On the Resolution of the System of Fuzzy Integer Inequalities

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Abstract

This work considers the resolution of the system of fuzzy integer inequalities. It is shown that a system of fuzzy integer inequalities with concave membership functions can be reduced to a regular convex integer programming problem. A modified solution algorithm of the p-th power Lagrangian method is introduced to deal with the resulting convex integer programming problem as a sequence of linearly constrained convex integer programming problems. Some computational results are included.

Keywords: Convex integer programming, fuzzy integer programming, membership functions, systems of fuzzy integer inequalities.

1. Introduction

Many decision making problems in real applications naturally result in optimization formulations in a form of nonlinear integer programming. However, facing uncertainty is a constant challenge for optimal decision making. Treating uncertainty by fuzzy mathematics results in the study of fuzzy optimization and decision making. Recently, Herrera and Verdegay [6] introduced three models of fuzzy integer programming, Wang and Liao [20] considered a differentiable nonlinear integer program with the fuzzy inequality constraint. From optimization theory, we know that solving a mathematical programming problem essentially can be reduced to solving a system of inequalities [14], [16], [17]. In this study we extend the idea and consider a fuzzy integer programming problem in view of the following system of fuzzy integer inequalities.

$$
\begin{aligned}
&f_i(x) \underset{\sim}{\le} 0,\ i = 1,2,\cdots,m,\\
&g_j(x) \le 0,\ j = 1,2,\cdots,l, \qquad (1)\\
&x \text{ is integer,}
\end{aligned}
$$

where $g_j(x) \le 0,\ j = 1,2,\cdots,l,$ are regular inequalities, $f_i(x) \underset{\sim}{\le} 0,\ i = 1,2,...,m,$ are fuzzy inequalities and "$\underset{\sim}{\le}$" denotes the fuzzified version of $\le$ with the linguistic interpretation "approximately less than or equal to". Each fuzzy inequality $f_i(x) \underset{\sim}{\le} 0$ actually determines a fuzzy set $\tilde{C}_i$, whose membership function is denoted by μ_{f_i}. The membership grade $\mu_{f_i}(x)$ can be interpreted as the degree to which the regular inequality $f_i(x) \le 0, i = 1,2,\cdots,m,$ is satisfied. To specify the membership functions μ_{f_i}, it is commonly assumed that $\mu_{f_i}(x)$ should be 0 if the regular linear inequality $f_i(x) \le 0$ is strongly violated, and 1 if it is satisfied. This leads to a membership function in the following form:

$$
\mu_{f_i}(x) = \begin{cases} 1, & \text{if } f_i(x) \le 0,\\ \mu_i(f_i(x)), & \text{if } 0 < f_i(x) \le t_i,\\ 0, & \text{if } f_i(x) > t_i, \end{cases} \qquad (2)
$$

for $i = 1,2,\cdots,m,$ where $t_i \ge 0$ is the tolerance level which a decision maker can tolerate in the accomplishment of the fuzzy inequality $f_i(x) \underset{\sim}{\le} 0$. We usually assume that $\mu_i(f_i(x)) \in 0,1$ and it is continuous and strictly decreasing over $[0, t_i]$. Figure 1 shows different shapes of such membership functions.

To solve the problem (1), we follow the "tolerance approach" and consider the following model [11] ,[21].

$$
\begin{aligned}
&\max_{x \in \mathrm{R}^n} \min_{i=1,2,\cdots,m} \{\mu_{f_i}(x)\}\\
&\text{s.t.} \qquad g_j(x) \le 0, \quad j = 1,2,\cdots,l, \qquad (3)\\
&\qquad\qquad x \text{ is integer.}
\end{aligned}
$$

Introducing one new variable α results in an equivalen problem.

$$
\begin{aligned}
&\max \quad \alpha\\
&\text{s.t.} \quad \mu_{f_i}(x) \ge \alpha, \quad i = 1,2,\cdots,m,\\
&\qquad g_j(x) \le 0, \quad j = 1,2,\cdots,l, \qquad (4)\\
&\qquad 0 \le \alpha \le 1, \quad x \text{ is integer.}
\end{aligned}
$$

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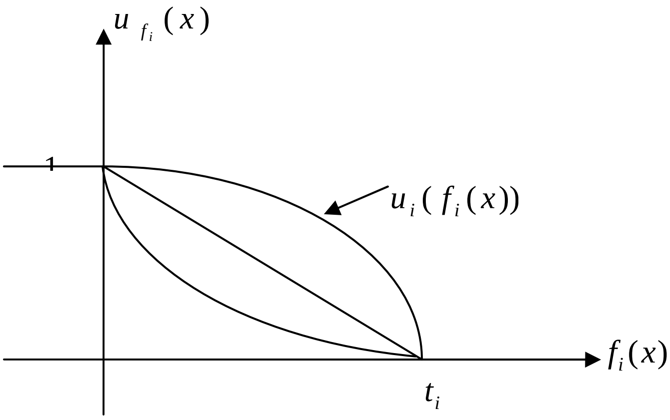

Figure 1. The membership function $\mu_{f_i}(x)$ of the fuzzy inequality $f_i(x)\underset{\sim}{\leq}0$.

Moreover, when μ_{f_i}, $i=1,2,\cdots,m$, are invertible, we have

$$\begin{aligned} &\max \quad \alpha \\ &\text{s.t.} \quad f_i(x)\leq \mu_{f_i}^{-1}(\alpha), \quad i=1,2,\cdots,m, \\ &\qquad g_j(x)\leq 0, \qquad j=1,2,\cdots,l, \qquad (5) \\ &\qquad 0\leq\alpha\leq 1, \qquad x \text{ is integer,} \end{aligned}$$

where $\mu_{f_i}^{-1}$ is the inverse function of μ_{f_i}, $i=1,2,\cdots,m.$ From the above procedure, we see that a system of fuzzy integer inequalities (1) can eventually be reduced to a nonlinear integer programming problem (5). Depending on the type of membership functions chosen and the properties of $f_i(x), i=1,2,\cdots,m,$ and $g_j(x),\ j=1,2,\cdots,l,$ the resulting optimization problem turns out to be either a linear or a nonlinear integer program. Recently, Hannan [5] solved the system of fuzzy linear inequalities with all membership functions being concave piecewise linear in the range of $[0,1]$, and Inuiguchi et al. [7] dealt with the fuzzy linear programming problems whose membership functions are strictly quasiconcave. Inspired and motivated by the recent research, this work focuses on developing solution methods for solving a system of fuzzy integer inequalities with concave membership functions.

2. System of Fuzzy Integer Inequalities with Concave Membership Functions

Consider the case that the membership function of each fuzzy inequality $f_i(x)\underset{\sim}{\leq}0$ in (1) is continuous, strictly decreasing, and concave over the tolerance interval $[0,t_i]$, for $i=1,2,\cdots,m.$ A commonly used example in fuzzy set theory is that $f(x)=1-ax^k$, with $k>1$. In this case, we have the following simple result.

Lemma 1. If $h(x)$ is continuous, strictly decreasing and concave over a convex set Ω in R^n, then its inverse $h^{-1}(y)$ is concave.

proof: Because $h(x)$ is continuous and strictly decreasing over Ω, its inverse function is well-defined. When $h(x)$ is concave over the convex set Ω, $\forall\ x^1,x^2\in\Omega$ and $0\leq\theta\leq 1$, we have

$$h(\theta x^1+(1-\theta)x^2)\geq\theta h(x^1)+(1-\theta)h(x^2),$$

or, equivalently,

$$h(\theta x^1+(1-\theta)x^2)\geq h\{h^{-1}[\theta h(x^1)+(1-\theta)h(x^2)]\}.$$

Since $h(x)$ is strictly decreasing, we further have

$$\theta x^1+(1-\theta)x^2\leq h^{-1}[\theta h(x^1)+(1-\theta)h(x^2)].$$

Consider $x^1=h^{-1}(y^1)$ and $x^2=h^{-1}(y^2)$, we see that

$$\theta h^{-1}(y^1)+(1-\theta)h^{-1}(y^2)\leq h^{-1}(\theta y^1+(1-\theta)y^2).$$

Hence $h^{-1}(y)$ is concave. q.e.d.

Lemma 1 and Problem (5) directly lead to the following result.

Theorem 1. For the system of fuzzy inequalities (1), if $f_i(x), i=1,2,\cdots,m,\ \ g_j(x),\ j=1,2,\cdots,l,$ are convex, and $\mu_{f_i}(x), i=1,2,\cdots,m,$ are continuous, strictly decreasing, and concave, then we can find a solution to (1) by solving the following convex integer programming problem.

$$\begin{aligned} &\max \qquad \alpha \\ &s.t. \quad f_i(x)-\mu_{f_i}^{-1}(\alpha)\leq 0, \quad i=1,2,\cdots,m, \\ &\qquad g_j(x)\leq 0, \qquad j=1,2,\cdots,l, \\ &\qquad 0\leq\alpha\leq 1, \\ &\qquad x \text{ is integer.} \end{aligned}$$

or equivalently,
(P)

$$-\min \quad F(x,\alpha)\overset{\Delta}{=}-\alpha \qquad (6a)$$

$$\text{s.t.} \quad f_i(x)-\mu_{f_i}^{-1}(\alpha)\leq 0,\ \ i=1,2,\cdots,m, \qquad (6b)$$

$$g_j(x)\leq 0,\ \ j=1,2,\cdots,l, \qquad (6c)$$

$$0\leq\alpha\leq 1, \qquad (6d)$$

$$x \text{ is integer.} \tag{6e}$$

For the numerical treatment of the convex integer programming problems, various methods have been proposed in the literature [1], [10], [19]. Computational difficulty, however, may be caused by the nonlinear constraints (6b) and (6c) when outer approximation algorithm [2] or branch and bound method [4] is adopted to solve the problem (P). In this work, a modified version of the p -th power Lagrangian method, which provides an approach to reduce the problem (P) to a sequence of linearly constrained convex integer programming problems, will be presented in the next section.

3. Solution Procedure

It is well known that the Lagrangian method is a powerful dual search approach in integer programming. Define F to be the feasible region of the decision vector (x,α) in (P),

$$F \overset{\Delta}{=} \{(x,\alpha) \mid f_i(x) - \mu_{f_i}^{-1}(\alpha) \le 0, i = 1,2,\cdots,m,$$

$$g_j(x) \le 0,\ j = 1,2,\cdots,l, 0 \le \alpha \le 1, \text{and x is integer.}\}$$

Constraints in (6b) and (6c) are called Lagrangian constraints. Denote the optimal value of the primal problem (P) is $v(P)$. In most situations, the Lagrangian methods provide a lower bound for $v(P)$. Moreover, incorporating the set of Lagrangi- an constraints into the objective function by introducing a nonnegative Lagrangian multiplier vector, $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{m+l}) \in R_+^{m+l}$, yields a Lagrangian relaxation.

$$(PR_\lambda) \qquad \min_{(x,\alpha)\in X} L((x,\alpha),\lambda) = F(x,\alpha) +$$

$$\sum_{i=1}^{m} \lambda_i [f_i(x) - \mu_{f_i}^{-1}(\alpha)] + \sum_{j=1}^{l} \lambda_{j+m}[g_j(x)],$$

where $X \overset{\Delta}{=} \{0 \le \alpha \le 1, \text{and x is integer.}\}$. The Lagrangian dual is an optimization problem (D) in λ,

$$(D) \qquad \max_{\lambda \in R_+^{m+l}} [v(PR_\lambda)],$$

where $v(PR_\lambda)$ is the optimal value of the problem (PR_λ). The following is well known,

$$v(PR_\lambda) \le v(P), \quad \forall\, \lambda \in R_+^{m+l}.$$

The Lagrangian method searches for an optimal solution of (P) via maximizing the dual function $v(PR_\lambda)$.

If $(\hat{x},\hat{\alpha})$ solves both (P) and $(PR_{\hat{\lambda}})$ with $\hat{\lambda} \in R_+^{m+l}$, then $\hat{\lambda}$ is said to be an optimal generating Lagrangian multiplier vector. If $(\hat{x},\hat{\alpha})$ solves both (P) and $(PR_{\hat{\lambda}})$ with $\hat{\lambda} \in R_+^{m+l}$, and $\hat{\lambda}$ solves the dual problem (D), then $\{(\hat{x},\hat{\alpha}),\hat{\lambda}\}$ is said to be an optimal primal-dual pair of (P).

While the Lagrangian method is a powerful constructive dual search method, it often fails to identify an optimal solution of the primal integer optimization problem. Two critical situations could be present that prevent the Lagrangian method from succeeding in the dual search. Firstly, the optimal solution of (P) may not even be generated by solving (PR_λ) for any $\lambda \ge 0$. Secondly, the optimal solution to (PR_{λ^*}), with λ^* being a solution to the dual problem (D), is not necessarily an optimal solution to (P), or even not feasible. The first situation mentioned above is associated with the existence of an optimal generating Lagrangian multiplier vector. The second situation is related to the existence of an optimal primal-dual pair.

A modified version of the p -th power Lagrangian method is introduced in this work to integrate two equivalent transformations that ensure the existence of an optimal primal-dual pair in an equivalent problem setting, thus offering a success guarantee for the dual search in generating an optimal solution of the primal integer programming problem. The detail discussion of the p -th power Lagrangian method is presented as follows. In Section 3.1, a $t-$norm surrogate constraint method is adopted to construct a single-constraint surrogate model that is exactly equivalent to the primal problem. In Section 3.2, a $p-$th power transformation is then applied to guarantee the existence of an optimal primal-dual pair, thus ensuring the success of the dual search. A modified solution algorithm of the $p-$th power Lagrangian method is described in Section 3.3.

3.1. The $t-$norm Surrogate Constraint Formulation

Considering the structure of the primal problem (P), the presence of many nonlinear inequality constraints in (P) causes difficulties in finding an optimal solution of (P). To handle these constraints, the use of the surrogate constraint formulation in integer programming was investigated in [3], [8], [9]. The surrogate constraint method converts a mathematical programming problem with multiple constraints into a one with a single aggregated constraint using a multiplier vector. The multiplier vector is successively adjusted such that a surrogare dual is maximized. The surrogate dual in general, however, does not guarantee the generation of an optimal solution of the primal problem. A surrogate strategy termed $p-$ norm surrogate constraint method was recently developed in [12] for general integer programming problems that yields an exact equivalence between the primal problem and the surrogated one . In this work a revised version of the $p-$norm surrogate formulation is introduced. Consider the primal problem (P) is equivalent to the following single-constraint problem.

$$\begin{aligned} -\min \quad & F(x,\alpha) \\ s.t. \quad & g_M(x,\alpha) \overset{\Delta}{=} \max\{f_1(x)-\mu_{f_1}^{-1}(\alpha),\cdots, \\ & f_m(x)-\mu_{f_m}^{-1}(\alpha), g_1(x),\cdots,g_l(x)\} \le 0, \\ & (x,\alpha)\in X. \end{aligned} \tag{7}$$

Let

$$g(x,\alpha) \overset{\Delta}{=} (f_1(x)-\mu_{f_1}^{-1}(\alpha),\cdots,f_m(x)-\mu_{f_m}^{-1}(\alpha),g_1(x),\cdots,g_l(x)).$$

Note that the nonsmooth function $g_M(x,\alpha)$ is exactly the infinite norm $\| g(x,\alpha)\|_\infty$, which can be approximated by the $t-$th norm

$$\| g(x,\alpha)\|_t = (\sum_{i=1}^{m}[f_i(x)-\mu_{f_i}^{-1}(\alpha)]^t + \sum_{j=1}^{l}[g_j(x)]^t)^{1/t}$$

as t tends to infinity. We further have

$$\frac{g_M(x,\alpha)}{(m+l)^{1/t}} \le \frac{\| g(x,\alpha)\|_t}{(m+l)^{1/t}} \le g_M(x,\alpha). \tag{8}$$

A t-norm surrogate constraint formulation of the problem (P) is then formed by replacing $g_M(x,\alpha)$ in (7) by

$$G_t(x,\alpha) \overset{\Delta}{=} \| g(x,\alpha)\|_t /(m+l)^{1/t} \quad \text{for} \quad t>0,$$

$$\begin{aligned} -\min \quad & F(x,\alpha) \\ s.t. \quad & G_t(x,\alpha)\le 0, \\ & (x,\alpha)\in X. \end{aligned} \tag{9}$$

Define F_t to be the feasible region of (9). It is clear from (8) that $F \subseteq F_t$ for any $t>0$. The single-constraint surrogate problem (9) is thus a relation of (P) when $t \ge 1$. Moreover, it is shown that the sets F_t and F will be identical if t is chosen sufficiently large [13]. In other words, the solution of the primal problem (P) can be obtained by solving the single-constraint surrogate problem (9) for a sufficiently large t.

3.2. The p -th Power Transformation

As mentioned above, the single-constraint surrogate problem (9) and the primal problem (P) are equivalent for a sufficiently large t. In this section a dual search scheme using the $p-$th power transformation for the problem (9) with a fixed $t>0$ will be discussed.

To describe this approach, we first impose a p-th power on the objective function of the problem (9). The problem (9) can then be represented by the following equivalent form with $p>0$.

$$\begin{aligned} -\min \quad & F_p(x,\alpha) \overset{\Delta}{=} [F(x,\alpha)]^p \\ s.t. \quad & G_t(x,\alpha)\le 0, \\ & (x,\alpha)\in X, \end{aligned}$$

or

$$\begin{aligned} -\min \quad & exp(-p\alpha) \\ s.t. \quad & \sum_{i=1}^{m} exp(t(f_i(x)-\mu_{f_i}^{-1}(\alpha))) \\ & +\sum_{j=1}^{l} exp(tg_j(x)) \le m+l \\ & (x,\alpha)\in X, \end{aligned} \tag{10}$$

where we have taken exponential transformations to the objective function and the constraints of the primal problem (P) for computational simplification. The Lagrangian relaxation of the problem (10) is given as follows with a Lagrangian multiplier $\gamma \ge 0$.

$$\begin{aligned} (P_pR_\gamma) \quad & -\min_{(x,\alpha)\in X} L_p((x,\alpha),\gamma) \overset{\Delta}{=} \\ & exp(-p\alpha)+\gamma[\sum_{i=1}^{m} exp(t(f_i(x)-\mu_{f_i}^{-1}(\alpha))) \\ & +\sum_{j=1}^{l} exp(tg_j(x)) - m - l]. \end{aligned}$$

It is shown that an optimal solution of the problem (10) is guaranteed to be generated by the dual search when the value of p is selected to be sufficiently large[13]. In other words, an optimal solution of the problem (P) can be generated by applying the conventional Lagrangian method to the problem (10). This leads to the following result.

Theorem 2. Let $(\hat{x},\hat{\alpha})$ solves (P), or equivalently $(\hat{x},\hat{\alpha})$ solves the problem (10). Then there exists $\{(\hat{x},\hat{\alpha}),\gamma(p)\}$ being an optimal primal-dual pair of the problem (10) for a sufficiently large p.

3.3. A Solution Algorithm

Recognizing prominent features of the problem (10), the following special dual search method is considered to facilitate the solution process. Set

$$w \overset{\Delta}{=} \frac{\gamma}{1+\gamma}.$$

The problem ($P_p R_\gamma$) can be recast to the following equivalent form.

$$(A_w) \quad -\min_{(x,\alpha)\in X} \; l((x,\alpha),w) \overset{\Delta}{=}$$
$$(1-w)exp(-p\alpha)+w(\sum_{i=1}^{m} exp(t(f_i(x)-\mu_{f_i}^{-1}(\alpha)))$$
$$+\sum_{j=1}^{l} exp(tg_j(x))-m-l),$$

where $w\in[0,1]$. It is clear that for any $p>0$ and $\lambda>0$ the problem (A_w) is a linearly constrained convex integer programming. A branch-and-bound procedure [15,18] for solving the sequence of linearly constrained convex integer programming problems will be considered by exploiting the polyhedral nature of the constraint set $X=\{0\le\alpha\le 1, x \text{ is integer.}\}$

Moreover, according to the geometric discussion in [13], a modified solution algorithm of the $p-$th power Lagrangian method for solving the system of fuzzy integer inequalities (1) with concave membership functions can be described as follows.

$p-$th Power Lagrangian Method ($pPLM$)

Step 1. Set $w=1$. Solve (A_1). Denote the optimal solution by (x^0,α^0). If $\sum_{i=1}^{m} exp(t(f_i(x^0)-\mu_{f_i}^{-1}(\alpha^0)))+\sum_{j=1}^{l} exp(tg_j(x^0))>m+l$, stop. There is no feasible solution. Otherwise, go to Step 2.

Step 2. Set $w=0$. Solve (A_0). Denote the optimal solution by (z^0,β^0). If $\sum_{i=1}^{m} exp(t(f_i(z^0)-\mu_{f_i}^{-1}(\beta^0)))+\sum_{j=1}^{l} exp(tg_j(z^0))\le m+l$, stop, (z^0,β^0) is the optimal solution. Otherwise, go to Step 3.

Step 3. Set $k=0$.

Step 4. Compute a w_k satisfying

$$l((x^k,\alpha^k),w_k)=l((z^k,\beta^k),w_k).$$

Step 5. Solve (A_{w_k}). Denote the optimal solution by (y^k,θ^k). If (x^k,α^k) solves (A_{w_k}), stop, (x^k, α^k) is an optimal solution to the problem (P). Otherwise, go to Step 6.

Step 6. If $\sum_{i=1}^{m} exp(t(f_i(y^k)-\mu_{f_i}^{-1}(\theta^k)))+\sum_{j=1}^{l} exp(tg_j(y^k))\le m+l$, set $(x^{k+1},\alpha^{k+1})=(y^k,\theta^k),(z^{k+1},\beta^{k+1})=(z^k,\beta^k)$. Otherwise, if $\sum_{i=1}^{m} exp(t(f_i(y^k)-\mu_{f_i}^{-1}(\theta^k)))+\sum_{j=1}^{l} exp(tg_j(y^k))>m+l$, set $(x^{k+1},\alpha^{k+1})=(x^k,\alpha^k),(z^{k+1},\beta^{k+1})=(y^k,\theta^k)$. Set $k:=k+1$. Go to Step 4.

4. Numerical Example

Consider the following system of fuzzy integer inequalities.

$$\begin{aligned} &f_1(x)=3x_1+x_2+2\underset{\sim}{\le}0,\\ &f_2(x)=2x_1+5x_2-5\underset{\sim}{\le}0,\\ &g_1(x)=-x_1\le 0,\\ &g_2(x)=-x_2\le 0,\\ &x_1,x_2 \text{ integer,}\end{aligned} \tag{11}$$

where the membership function $\mu_{f_i}(x), i=1,2,$ are specified as follows.

$$\mu_{f_1}(x)=\begin{cases}1, & if\ f_1(x)\le 0,\\ 1-(\frac{f_1(x)}{4})^2, & if\ 0< f_1(x)\le 4,\\ 0, & if\ f_1(x)>4,\end{cases}$$
$$\mu_{f_2}(x)=\begin{cases}1, & if\ f_2(x)\le 0,\\ 1-(\frac{f_2(x)}{6})^2, & if\ 0< f_2(x)\le 6,\\ 0, & if\ f_2(x)>6.\end{cases} \tag{12}$$

To solve the system of fuzzy integer inequalities (11), we follow the tolerance approach to consider the following convex integer programming problem.

$$\begin{aligned} \max \quad & \alpha\\ s.t. \quad & -4\sqrt{1-\alpha}+3x_1+x_2+2\le 0,\\ & -6\sqrt{1-\alpha}+2x_1+5x_2-5\le 0,\\ & -x_1\le 0,\\ & -x_2\le 0,\\ & 0\le\alpha\le 1,\\ & x_1,x_2 \text{ integer.}\end{aligned} \tag{13}$$

Using the proposed algorithm to solve this convex program, the problem (13) has the following equivalent form for a suitable $t > 0$.

$$\begin{aligned} -\min \quad & \exp(-p\alpha) \\ s.t. \quad & \exp(t(-4\sqrt{1-\alpha}+3x_1+x_2+2)) + \\ & \exp(t(-6\sqrt{1-\alpha}+2x_1+5x_2-5)) + \\ & \exp(t(-x_1)) + \exp(t(-x_2)) \le 4 \\ & (x,\alpha) \in X, \end{aligned} \tag{14}$$

where $X = \{(x,\alpha), 0 \le \alpha \le 1, x \text{ is integer.}\}$ The Lagrangian relaxation (A_w) of the problem (14) is

$$\begin{aligned} \min_{(x,\alpha)\in X} \quad & (1-w)\exp(-p\alpha) \\ & +w(\exp(t(-4\sqrt{1-\alpha}+3x_1+x_2+2)) + \\ & \exp(t(-6\sqrt{1-\alpha}+2x_1+5x_2-5)) \\ & +\exp(t(-x_1)) + \exp(t(-x_2)) - 4), \end{aligned} \tag{15}$$

where $w \in [0,1]$.

The optimal solution of this example is $(x^*, \alpha^*) = ((0,0), 0.7502)$. We take $p = 6$ and $t = 1000$ in the algorithm ($pPLM$). The linearly constrained convex integer programming (A_w) at each iteration is solved by a branch and bound procedure. In addition to the existence guarantee of a primal-dual pair and the success guarantee of the dual search associated with the $p-$th power Lagrangian method, the reduction in the dimension of the Lagrangian multiplier greatly facilitates the solution process. The Lagrangian multiplier is a scalar in the $p-$th power Lagrangian method, while it is of an $m-$dimension in the conventional Lagrangian method.

5. Concluding Remarks

In this work we study a system of fuzzy integer inequalities with concave membership functions. Such a system of fuzzy integer inequalities can be converted to a regular convex integer programming problem. A modified version of the p -th power Lagrangian method has been introduced to solve the resulting convex program. Due to the polyhedral nature of the constraint set, only a branch-and-bound procedure is required in our implementation for solving the sequence of linearly constrained convex integer programming problems. An example is used to illustrate the proposed algorithm. Compared to other approaches to solving general convex integer programming problems, the p -th power Lagrangian method essentially provides a theoretical foundation in characterizing the existence of optimal generating Lagrangian multiplier vectors and the existence of optimal primal-dual pairs.

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