-hand side of (6) is valid because $\mu_A(x) \le 1$, but the left-hand side of (6) is not correct in the cases that $\mu_A(x) < \mu_B(x)$ or $\mu_B(x) = 1$. To fulfill the inequality in (6), the conditions that $\mu_A(x) \ge \mu_B(x)$ and $\mu_B(x) \ne 1$ are required. Similarly,

$$0 \le \frac{\nu_A(x)}{\nu_B(x)} \le 1. \tag{7}$$

It is obvious that the left-hand side of (7) is correct because $\nu_A(x), \nu_B(x) \ge 0$ but $\nu_B(x) \ne 0$. The satisfied condition of the right-hand side of (7) is that $\nu_A(x) \le \nu_B(x)$. Combining $\mu_A(x) \ge \mu_B(x)$ and $\nu_A(x) \le \nu_B(x)$, we know that $A \ge B$. Therefore, the inequalities (6) and (7) hold only if $A \ge B$, $\mu_B(x) \ne 1$ and $\nu_B(x) \ne 0$.

Moreover, C is an IFS and thus

$$\mu_C(x) + \nu_C(x) = \frac{\mu_A(x) - \mu_B(x)}{1 - \mu_B(x)} + \frac{\nu_A(x)}{\nu_B(x)} \le 1. \tag{8}$$

Then,

$\mu_A(x) \cdot \nu_B(x) - \mu_B(x) \cdot \nu_A(x) \le \nu_B(x) - \nu_A(x)$.

From the discussion of the above possible cases, we can now summarize the conditions for obtaining a solution of (3): $A \ge B$, $\mu_A(x) \cdot \nu_B(x) - \mu_B(x) \cdot \nu_A(x) \le \nu_B(x) - \nu_A(x)$, $\mu_B(x) \ne 1$ and $\nu_B(x) \ne 0$.

Example 1. Consider a degenerated universe of discourse $X=\{x\}$. Let A, B, and C be the three IFSs of X given by

$A = \{\langle x, 0.3, 0.2 \rangle\}$, $B = \{\langle x, 0.1, 0.5 \rangle\}$,

and $C = \{\langle x, 0.6, 0.3 \rangle\}$.

Then we see that

$$A - B = \left\{\left\langle x, \frac{0.3 - 0.1}{1 - 0.1}, \frac{0.2}{0.5} \right\rangle\right\} = \{\langle x, 0.22, 0.40 \rangle\}.$$

There is no solution of $A - C$ since $A \precsim C$.

There is no solution of $B - C$ since $B \le C$.

There is no solution of $C - A$ since $C \succsim A$.

There is no solution of $B - A$ since $B \le A$.

There is no solution of $C - B$ since $(0.6 \times 0.5 - 0.1 \times 0.3) > (0.5 - 0.3)$.

3. Division Operation in IFSs

Next we will introduce the division operation. Consider an equation of the following type:

$$D \cdot B = A. \tag{9}$$

If the IFSs A and B are given, then we can find the unknown IFS D which satisfies $0 \le \mu_D(x) + \nu_D(x) \le 1$. Using (2), we know that

$\mu_D(x) \cdot \mu_B(x) = \mu_A(x)$,

$\nu_D(x) + \nu_B(x) - \nu_D(x) \cdot \nu_B(x) = \nu_A(x)$.

Then

$$\mu_D(x) = \frac{\mu_A(x)}{\mu_B(x)}, \tag{10}$$

$$\nu_D(x) = \frac{\nu_A(x) - \nu_B(x)}{1 - \nu_B(x)}. \tag{11}$$

However, D with (10) and (11) may not be an IFS. Similar to (6)-(8), a solution for IFS D exists only if $0 \le \mu_A(x) / \mu_B(x) \le 1$, $0 \le (\nu_A(x) - \nu_B(x)) / (1 - \nu_B(x)) \le 1$, and $\mu_A(x) / \mu_B(x) + (\nu_A(x) - \nu_B(x)) / (1 - \nu_B(x)) \le 1$. From the first inequality, we obtain the conditions that $\mu_A(x) \le \mu_B(x)$ and $\mu_B(x) \ne 0$. The conditions derived from the second inequality include that $\nu_A(x) \ge \nu_B(x)$ and $\nu_B(x) \ne 1$. Finally, the condition according to the last inequality is that $\mu_A(x) \cdot \nu_B(x) - \mu_B(x) \cdot \nu_A(x) \ge \mu_A(x) - \mu_B(x)$.

Therefore, we summarize the conditions for obtaining an IFS solution of (9): $A \le B$, $\mu_A(x) \cdot \nu_B(x) - \mu_A(x) \cdot \nu_B(x) - \mu_B(x) \cdot \nu_A(x) \ge \mu_A(x) - \mu_B(x)$, $\mu_B(x) \ne 0$

and $\nu_B(x) \neq 1$.

Example 2. Let us consider following IFSs *A*, *B*, $C \in X=\{x\}$:
$A = \{\langle x, 0.2, 0.4\rangle\}$, $B = \{\langle x, 0.6, 0.3\rangle\}$,
and $C = \{\langle x, 0.2, 0.2\rangle\}$.
According to the above discussions regarding the division operation, we know that
$$A/B = \left\{\left\langle x, \frac{0.2}{0.6}, \frac{0.4-0.3}{1-0.3}\right\rangle\right\} = \{\langle x, 0.33, 0.14\rangle\}.$$
We cannot give the value of A/C because $(0.2\times 0.2 - 0.2\times 0.4) < (0.2 - 0.2)$.
We cannot give the value of B/C because $B \succsim C$.
We cannot give the value of C/A because $C \geq A$.
We cannot give the value of B/A because $B \geq A$.
We cannot give the value of C/B because $C \precsim B$.

4. Subtraction and Division in IVFSs

What has to be noticed is the applicability of the aforementioned discussions in interval-valued fuzzy sets (IVFSs). IVFS is defined by an interval-valued membership function [12], [13]; that is, the degree of membership of an element to a set is characterized by a closed subinterval of [0,1]. Let Int([0,1]) stand for the set of all closed subintervals of [0,1]. An IVFS *A* on *X* is given by:
$$A = \{\langle x, M_A(x)\rangle \mid x \in X\},$$
where M_A : $X \to$ Int([0,1]), such that $x \to M_A(x) = [M_{AL}(x), M_{AU}(x)]$. $M_{AL}(x)$ and $M_{AU}(x)$ are the lower bound and the upper bound, respectively, of the interval $M_A(x)$. In addition, let $W_A(x)$ be the width of the interval $M_A(x)$, and $W_A(x) = M_{AU}(x) - M_{AL}(x)$.

Burillo and Bustince [6] defined the following expressions for all *A*, $B \in$ IVFSs(*X*):

- $A \leq B$ if and only if $M_{AL}(x) \leq M_{BL}(x)$ and $M_{AU}(x) \leq M_{BU}(x)$ for all $x \in X$.
- $A \precsim B$ if and only if $M_{AL}(x) \leq M_{BL}(x)$ and $M_{AU}(x) \geq M_{BU}(x)$ for all $x \in X$.

Atanassov and Gargov [4] showed that IFS and IVFS are equipollent generalizations of ordinary fuzzy sets. Some researches mentioned that IFS theory is mathematically equivalent to IVFS theory [8], [9]. An interval $[M_{AL}(x), M_{AU}(x)]$ can be mapped bijectively onto a couple $(\mu_A(x), \nu_A(x))$ [11]. Thus, $M_{AL}(x) = \mu_A(x)$ and $M_{AU}(x) = 1 - \nu_A(x)$. The discussions concerning subtraction and division operations for IFSs can be extended to IVFSs. A solution of $A - B$ exists if the following conditions are satisfied, including $A \geq B$, $M_{BL}(x), M_{BU}(x) \neq 1$, and $M_{AU}(x)\cdot M_{BL}(x) - M_{AL}(x)\cdot M_{BU}(x) \leq W_A(x) - W_B(x)$. Here we have
$$A - B = \{\langle x, M_{A-B}(x)\rangle \mid x \in X\},$$
where
$$M_{A-B}(x) = \left[\frac{M_{AL}(x) - M_{BL}(x)}{1 - M_{BL}(x)}, \frac{M_{AU}(x) - M_{BU}(x)}{1 - M_{BU}(x)}\right].$$

If the conditions that $A \leq B$, $M_{AL}(x), M_{AU}(x) \neq 0$, and $M_{BL}(x)\cdot M_{BU}(x) \geq M_{AL}(x)\cdot M_{AU}(x)$ are fulfilled, we have
$$A/B = \{\langle x, M_{A/B}(x)\rangle \mid x \in X\},$$
where $M_{A/B}(x) = \left[\frac{M_{AL}(x)}{M_{BL}(x)}, \frac{M_{AU}(x)}{M_{BU}(x)}\right]$.

5. Conclusions

In the present paper, we have presented a simple method for the solution of equations of the type $C + B = A$, and then for equations of the type $D \cdot B = A$ for given *A* and *B*, and unknown *C* and *D*. To know if a valid solution exists, we have suggested several conditions to examine the existence of the solution. From the deconvolution results with corresponding conditions, we define subtraction and division operations over IFSs and IVFSs. Finally, it is important to recall that the subtraction and division operations over IFSs and IVFSs are not applicable except in some particular cases.

We summarize the results from deconvolution solutions in Tables 1 and 2.

Table 1. Subtraction operations over IFSs and IVFSs derived from the deconvolution solution $C + B = A$.

$A - B$	Subtraction Operation
IFS notation	$\left\{\left\langle x, \frac{\mu_A(x) - \mu_B(x)}{1-\mu_B(x)}, \frac{\nu_A(x)}{\nu_B(x)}\right\rangle \middle| x \in X\right\}$
	Conditions:
	$A \geq B$, $\mu_B(x) \neq 1$, $\nu_B(x) \neq 0$, and $\mu_A(x)\cdot\nu_B(x) - \mu_B(x)\cdot\nu_A(x) \leq \nu_B(x) - \nu_A(x)$
IVFS notation	$\left\{\left\langle x, \frac{M_{AL}(x) - M_{BL}(x)}{1-M_{BL}(x)}, \frac{1 - M_{AU}(x)}{1-M_{BU}(x)}\right\rangle \middle| x \in X\right\}$
	Conditions:

$A \geq B$, $M_{BL}(x), M_{BU}(x) \neq 1$, and
$M_{AU}(x) \cdot M_{BL}(x) - M_{AL}(x) \cdot M_{BU}(x) \leq W_A(x) - W_B(x)$

Table 2. Division operations over IFSs and IVFSs derived from the deconvolution solution $D \cdot B = A$.

A/B	Division Operation
IFS notation	$\left\{\left\langle x, \dfrac{\mu_A(x)}{\mu_B(x)}, \dfrac{\nu_A(x)-\nu_B(x)}{1-\nu_B(x)} \right\rangle \middle| x \in X \right\}$ Conditions: $A \leq B$, $\mu_B(x) \neq 0$, $\nu_B(x) \neq 1$, and $\mu_A(x) \cdot \nu_B(x) - \mu_B(x) \cdot \nu_A(x) \geq \mu_A(x) - \mu_B(x)$
IVFS notation	$\left\{\left\langle x, \dfrac{M_{AL}(x)}{M_{BL}(x)}, \dfrac{M_{BU}(x)-M_{AU}(x)}{M_{BU}(x)} \right\rangle \middle| x \in X \right\}$ Conditions: $A \leq B$, $M_{AL}(x), M_{AU}(x) \neq 0$, and $M_{BL}(x) \cdot M_{BU}(x) \geq M_{AL}(x) \cdot M_{AU}(x)$

These basic operations, including addition, subtraction, multiplication, and division, can be applied on a lattice of both IFS theory and IVFS theory. In addition, the four operations can be used to extend several important concepts. For example, using these standard operations can produce additive and multiplicative generators required in triangular norms over IFSs or IVFSs. Moreover, we can determine the cardinality of IFSs or IVFSs that can be utilized on fuzzy quantifiers in natural languages. On the other hand, similarity and inclusion measures can be defined by these operations, and further be used to propound the concepts of fuzzy entropy and conditioning. Finally, it should also be added that the basic operations can be taken to advance vague quantifiers which can be applied on antonyms and linguistic quantifiers in intuitionistic or interval-valued fuzzy logic.

6. References

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