Chaotic Synchronization Using Fuzzy Control Approach

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Abstract

This paper presents the synchronization of chaotic systems subject to parameter uncertainties. Based on the fuzzy models of chaotic systems, a fuzzy controller is designed to realize chaotic synchronization. A design criterion of the membership functions of fuzzy controller is proposed to facilitate the stability design towards chaotic synchronization when parameter uncertainties are under consideration. LMI-based stability conditions are derived to guarantee the system stability using Lyapunov-based approach. Simulation examples are given to illustrate the merits of the proposed fuzzy-model-based control approach.

1. Introduction

Chaotic synchronization has drawn the researchers' attention for many years due to its practical applications such as secure communication. The highly nonlinear nature of the chaotic systems and its sensitivity to initial conditions make the system analysis and controller design for chaotic synchronization challenging. The situation becomes further complicated when chaotic systems are subject to parameter uncertainties, which is inevitable in most practical applications.

Fuzzy-model-based control approach has been shown to be beneficial in dealing with ill-defined and nonlinear systems. Recently, various fuzzy-model-control approaches have been proposed to realize chaotic synchronization and promising stability analysis results have been achieved. In general, under the fuzzy-model-based control approach, a TS-fuzzy model [1], [2], which exhibits favourable properties to facilitate the stability analysis and controller design, is employed to provide a general and systematical framework to represent the dynamics of the chaotic systems. It was shown in [3]-[5] that some common chaotic systems can be represented by fuzzy models with simple rules. Based on the fuzzy models, a fuzzy controller is then designed to realize chaotic synchronization. In [3], [6], by taking advantage of the identical structure of the chaotic systems and the favourable property given by sharing the same premises between fuzzy model and controller, an exact-linearization fuzzy control approach was proposed and stability conditions in terms of linear matrix inequalities (LMIs) were derived. By employing some convex programming techniques, the solution, which includes the feedback gains of the fuzzy controller, to the LMI-based conditions can be solved numerically and efficiently. This idea was extended to H_∞ approach of which the synchronization performance is guaranteed by an H_∞ performance index [4], [5].

In [3]-[6], only uncertainty-free chaotic systems were considered. When the chaotic systems are subject to parameter uncertainties, the stability conditions in [3]-[6] are not applicable to reach a stable design of fuzzy controller to realize chaotic synchronization. To deal with the parameter uncertainties, adaptation ability [7] was endowed to the fuzzy controller. By taking advantage of the superior approximation ability of the fuzzy system, the values of parameter uncertainties can be estimated in an online manner for the fuzzy controller to realize synchronization. Consequently, compared with the fuzzy-model-based control approach in [3]-[6], the adaptive fuzzy controller offers an outstanding robustness property to handle parameter uncertainties at the cost of high structural complexity and computational demand. Various adaptive fuzzy control approaches were reported in [8], [9].

In this paper, a fuzzy controller is employed to synchronize chaotic systems subject to parameter uncertainties. As parameter uncertainties are considered, the favourable property given by sharing the same premises between the fuzzy model and controller [3]-[6] cannot facilitate the stability analysis and design. Instead, by designing properly the membership functions of the fuzzy controller, some arbitrary matrices can be introduced to ease the stability analysis. LMI-based stability conditions are derived using Lyapunov-based approach to aid the design of fuzzy controllers.

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2. FUZZY MODEL AND FUZZY CONTROLLER

Fuzzy models are employed to represent the dynamical behavior of the response and drive chaotic systems. A fuzzy controller is designed accordingly to drive the system state of the response chaotic system to follow those of the drive chaotic system.

A. Fuzzy Model

Let p be the number of fuzzy rules describing the chaotic system with control input term. The i-th rule is of the following format,

Rule i: IF $f_1(\mathbf{x}(t))$ is M_1^i AND … AND $f_\Psi(\mathbf{x}(t))$ is M_Ψ^i

THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$, $i = 1, 2, ..., p$ (1)

where M_α^i is a fuzzy term of rule i corresponding to the function $f_\alpha(\mathbf{x}(t))$ with known form, $\alpha = 1, 2, ..., \Psi$, $i = 1, 2, ..., p$, Ψ is a positive integer; $\mathbf{x}(t) \in \Re^{n\times 1}$ is the system state vector; $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{B} \in \Re^{n\times m}$ are the known constant system and input matrices respectively; $\mathbf{u}(t) \in \Re^{m\times 1}$ is the input vector. The system dynamics of the chaotic system are described by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\big(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\big) \quad (2)$$

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1,\ \ w_i(\mathbf{x}(t)) \in [0\ \ 1] \ \text{ for all } i \quad (3)$$

$$w_i(\mathbf{x}(t)) = \frac{\mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t)))\times\mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{M}_\Psi^i}(f_\Psi(\mathbf{x}(t)))}{\sum_{k=1}^{p}\Big(\mu_{\mathrm{M}_1^k}(f_1(\mathbf{x}(t)))\times\mu_{\mathrm{M}_2^k}(f_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{M}_\Psi^k}(f_\Psi(\mathbf{x}(t)))\Big)} \quad (4)$$

is a nonlinear function of $\mathbf{x}(t)$ and $\mu_{\mathrm{M}_\alpha^i}(f_\alpha(\mathbf{x}(t)))$ is the grade of membership corresponding to the fuzzy terms M_α^i. It should be noted that the grades of membership are uncertain if the nonlinear plant is subject to parameter uncertainties. Let the chaotic system of (2) be the response system. The dynamics of the drive chaotic system is represented in the following general form.

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}_j\big(\hat{\mathbf{x}}(t)\big)\hat{\mathbf{x}}(t) \quad (5)$$

where $\hat{\mathbf{x}}(t) \in \Re^{n\times 1}$ is the system state vector, $\hat{\mathbf{A}}\big(\hat{\mathbf{x}}(t)\big) \in \Re^{n\times n}$ is the system matrix. It should be noted that there is no control input term for the drive chaotic system.

B. Fuzzy Controller

A fuzzy controller with p rules is considered to drive the system states of the response chaotic system to follow those of the drive chaotic system. The j-th rule of the fuzzy controller is defined as follows.

Rule j: IF $g_1(\mathbf{x}(t))$ is N_1^j AND … AND $g_\Omega(\mathbf{x}(t))$ is N_Ω^j

THEN $\mathbf{u}(t) = \mathbf{G}_j\mathbf{e}(t)$, $j = 1, 2, ..., p$ (6)

where N_β^j is a fuzzy term of rule j corresponding to the function $g_\beta(\mathbf{x}(t))$, $\beta = 1, 2, ..., \Omega$; $j = 1, 2, ..., p$; Ω is a positive integer; $\mathbf{G}_j \in \Re^{m\times n}$ is the feedback gain of rule j to be designed; $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$. The inferred output of the fuzzy controller is given by,

$$\mathbf{u}(t) = \sum_{j=1}^{p} m_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{e}(t) \quad (7)$$

$$\sum_{j=1}^{p} m_j(\mathbf{x}(t)) = 1,\ \ m_j(\mathbf{x}(t)) \in [0\ \ 1] \ \text{ for all } j \quad (8)$$

$$m_j(\mathbf{x}(t)) = \frac{\mu_{\mathrm{N}_1^j}(g_1(\mathbf{x}(t)))\times\mu_{\mathrm{N}_2^j}(g_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{N}_\Omega^j}(g_\Omega(\mathbf{x}(t)))}{\sum_{k=1}^{p}\Big(\mu_{\mathrm{N}_1^k}(g_1(\mathbf{x}(t)))\times\mu_{\mathrm{N}_2^k}(g_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{N}_\Omega^k}(g_\Omega(\mathbf{x}(t)))\Big)} \quad (9)$$

is a nonlinear function of $\mathbf{x}(t)$ and $\mu_{\mathrm{N}_\beta^j}(g_\beta(\mathbf{x}(t)))$ is the grade of membership corresponding to the fuzzy term N_β^j.

3. STABILITY ANALYSIS

In this section, stability analysis is proceeded to achieve stability conditions to aid the design of fuzzy controller. In the following analysis, $w_i(\mathbf{x}(t))$ and $\hat{w}_j(\hat{\mathbf{x}}(t))$ are denoted as w_i and $\hat{w}_j$ for simplicity. Furthermore, the inequality of $\sum_{i=1}^{p} w_i = \sum_{j=1}^{p} m_j = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i m_j = 1$ is applied. From (2), (5) and (7), the error system is defined as follows.

$$\dot{\mathbf{e}}(t) = \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t)$$
$$= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i m_j\big(\mathbf{A}_i + \mathbf{B}\mathbf{G}_j\big)\mathbf{e}(t) + \mathbf{m}_e(t) \quad (10)$$

where $\mathbf{m}_e(t) = \sum_{i=1}^{p} w_i\mathbf{A}_i\hat{\mathbf{x}}(t) - \hat{\mathbf{A}}\big(\hat{\mathbf{x}}(t)\big)\hat{\mathbf{x}}(t)$. It should be noted that $\mathbf{m}_e(t)$ is bounded due to $\hat{\mathbf{x}}(t)$, w_i and $\hat{w}_j$ are bounded. To investigate the system stability [10] of (10), the following Lyapunov function is considered.

$$V(t) = \mathbf{e}(t)^{\mathrm{T}}\mathbf{P}\mathbf{e}(t)$$

where $\mathbf{P} = \mathbf{P}^{\mathrm{T}} \in \Re^{n\times n} > 0$. The time derivative of the Lyapunov function is as follows.

$$\dot{V}(t) = \dot{\mathbf{e}}(t)^{\mathrm{T}}\mathbf{P}\mathbf{e}(t) + \mathbf{e}(t)^{\mathrm{T}}\mathbf{P}\dot{\mathbf{e}}(t)$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i m_j \mathbf{e}(t)^{\mathrm{T}} \begin{pmatrix} (\mathbf{A}_i + \mathbf{B}\mathbf{G}_j)^{\mathrm{T}} \mathbf{P} \\ + \mathbf{P}(\mathbf{A}_i + \mathbf{B}\mathbf{G}_j) \end{pmatrix} \mathbf{e}(t)$$
$$+ \mathbf{m}_e(t)^{\mathrm{T}} \mathbf{P}\mathbf{e}(t) + \mathbf{e}(t)^{\mathrm{T}} \mathbf{P}\mathbf{m}_e(t) \quad (11)$$
$$+ \mathbf{e}(t)^{\mathrm{T}} \mathbf{PUPe}(t) - \mathbf{e}(t)^{\mathrm{T}} \mathbf{PUPe}(t)$$
$$+ \sigma^2 \mathbf{m}_e(t)^{\mathrm{T}} \mathbf{PUPm}_e(t) - \sigma^2 \mathbf{m}_e(t)^{\mathrm{T}} \mathbf{PUPm}_e(t)$$

where $\mathbf{U} = \mathbf{U}^{\mathrm{T}} = \sum_{i=1}^{p} w_i \mathbf{U}_i > 0$, $\mathbf{U}_i = \mathbf{U}_i^{\mathrm{T}} \in \Re^{n \times n} > 0$, $i = 1, 2, ..., p$, σ is a non-zero positive scalar. From (11), let $\mathbf{X} = \mathbf{X}^{\mathrm{T}} = \mathbf{P}^{-1} > 0$, we have,

$$\dot{V}(t) = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i m_j \mathbf{z}(t)^{\mathrm{T}} \times \begin{bmatrix} \mathbf{X}(\mathbf{A}_i + \mathbf{B}\mathbf{G}_j)^{\mathrm{T}} + (\mathbf{A}_i + \mathbf{B}\mathbf{G}_j)\mathbf{X} + \mathbf{U}_i & \mathbf{X} \\ \mathbf{X} & -\sigma^2 \mathbf{U}_i \end{bmatrix} \mathbf{z}(t)$$
$$- \mathbf{e}(t)^{\mathrm{T}} \mathbf{PUPe}(t) + \sigma^2 \mathbf{m}_e(t)^{\mathrm{T}} \mathbf{PUPm}_e(t) \quad (12)$$

where $\mathbf{z}(t) = \begin{bmatrix} \mathbf{X}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{e}(t) \\ \mathbf{m}_e(t) \end{bmatrix}$. From (12), let $\mathbf{G}_j = \mathbf{N}_j \mathbf{X}^{-1}$ where $\mathbf{N}_j \in \Re^{m \times n}$, we have,

$$\dot{V}(t) = \frac{1}{\rho} \sum_{i=1}^{p}\sum_{j=1}^{p} w_i (\rho m_j - w_j + w_j) \mathbf{z}(t)^{\mathrm{T}} \times \begin{bmatrix} \mathbf{X}\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i \mathbf{X} + \mathbf{N}_j^{\mathrm{T}} \mathbf{B}^{\mathrm{T}} + \mathbf{B}\mathbf{N}_j + \mathbf{U}_i & \mathbf{X} \\ \mathbf{X} & -\sigma^2 \mathbf{U}_i \end{bmatrix} \mathbf{z}(t)$$
$$- \mathbf{e}(t)^{\mathrm{T}} \mathbf{PUPe}(t) + \sigma^2 \mathbf{m}_e(t)^{\mathrm{T}} \mathbf{PUPm}_e(t)$$
$$= \frac{1}{\rho} \sum_{i=1}^{p} w_i \mathbf{z}(t)^{\mathrm{T}} \mathbf{V}_{ii} \mathbf{z}(t) + \frac{1}{\rho} \sum_{i=1}^{p}\sum_{j=1}^{p} w_i (\rho m_j - w_j) \mathbf{z}(t)^{\mathrm{T}} \mathbf{V}_{ij} \mathbf{z}(t)$$
$$- \mathbf{e}(t)^{\mathrm{T}} \mathbf{PUPe}(t) + \sigma^2 \mathbf{m}_e(t)^{\mathrm{T}} \mathbf{PUPm}_e(t) \quad (13)$$

where

$$\mathbf{V}_{ij} = \begin{bmatrix} \mathbf{X}\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i \mathbf{X} + \mathbf{N}_j^{\mathrm{T}} \mathbf{B}^{\mathrm{T}} + \mathbf{B}\mathbf{N}_j + \mathbf{U}_i & \mathbf{X} \\ \mathbf{X} & -\sigma^2 \mathbf{U}_i \end{bmatrix},$$

$i, j = 1, 2, ..., p$. Let $\rho m_j - w_j > 0$, $j = 1, 2, ..., p$ where $\rho > 1$. From (13), we have,

$$\dot{V}(t) = \frac{1}{\rho} \sum_{i=1}^{p} w_i \mathbf{z}(t)^{\mathrm{T}} \mathbf{V}_{ii} \mathbf{z}(t)$$
$$+ \frac{1}{\rho} \sum_{i=1}^{p}\sum_{j=1}^{p} w_i (\rho m_j - w_j) \mathbf{z}(t)^{\mathrm{T}} \mathbf{V}_{ij} \mathbf{z}(t)$$
$$+ \sum_{i=1}^{p}\sum_{j=1}^{p} w_i (\rho m_j - w_j) \mathbf{z}(t)^{\mathrm{T}} (\mathbf{\Lambda}_i - \mathbf{\Lambda}_i) \mathbf{z}(t)$$
$$- \mathbf{e}(t)^{\mathrm{T}} \mathbf{PUPe}(t) + \sigma^2 \mathbf{m}_e(t)^{\mathrm{T}} \mathbf{PUPm}_e(t)$$

$$= \frac{1}{\rho} \sum_{i=1}^{p} w_i \mathbf{z}(t)^{\mathrm{T}} (\mathbf{V}_{ii} - (\rho - 1)\mathbf{\Lambda}_i) \mathbf{z}(t)$$
$$+ \frac{1}{\rho} \sum_{i=1}^{p}\sum_{j=1}^{p} w_i (\rho m_j - w_j) \mathbf{z}(t)^{\mathrm{T}} (\mathbf{V}_{ij} + \mathbf{\Lambda}_i) \mathbf{z}(t) \quad (14)$$
$$- \mathbf{e}(t)^{\mathrm{T}} \mathbf{PUPe}(t) + \sigma^2 \mathbf{m}_e(t)^{\mathrm{T}} \mathbf{PUPm}_e(t)$$

where $\mathbf{\Lambda}_i = \mathbf{\Lambda}_i^{\mathrm{T}} \in \Re^{2n \times 2n}$, $i = 1, 2, ..., p$. It can be seen in (14) that the arbitrary matrices $\mathbf{\Lambda}_i$ effective share unstable elements between $\mathbf{V}_{ii}$ and $\mathbf{V}_{ij}$ to ease the satisfaction of stability conditions. If the stability conditions $\mathbf{V}_{ii} - (\rho - 1)\mathbf{\Lambda}_i < 0$ and $\mathbf{V}_{ij} + \mathbf{\Lambda}_i < 0$ for all i, j, we have,

$$\dot{V}(t) \le -\mathbf{e}(t)^{\mathrm{T}} \mathbf{PUPe}(t) + \sigma^2 \mathbf{m}_e(t)^{\mathrm{T}} \mathbf{PUPm}_e(t) \quad (15)$$

Taking integration on both sides of (15), we have,

$$\int_0^\infty \dot{V}(t)dt \le \int_0^\infty \left(-\mathbf{e}(t)^{\mathrm{T}} \mathbf{PUPe}(t) + \sigma^2 \mathbf{m}_e(t)^{\mathrm{T}} \mathbf{PUPm}_e(t)\right)dt$$
$$\int_0^\infty \mathbf{e}(t)^{\mathrm{T}} \mathbf{PUPe}(t)dt \le V(0) + \int_0^\infty \sigma^2 \mathbf{m}_e(t)^{\mathrm{T}} \mathbf{PUPm}_e(t)dt \quad (16)$$

With the facts that $V(\infty) \ge 0$ and $\sigma^2 > 0$, the H_∞ tracking performance of (16) is achieved to guarantee the tracking performance. It can be seen that a good tracking performance is ensured by a small value of σ. The stability analysis result is summarized in the following theorem.

Theorem 1: The error system of (10), formed by the response chaotic system in the form of (2), the drive chaotic system of (5) and the fuzzy controller of (7), satisfies the following H_∞ tracking performance for a prescribed attenuation level $\sigma > 0$,

$$\int_0^\infty \mathbf{e}(t)^{\mathrm{T}} \mathbf{PUPe}(t)dt \le V(0) + \int_0^\infty \sigma^2 \mathbf{m}_e(t)^{\mathrm{T}} \mathbf{PUPm}_e(t)dt,$$

if the membership functions of the fuzzy controller are designed such that $\rho m_j(\mathbf{x}(t)) - w_j(\mathbf{x}(t)) > 0$, *$j = 1, 2, ..., p$ where $\rho > 1$, and there exist constant matrices* $\mathbf{X} = \mathbf{X}^{\mathrm{T}} \in \Re^{n \times n}$, $\mathbf{N}_j \in \Re^{m \times n}$, $\mathbf{U}_i = \mathbf{U}_i^{\mathrm{T}} \in \Re^{n \times n}$ *and* $\mathbf{\Lambda}_i = \mathbf{\Lambda}_i^{\mathrm{T}} \in \Re^{2n \times 2n}$ *such that the following LMIs hold.*

$\mathbf{X} \in \Re^{n \times n}$; $\mathbf{V}_{ii} - (\rho - 1)\mathbf{\Lambda}_i < 0$, *$i = 1, 2, ..., p$;*
$\mathbf{V}_{ij} + \mathbf{\Lambda}_i < 0$, *$i, j = 1, 2, ..., p$;*

and the feedback gains are defined as $\mathbf{G}_i = \mathbf{N}_i \mathbf{X}_1^{-1}$, *$i = 1, 2, ..., p$.*

4. Simulation Examples

Two simulation examples are given to illustrate the merits of the proposed approach for chaotic synchronization subject to parameter uncertainties. In the first simulation example, two Rössler systems with identical structure are considered. In the second simulation example, two chaotic systems, Rössler and Chua's systems, with non-identical structure are considered. The stability conditions in Theorem 1 are employed to aid the design of fuzzy controllers to synchronize the chaotic systems in both simulation examples.

A. Rössler Systems

Two Rössler systems subject to parameter uncertainties will be taken as the drive and response chaotic systems. A fuzzy controller will be designed to synchronize both chaotic systems.

A1) The dynamics of the response Rössler's system [3] with input term are described as follows,

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}u(t) \tag{17}$$

where $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$, $\mathbf{A}(\mathbf{x}(t)) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ b & 0 & -(c(t) - x_1(t)) \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; a = 0.34, b = 0.4, $c(t) = \frac{c_{\max} + c_{\min}}{2} + \frac{c_{\max} - c_{\min}}{2}\cos(t) \in [c_{\min} \quad c_{\max}] > 0$ is the uncertain parameter, $c_{\min}$ = 4.5, and $c_{\max}$ = 7.7. It is assumed that $x_1(t) \in [c_{\max} - d \quad c_{\min} + d]$ and d = 25. The response Rössler system can be exactly represented by a fuzzy model with the following fuzzy rules [3].

Rule i: IF $x_1(t)$ is M^i

THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{Bu}(t)$, i = 1, 2. (18)

The inferred response Rössler system is defined as

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{2} w_i(x_1(t))(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}u(t)) \tag{19}$$

where $\mathbf{A}_1 = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ b & 0 & -d \end{bmatrix}$ and $\mathbf{A}_2 = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ b & 0 & d \end{bmatrix}$; $w_1(x_1(t)) = \mu_{\mathrm{M}^1}(x_1(t)) = \frac{1}{2}\left(1 + \frac{c(t) - x_1(t)}{d}\right)$ and $w_2(x_1(t)) = \mu_{\mathrm{M}^2}(x_1(t)) = 1 - \mu_{\mathrm{M}^1}(x_1(t))$. It can be seen that the grades of membership are unknown as the value of $c(t)$ is uncertain.

A2) The dynamics of the drive Rössler's system are described as follows,

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}(\hat{\mathbf{x}}(t))\hat{\mathbf{x}}(t) \tag{20}$$

where

$\hat{\mathbf{x}}(t) = \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \hat{x}_3(t) \end{bmatrix}$, $\hat{\mathbf{A}}(\hat{\mathbf{x}}(t)) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ b & 0 & -(\hat{c}(t) - \hat{x}_1(t)) \end{bmatrix}$, $\hat{c}(t) = \frac{c_{\max} + c_{\min}}{2} + \frac{c_{\max} - c_{\min}}{2}\sin(t) \in [c_{\min} \quad c_{\max}] > 0$ is the uncertain parameter. It is assumed that $\hat{x}_1(t) \in [c_{\max} - d \quad c_{\min} + d]$. It can be seen from (17) and (20) that both response and drive chaotic systems are subject to identical structure which can ease the realization of chaotic synchronization.

A3) A two-rule fuzzy controller is employed to synchronize both Rössler's systems. The fuzzy rules are designed as follows.

Rule j: IF $x_1(t)$ is N_1^j

THEN $u(t) = \mathbf{G}_j\mathbf{e}(t)$, j = 1, 2 (21)

The inferred fuzzy controller is defined as

$$u(t) = \sum_{j=1}^{2} m_j(x_1(t))\mathbf{G}_j\mathbf{e}(t) \tag{22}$$

where the membership functions are designed as $m_1(x_1(t)) = \mu_{\mathrm{N}_1^1}(x_1(t)) = -0.0179(x_1(t) - 29.5) + 0.0804$ and $m_2(x_1(t)) = 1 - \mu_{\mathrm{N}_1^2}(x_1(t)) = 1 - m_1(x_1(t))$ for $x_1(t) \in [c_{\max} - d \quad c_{\min} + d]$. It can be seen that the condition of $\rho m_j(x_1(t)) - w_j(x_1(t)) > 0$, j = 1, 2, with ρ = 1.1 is satisfied. With the aid of MATLAB LMI toolbox to solve the solution to the stability conditions in Theorem 1, we obtain $\mathbf{G}_1$ = [508.8457 895.4915 –83.0081] and $\mathbf{G}_2$ = [51.2570 89.0173 –34.9030] with σ = 0.5.

Fig. 1 and Fig. 2 show the system state responses and tracking error under the initial state conditions of $\mathbf{x}(0) = [1 \quad 1 \quad 1]^{\mathrm{T}}$ and $\hat{\mathbf{x}}(0) = [-1 \quad -1 \quad -1]^{\mathrm{T}}$. In this simulation, $u(t)$ = 0 is employed for $0 \le t < 50$s and the fuzzy controller of (22) is applied for $t \ge 50$s. Referring to these figures, it can be seen that the proposed fuzzy controller, which is applied for $t \ge 50$s, is able to drive the system states of the response Rössler system to follow those of the drive Rössler system, both of them are subject to parameter uncertainties, with a sufficiently small tracking error.

B. Rössler and Chua's Systems

In this simulation example, the Rössler and Chua's systems are taken as the response and drive chaotic systems respectively.

B1) The same Rössler system of (17) subject to parameter uncertainty is considered as the response system. In this simulation example, we take $\mathbf{B}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$. The response Rössler system can be represented by the fuzzy model of (19) to facilitate the design of fuzzy controller.

B2) The dynamics of the drive Chua's system [4]-[5] is defended as follows.

$$\dot{\hat{\mathbf{x}}}(t)=\hat{\mathbf{A}}(\hat{\mathbf{x}}(t))\hat{\mathbf{x}}(t)+\mathbf{E} \tag{23}$$

where

$$\hat{\mathbf{x}}(t)=\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\\ \hat{x}_3(t)\end{bmatrix},\quad \mathbf{A}(\mathbf{x}(t))=\begin{bmatrix}-\hat{a} & \hat{a} & 0\\ 1 & -1 & 1\\ 0 & -\hat{b} & 0\end{bmatrix},\quad \mathbf{E}=\begin{bmatrix}\hat{c}(t)\\ 0\\ 0\end{bmatrix};$$

$\hat{a}=9$, $\hat{b}=14.2850$, $\hat{c}(t)=g_b\hat{x}_1(t)+\frac{1}{2}(g_a(t)-g_b)(|\hat{x}_1(t)+1|-|\hat{x}_1(t)-1|)$ is the parameter uncertainty, $g_a(t)=\frac{g_{a_1}+g_{a2}}{2}+\frac{g_{a_1}-g_{a2}}{2}\sin(t)$, $g_{a_1}=-1.3428$, $g_{a2}=-1.5285$ and g_b = 0.7143. It is assumed that $\hat{x}_1(t)\in[-25\quad 25]$. It can be seen from (17) and (23) that both Rössler and Chua's systems are subject to non-identical structure.

B3) A two-rule fuzzy controller in the same form of (22) is employed to synchronize the chaotic systems. With the aid of MATLAB LMI toolbox to solve the stability conditions in Theorem 1, we obtain $\mathbf{G}_1=\begin{bmatrix}-327.6863 & -0.5820 & 5.5793\\ 0.5820 & -328.0263 & -0.4472\\ -5.6224 & 0.4955 & -304.6995\end{bmatrix}$ and $\mathbf{G}_2=\begin{bmatrix}-327.6863 & -1.5138 & 9.5811\\ 1.5138 & -328.0263 & 0.0675\\ -10.0579 & 0.0748 & -306.2915\end{bmatrix}$ with $\sigma=$ 0.01.

Fig. 3 and Fig. 4 show the system state responses and tracking error under the initial state conditions of $\mathbf{x}(0)=[1\quad 1\quad 1]^{\mathrm{T}}$ and $\hat{\mathbf{x}}(0)=[-1\quad -1\quad -1]^{\mathrm{T}}$. In this simulation, $\mathbf{u}(t)$ = $\mathbf{0}$ is employed for $0 \le t < 50$s and the fuzzy controller is applied for $t \ge 50$s. It can be seen that the proposed fuzzy controller is able to synchronize the non-identical-structured response and drive chaotic systems subject to parameter uncertainties, with a sufficiently small tracking error.

In both simulation examples, due to the existence of parameter uncertainties, the stability conditions in [3]-[6] for uncertainty-free chaotic systems cannot be applied to aid the design of fuzzy controller. Furthermore, compared to the adaptive fuzzy controller [7]-[9], it can be seen that the proposed fuzzy controller offers lower structural complexity and requires lower computational demand.

5. Conclusions

A fuzzy controller has been employed to synchronize chaotic systems subject to parameter uncertainties. A design criterion of membership functions for the fuzzy controller has been proposed to ease the stability analysis. LMI-based stability conditions have been derived using Lyapunov-based approach to aid the design of the fuzzy controller. Simulation examples have been given to illustrate the merits of the proposed approach.

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7. References

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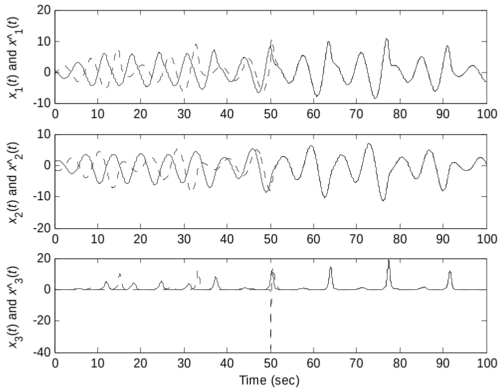

Fig. 1. System state responses of the drive (solid lines) and response (dotted lines) Rössler systems with $u(t) = 0$ for $0 \le t < 50$s and the proposed fuzzy controller applied for $t \ge 50$s.

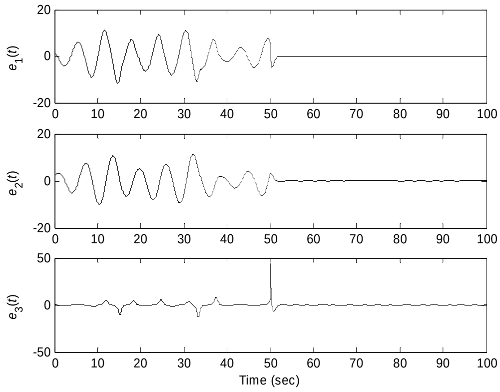

Fig. 2. Tracking error of the drive and response Rössler systems.

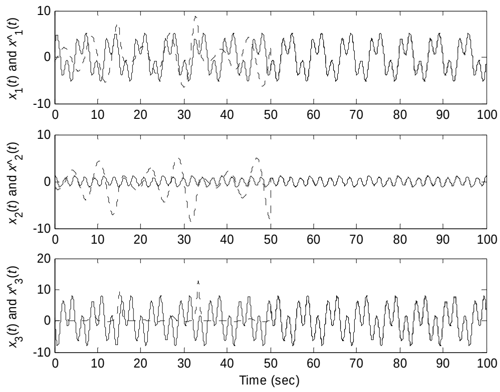

Fig. 3. System state responses of the drive Rössler (solid lines) and response Chua's (dotted lines) systems with $\mathbf{u}(t) = \mathbf{0}$ for $0 \le t < 50$s and the proposed fuzzy controller applied for $t \ge 50$s.

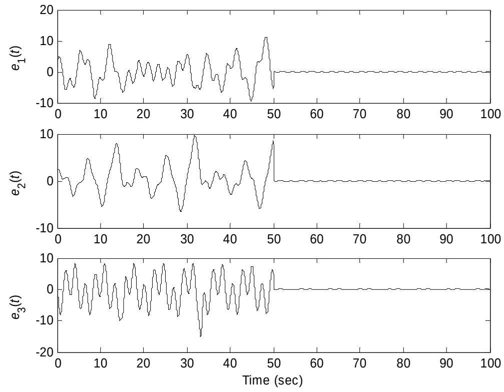

Fig. 4. Tracking error of the drive Rössler and response Chua's systems.