Auxiliary-state-driven Fuzzy Controller for Nonlinear Systems Based on TS-fuzzy-model-based Approach

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Abstract

This paper investigates the system stability of fuzzy-model-based control system. An auxiliary-state-driven fuzzy controller is proposed to deal with nonlinear systems. With a particular and systematic design, the system stability of fuzzy control systems is implied by the auxiliary and error systems. Instead of studying the system stability of the fuzzy control systems directly, the system stability of the auxiliary system and the error model is investigated. Furthermore, the auxiliary system offers a favourable property that some system parameters can be freely designed. As a result, except the feedback gains of the fuzzy controller, these freely designed parameters can also contribute themselves to relax stability conditions. Based on the Lyapunov-based approach, the stability and performance conditions in terms of linear matrix inequalities are derived. Simulation examples are given to illustrate the effectiveness and design procedure of the proposed approach.

1. Introduction

Fuzzy-model-based control approach offers a systematic and effective framework to stability analysis of fuzzy control systems. Based on the TS-fuzzy model [1], [2], a nonlinear system can be represented as a weighted average of some linear sub-systems. The nonlinear and linear characteristics of the nonlinear system can be extracted. This particular structure offers a favourable framework to facilitate the system analysis. In [3], [4], a fuzzy controller was proposed to handle the nonlinear system represented by the fuzzy model. With a similar structure as the fuzzy model, a fuzzy controller which is a weighted average of some linear sub-controllers is proposed. As a result, the fuzzy-model-based control system can be represented as a weighted average of some linear sub-control systems. The fuzzy-model-based control is guaranteed to be stable if there exists a solution to a set of Lyapunov inequalities [3], [4]. The Lyapunov inequalities can be cast as a linear-matrix-inequality (LMI) problem [5] which can be solved numerically using some convex programming techniques. It was reported in [6] that the stability conditions can be relaxed when the fuzzy controller shares the same premises as those of the fuzzy model. Under this design criterion, further relaxed stability conditions were reported [7]-[11]. The stability analysis was further extended to multiple Lyapunov function approach [12] which displays potential to further relax the stability conditions. However, derivative information of membership functions is needed which makes the analysis more complicated. In [13], a switching Lyapunov function approach was proposed to relax the stability conditions under some particular cases.

In this paper, an auxiliary-state-driven (ASD) fuzzy control approach is proposed. An auxiliary system is proposed for the purpose of relaxation of stability conditions. Traditionally, the control signals of fuzzy controller [3], [4], [6]-[10], depend on the system states of the nonlinear plant only. In the proposed approach, the control signals depend on both the system states of the nonlinear plant and the auxiliary system. Consequently, richer dynamics enhancing stabilization ability can be introduced to the fuzzy controller to carry out control process. Lyapunov-based approach is employed to perform stability analysis. Under such a design, the system stability of fuzzy control system is implied by the auxiliary system and the error system. Hence, the system stability of auxiliary system is investigated instead of the fuzzy control system directly. Stability and performance conditions in terms of the system matrices of the nonlinear plant and the auxiliary system are derived. As some matrices of the auxiliary system can be freely designed, the stability conditions of the fuzzy-model-based controller can be relaxed by this favourable property. It iss shown that the proposed ASD approach can offer relaxed stability conditions than those of the existing ones [3], [4], [6]-[10]. Consequently, some applications that stable design of fuzzy controllers cannot be achieved by the existing stability conditions can benefit from the proposed ASD control approach.

This paper is organized as follows. In section 2, the fuzzy model, ASD fuzzy controller and auxiliary system are presented. In section 3, stability of fuzzy-model-based control system is analyzed. LMI-based stability conditions are derived to guarantee the stability of the fuzzy-model-based control systems.

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In section 4, the design of the feedback gains and the LMI-based performance conditions are presented. In section 5, illustrative examples are given to show the merits of the proposed approach. A conclusion is drawn in section 6.

2. Fuzzy model, Auxiliary-State-Driven Fuzzy Controller and Auxiliary system

A multivariable nonlinear control system comprising a fuzzy model and an ASD fuzzy controller connected in closed loop is considered.

A. Fuzzy Model

Let p be the number of fuzzy rules describing the nonlinear plant. The i-th rule is of the following format:

Rule i: IF $f_1(\mathbf{x}(t))$ is M_1^i AND … AND $f_\Psi(\mathbf{x}(t))$ is M_Ψ^i

$$\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t) \tag{1}$$

where M_α^i is a fuzzy term of rule i corresponding to the known function $f_\alpha(\mathbf{x}(t))$, $\alpha = 1, 2, ..., \Psi$; $i = 1, 2, ..., p$; Ψ is a positive integer; $\mathbf{A}_i \in \Re^{n\times n}$ and $\mathbf{B}_i \in \Re^{n\times m}$ are known constant system and input matrices respectively; $\mathbf{x}(t) \in \Re^{n\times 1}$ is the system state vector and $\mathbf{u}(t) \in \Re^{m\times 1}$ is the input vector. The system dynamics are described by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\right) \tag{2}$$

where,

$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1, \quad w_i(\mathbf{x}(t)) \in [0 \quad 1] \quad \text{for all } I \tag{3}$$

$$w_i(\mathbf{x}(t)) = \frac{\mu_{\mathrm{M}_1^i}(f_1(\mathbf{x}(t)))\times\mu_{\mathrm{M}_2^i}(f_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{M}_\Psi^i}(f_\Psi(\mathbf{x}(t)))}{\sum_{k=1}^{p}\left(\mu_{\mathrm{M}_1^k}(f_1(\mathbf{x}(t)))\times\mu_{\mathrm{M}_2^k}(f_2(\mathbf{x}(t)))\times\cdots\times\mu_{\mathrm{M}_\Psi^k}(f_\Psi(\mathbf{x}(t)))\right)} \tag{4}$$

is a nonlinear function of $\mathbf{x}(t)$ and $\mu_{\mathrm{M}_\alpha^i}\left(f_\alpha(\mathbf{x}(t))\right)$ is the grade of membership corresponding to the fuzzy term M_α^i.

B. Auxiliary-State-Driven Fuzzy Controller

A fuzzy controller with p fuzzy rules is to be designed for the nonlinear plant. The j-th rule of the fuzzy controller is of the following format:

IF $f_1(\mathbf{x}(t))$ is M_1^i AND … AND $f_\Psi(\mathbf{x}(t))$ is M_Ψ^i

$$\text{THEN } \mathbf{u}(t) = \mathbf{G}_j\hat{\mathbf{x}}(t) + \hat{\mathbf{G}}_j\mathbf{e}(t) \tag{5}$$

where $\mathbf{G}_j \in \Re^{m\times n}$ and $\hat{\mathbf{G}}_j \in \Re^{m\times n}$ are the constant feedback gains of the j-th rule to be designed; $\hat{\mathbf{x}}(t) \in \Re^{n\times 1}$ is the system state vector of the auxiliary system discussed later and $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$. The inferred output of the ASD fuzzy controller is given by,

$$\mathbf{u}(t) = \sum_{j=1}^{p} w_j(\mathbf{x}(t))\left(\mathbf{G}_j\hat{\mathbf{x}}(t) + \hat{\mathbf{G}}_j\mathbf{e}(t)\right) \tag{6}$$

Remark 1: The ASD fuzzy controller is reduced to a traditional fuzzy controller [4] when $\mathbf{G}_j = \hat{\mathbf{G}}_j$, j = 1, 2, ..., p.

C. Auxiliary system

Let the auxiliary system be defined as,

$$\dot{\hat{\mathbf{x}}}(t) = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i(\mathbf{x}(t))w_j(\mathbf{x}(t))\left(\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j + \mathbf{E}_{ij}\right)\hat{\mathbf{x}}(t) - \mathbf{F}_{ij}\mathbf{e}(t)\right) \tag{7}$$

where $\hat{\mathbf{x}}(t) \in \Re^{n\times 1}$ is the system state vector of the auxiliary system; $\mathbf{E}_{ij} \in \Re^{n\times n}$ and $\mathbf{F}_{ij} \in \Re^{n\times n}$, i, j = 1, 2, ..., p, are the constant matrices to be designed.

Remark 2: The main idea of introducing the auxiliary system is to provide richer dynamical information, e.g., $\hat{\mathbf{x}}(t)$ other than that by the nonlinear plant, i.e., $\mathbf{x}(t)$ for the purpose of achieving stable fuzzy-model-based control systems. It can be shown later on that under such a particular structure the stability condition can be relaxed by properly choosing the values of the free matrices $\mathbf{E}_{ij}$ and $\mathbf{F}_{ij}$ for all i and j in the auxiliary system of (7).

D. Published Stability Conditions

Some stability conditions have been derived to test the system stability of the fuzzy control systems in the form of $\dot{\mathbf{x}}(t) = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right)\mathbf{x}(t)$. The stability conditions are summarized in the following theorems.

***Theorem 1* [6]**: The fuzzy-model-based control system, formed by the fuzzy model of (2) and the fuzzy controller in the form of $\mathbf{u}(t) = \sum_{j=1}^{p} w_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t)$, is asymptotically stable if there exist matrices $\mathbf{P} = \mathbf{P}^{\mathrm{T}} \in \Re^{n\times n}$ and $\mathbf{M} = \mathbf{M}^{\mathrm{T}} \in \Re^{n\times n}$ such that the following LMIs hold.

$\mathbf{P} > 0$; $\mathbf{M} \geq 0$;

$\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_i\right)^{\mathrm{T}}\mathbf{P} + \mathbf{P}\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_i\right) + (s-1)\mathbf{M} < 0$, i = 1, 2, ..., p; $1 \leq s \leq p$;

$$\frac{\left(\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right) + \left(\mathbf{A}_j + \mathbf{B}_j\mathbf{G}_i\right)\right)^{\mathrm{T}}}{2}\mathbf{P} + \mathbf{P}\frac{\left(\left(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j\right) + \left(\mathbf{A}_j + \mathbf{B}_j\mathbf{G}_i\right)\right)}{2} - \mathbf{M} \leq 0$$

, j = 1, 2, ..., p;

i < j; $w_i(\mathbf{x}(t))w_j(\mathbf{x}(t)) \neq 0$ where s is an integer denoting the maximum number of fired fuzzy subsystems at an instant.

Remark 3: The stability conditions in [4] are the particular case of [6]. When $\mathbf{M} = \mathbf{0}$, the stability condition of Theorem 1 [6] are reduced to that in [4].

Theorem 2* [9], [10]*: The fuzzy-model-based control system, formed by the fuzzy model of (2) and the fuzzy controller in the form of $\mathbf{u}(t) = \sum_{j=1}^{p} w_j(\mathbf{x}(t))\mathbf{G}_j\mathbf{x}(t)$ is asymptotically stable if there exist matrices $\mathbf{P} = \mathbf{P}^{\mathrm{T}} \in \Re^{n\times n}$ and $\mathbf{P}_{ij} = \mathbf{P}_{ji}{}^{\mathrm{T}} \in \Re^{n\times n}$ such that the following LMIs hold.

$\mathbf{P} > 0$; $(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_i)^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_i) + \mathbf{P}_{ii} < 0$, *i = 1, 2, ..., p;*

$$\begin{aligned}&\left((\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j) + (\mathbf{A}_j + \mathbf{B}_j\mathbf{G}_i)\right)^{\mathrm{T}}\mathbf{P}\\ &+ \mathbf{P}\left((\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j) + (\mathbf{A}_j + \mathbf{B}_j\mathbf{G}_i)\right) + \mathbf{P}_{ij} + \mathbf{P}_{ij}{}^{\mathrm{T}} \leq 0\end{aligned}, \; j = 1,$$

2, ..., p; i < j; $\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} & \cdots & \mathbf{P}_{1p} \\ \mathbf{P}_{21} & \mathbf{P}_{22} & \cdots & \mathbf{P}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}_{p1} & \mathbf{P}_{p2} & \cdots & \mathbf{P}_{pp} \end{bmatrix} > 0$.

Remark 4: The stability conditions in [8] are the particular case of [9], [10]. When $\mathbf{P}_{ij} = \mathbf{P}_{ij}{}^{\mathrm{T}} = \mathbf{P}_{ji}$, the stability condition of Theorem 2 in [9], [10] is reduced to that in [8].

3. Stability Analysis

A fuzzy-model-based control system is formed by a nonlinear system represented by the fuzzy model of (2) and the fuzzy controller of (6) connected in closed loop. In the following analysis, the inequality of $\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = \sum_{j=1}^{p} w_j(\mathbf{x}(t)) = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i(\mathbf{x}(t))w_j(\mathbf{x}(t)) = 1$ is used. For short, $w_i(\mathbf{x}(t))$ is denoted by w_i. From (2) and (6) we have,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\left(\mathbf{G}_j\hat{\mathbf{x}}(t) + \hat{\mathbf{G}}_j\mathbf{e}(t)\right)\right) \quad (8)$$

In this paper, the objective is to derive stability conditions to ensure the system stability of the fuzzy-model-based control system of (8). The system stability will not be studied directly as it is implied by the auxiliary system and the error system. From (7) and (8), the error system is defined as,

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t)\\ &= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \left((\mathbf{A}_i + \mathbf{B}_i\hat{\mathbf{G}}_j + \mathbf{F}_{ij})\mathbf{e}(t) - \mathbf{E}_{ij}\hat{\mathbf{x}}(t)\right) \end{aligned} \quad (9)$$

From (7) and (9), we have,

$$\dot{\mathbf{z}}(t) = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \mathbf{H}_{ij}\mathbf{z}(t) \quad (10)$$

Where $\mathbf{z}(t) = \begin{bmatrix} \hat{\mathbf{x}}(t) \\ \mathbf{e}(t) \end{bmatrix}$,

$$\mathbf{H}_{ij} = \begin{bmatrix} \mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j + \mathbf{E}_{ij} & -\mathbf{F}_{ij} \\ -\mathbf{E}_{ij} & \mathbf{A}_i + \mathbf{B}_i\hat{\mathbf{G}}_j + \mathbf{F}_{ij} \end{bmatrix}.$$

Remark 5: The system stability of (10) implies that of the fuzzy control system of (8). It can be seen that $\hat{\mathbf{x}}(t) \to \mathbf{0}$ and $\mathbf{e}(t) \to \mathbf{0}$ as $t \to \infty$ when $\mathbf{z}(t) \to \mathbf{0}$. It is because $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t) \to \mathbf{0}$ as $t \to \infty$, $\hat{\mathbf{x}}(t) \to \mathbf{0}$ as $t \to \infty$ implies $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$. Hence, the stability of auxiliary system of (7) and the error system of (9) implies the system stability of (8).

Remark 6: In the traditional fuzzy-model-based control systems [3, 4], [6]-[10], the system stability is governed by the feedback gains $\mathbf{G}_j$ of the fuzzy controller only. In the proposed ASD fuzzy controller, the system stability is governed by the feedback gains $\mathbf{G}_j$ and $\hat{\mathbf{G}}_j$, and the arbitrary matrices $\mathbf{E}_{ij}$ and $\mathbf{F}_{ij}$. As a result, the conservativeness of the stability conditions can be relaxed.

To investigate the system stability of (10), the following Lyapunov function candidate is considered.

$$V(t) = \mathbf{z}(t)^{\mathrm{T}}\mathbf{P}\mathbf{z}(t) \quad (11)$$

From (10) and (11), we have,

$$\begin{aligned}\dot{V}(t) &= \dot{\mathbf{z}}(t)^{\mathrm{T}}\mathbf{P}\mathbf{z}(t) + \mathbf{z}(t)^{\mathrm{T}}\mathbf{P}\dot{\mathbf{z}}(t)\\ &= \sum_{i=1}^{p} w_i^2\mathbf{z}(t)^{\mathrm{T}}\left(\mathbf{H}_{ii}{}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{ii}\right)\mathbf{z}(t)\\ &\quad + 2\sum_{j=1}^{p}\sum_{i<j} w_i w_j \mathbf{z}(t)^{\mathrm{T}}\begin{pmatrix}\mathbf{H}_{ij}{}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{ij} \\ + \mathbf{H}_{ji}{}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{ji}\end{pmatrix}\mathbf{z}(t)\end{aligned} \quad (12)$$

Let

$$\mathbf{H}_{ii}{}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{ii} < \mathbf{P}_{ii}, \; i = 1, 2, ..., p \quad (13)$$

$$\mathbf{H}_{ij}{}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{ij} + \mathbf{H}_{ji}{}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{H}_{ji} \leq \mathbf{P}_{ij} + \mathbf{P}_{ij}{}^{\mathrm{T}}, \; j = 1, 2, ..., p; \; i < j \quad (14)$$

where $\mathbf{P}_{ii} = \mathbf{P}_{ii}{}^{\mathrm{T}} \in \Re^{2n\times 2n}$ and $\mathbf{P}_{ij} = \mathbf{P}_{ji}{}^{\mathrm{T}} \in \Re^{2n\times 2n}$.

From (12) to (14), we have,

$$\dot{V}(t) < \hat{\mathbf{z}}(t)^{\mathrm{T}}\overline{\mathbf{P}}\hat{\mathbf{z}}(t) \quad (15)$$

where $\hat{\mathbf{z}}(t)=\begin{bmatrix} w_1\mathbf{z}(t) \\ w_2\mathbf{z}(t) \\ \vdots \\ w_n\mathbf{z}(t) \end{bmatrix}$ and $\overline{\mathbf{P}}=\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} & \cdots & \mathbf{P}_{1n} \\ \mathbf{P}_{21} & \mathbf{P}_{22} & \cdots & \mathbf{P}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{P}_{n1} & \mathbf{P}_{n2} & \cdots & \mathbf{P}_{nn} \end{bmatrix}$.

It can be seen from (15) that the system of (10) is asymptotically stable if $\overline{\mathbf{P}}<0$, i.e., $\mathbf{z}(t)\rightarrow\mathbf{0}$ as $t\rightarrow\infty$. Based on Remark 5, the fuzzy-model-based control system of (8) is guaranteed to be asymptotically stable, i.e., $\mathbf{x}(t)\rightarrow\mathbf{0}$ as $t\rightarrow\infty$. The analysis result is summarized in the following Theorem.

Theorem 3: The fuzzy-model-based control system formed by the nonlinear system in the form of (2) and the ASD fuzzy controller of (6) is asymptotically stable if there exist matrices $\mathbf{P}=\mathbf{P}^{\mathrm{T}}\in\Re^{2n\times 2n}$ and $\mathbf{P}_{ij}=\mathbf{P}_{ji}{}^{\mathrm{T}}\in\Re^{2n\times 2n}$ such that the following LMIs are satisfied.

$\mathbf{P}>0$; $\overline{\mathbf{P}}<0$; $\mathbf{H}_{ii}{}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\mathbf{H}_{ii}<\mathbf{P}_{ii}$, i = 1, 2, ..., p;
$\mathbf{H}_{ij}{}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\mathbf{H}_{ij}+\mathbf{H}_{ji}{}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\mathbf{H}_{ji}\le\mathbf{P}_{ij}+\mathbf{P}_{ij}{}^{\mathrm{T}}$, j = 1, 2, ..., p; $i<j$.

Remark 7: Theorem 3 is reduced to Theorem 2 [9], [10] when $\hat{\mathbf{G}}_{ij}=\mathbf{G}_{ij}$, and $\mathbf{F}_{ij}=\mathbf{E}_{ij}=\mathbf{0}$ for all i and j.

4. Design of Feedback Gains and System Performance

In this section, the design of feedback gains and system performance is cast as LMI-based conditions which can be solved effectively using some convex programming techniques, e.g., MATLAB LMI toolbox.

A. Feedback Gains Design

Let $\begin{bmatrix} \mathbf{X} & a\mathbf{X} \\ a\mathbf{X} & \mathbf{X} \end{bmatrix}=\mathbf{P}^{-1}$, $\mathbf{G}_i=\mathbf{N}_i\mathbf{X}^{-1}$, $\hat{\mathbf{G}}_i=\mathbf{M}_i\mathbf{X}^{-1}$, $\mathbf{E}_{ij}=\mathbf{R}_{ij}\mathbf{X}^{-1}$, $\mathbf{F}_{ij}=\mathbf{S}_{ij}\mathbf{X}^{-1}$ and $\mathbf{V}_{ij}=\begin{bmatrix} \mathbf{X} & a\mathbf{X} \\ a\mathbf{X} & \mathbf{X} \end{bmatrix}\mathbf{P}_{ij}\begin{bmatrix} \mathbf{X} & a\mathbf{X} \\ a\mathbf{X} & \mathbf{X} \end{bmatrix}$ where $\mathbf{X}=\mathbf{X}^{\mathrm{T}}\in\Re^{n\times n}>0$, $\mathbf{N}_i\in\Re^{m\times n}$, $\mathbf{M}_i\in\Re^{m\times n}$, $\mathbf{E}_{ij}\in\Re^{m\times n}$, $\mathbf{F}_{ij}\in\Re^{m\times n}$ and $0\le a<0$ is a pre-defined scalar. From (13), we have,

$$\begin{bmatrix} \mathbf{X} & a\mathbf{X} \\ a\mathbf{X} & \mathbf{X} \end{bmatrix}\begin{bmatrix} \mathbf{A}_i+\mathbf{B}_i\mathbf{G}_i+\mathbf{E}_{ii} & -\mathbf{F}_{ii} \\ -\mathbf{E}_{ii} & \mathbf{A}_i+\mathbf{B}_i\hat{\mathbf{G}}_i+\mathbf{F}_{ii} \end{bmatrix}^{\mathrm{T}}$$
$$+\begin{bmatrix} \mathbf{A}_i+\mathbf{B}_i\mathbf{G}_i+\mathbf{E}_{ii} & -\mathbf{F}_{ii} \\ -\mathbf{E}_{ii} & \mathbf{A}_i+\mathbf{B}_i\hat{\mathbf{G}}_i+\mathbf{F}_{ii} \end{bmatrix}\begin{bmatrix} \mathbf{X} & a\mathbf{X} \\ a\mathbf{X} & \mathbf{X} \end{bmatrix}$$
$$<\begin{bmatrix} \mathbf{X} & a\mathbf{X} \\ a\mathbf{X} & \mathbf{X} \end{bmatrix}\mathbf{P}_{ii}\begin{bmatrix} \mathbf{X} & a\mathbf{X} \\ a\mathbf{X} & \mathbf{X} \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{X}\mathbf{A}_i{}^{\mathrm{T}}+\mathbf{N}_i{}^{\mathrm{T}}\mathbf{B}_i{}^{\mathrm{T}}+\mathbf{R}_{ii}{}^{\mathrm{T}}-a\mathbf{S}_{ii}{}^{\mathrm{T}} & -\mathbf{R}_{ii}{}^{\mathrm{T}}+a\mathbf{X}\mathbf{A}_i{}^{\mathrm{T}}+a\mathbf{M}_i{}^{\mathrm{T}}\mathbf{B}_i{}^{\mathrm{T}}+a\mathbf{S}_{ii}{}^{\mathrm{T}} \\ a\mathbf{X}\mathbf{A}_i{}^{\mathrm{T}}+a\mathbf{N}_i{}^{\mathrm{T}}\mathbf{B}_i{}^{\mathrm{T}}+a\mathbf{R}_{ii}{}^{\mathrm{T}}-\mathbf{S}_{ii}{}^{\mathrm{T}} & -a\mathbf{R}_{ii}{}^{\mathrm{T}}+\mathbf{X}\mathbf{A}_i{}^{\mathrm{T}}+\mathbf{M}_i{}^{\mathrm{T}}\mathbf{B}_i{}^{\mathrm{T}}+\mathbf{S}_{ii}{}^{\mathrm{T}} \end{bmatrix}$$
$$+\begin{bmatrix} \mathbf{A}_i\mathbf{X}+\mathbf{B}_i\mathbf{N}_i+\mathbf{R}_{ii}-a\mathbf{S}_{ii} & a\mathbf{A}_i\mathbf{X}+a\mathbf{B}_i\mathbf{N}_i+a\mathbf{R}_{ii}-\mathbf{S}_{ii} \\ -\mathbf{R}_{ii}+a\mathbf{A}_i\mathbf{X}+a\mathbf{B}_i\mathbf{M}_i+a\mathbf{S}_{ii} & -a\mathbf{R}_{ii}+\mathbf{A}_i\mathbf{X}+\mathbf{B}_i\mathbf{M}_i+\mathbf{S}_{ii} \end{bmatrix}<\mathbf{V}_{ii}$$
$$,\ i=1,2,\ldots,p \tag{16}$$

Similarly, from (14), we have,

$$\begin{bmatrix} \mathbf{X}\mathbf{A}_i{}^{\mathrm{T}}+\mathbf{N}_j{}^{\mathrm{T}}\mathbf{B}_i{}^{\mathrm{T}}+\mathbf{R}_{ij}{}^{\mathrm{T}}-a\mathbf{S}_{ij}{}^{\mathrm{T}} & -\mathbf{R}_{ij}{}^{\mathrm{T}}+a\mathbf{X}\mathbf{A}_i{}^{\mathrm{T}}+a\mathbf{M}_j{}^{\mathrm{T}}\mathbf{B}_i{}^{\mathrm{T}}+a\mathbf{S}_{ij}{}^{\mathrm{T}} \\ a\mathbf{X}\mathbf{A}_i{}^{\mathrm{T}}+a\mathbf{N}_j{}^{\mathrm{T}}\mathbf{B}_i{}^{\mathrm{T}}+a\mathbf{R}_{ij}{}^{\mathrm{T}}-\mathbf{S}_{ij}{}^{\mathrm{T}} & -a\mathbf{R}_{ij}{}^{\mathrm{T}}+\mathbf{X}\mathbf{A}_i{}^{\mathrm{T}}+\mathbf{M}_j{}^{\mathrm{T}}\mathbf{B}_i{}^{\mathrm{T}}+\mathbf{S}_{ij}{}^{\mathrm{T}} \end{bmatrix}$$
$$+\begin{bmatrix} \mathbf{A}_i\mathbf{X}+\mathbf{B}_i\mathbf{N}_j+\mathbf{R}_{ij}-a\mathbf{S}_{ij} & a\mathbf{A}_i\mathbf{X}+a\mathbf{B}_i\mathbf{N}_j+a\mathbf{R}_{ij}-\mathbf{S}_{ij} \\ -\mathbf{R}_{ij}+a\mathbf{A}_i\mathbf{X}+a\mathbf{B}_i\mathbf{M}_j+a\mathbf{S}_{ij} & -a\mathbf{R}_{ij}+\mathbf{A}_i\mathbf{X}+\mathbf{B}_i\mathbf{M}_j+\mathbf{S}_{ij} \end{bmatrix},$$
$$\begin{bmatrix} \mathbf{X}\mathbf{A}_j{}^{\mathrm{T}}+\mathbf{N}_i{}^{\mathrm{T}}\mathbf{B}_j{}^{\mathrm{T}}+\mathbf{R}_{ji}{}^{\mathrm{T}}-a\mathbf{S}_{ji}{}^{\mathrm{T}} & -\mathbf{R}_{ji}{}^{\mathrm{T}}+a\mathbf{X}\mathbf{A}_j{}^{\mathrm{T}}+a\mathbf{M}_i{}^{\mathrm{T}}\mathbf{B}_j{}^{\mathrm{T}}+a\mathbf{S}_{ji}{}^{\mathrm{T}} \\ a\mathbf{X}\mathbf{A}_j{}^{\mathrm{T}}+a\mathbf{N}_i{}^{\mathrm{T}}\mathbf{B}_j{}^{\mathrm{T}}+a\mathbf{R}_{ji}{}^{\mathrm{T}}-\mathbf{S}_{ji}{}^{\mathrm{T}} & -a\mathbf{R}_{ji}{}^{\mathrm{T}}+\mathbf{X}\mathbf{A}_j{}^{\mathrm{T}}+\mathbf{M}_i{}^{\mathrm{T}}\mathbf{B}_j{}^{\mathrm{T}}+\mathbf{S}_{ji}{}^{\mathrm{T}} \end{bmatrix}$$
$$+\begin{bmatrix} \mathbf{A}_j\mathbf{X}+\mathbf{B}_j\mathbf{N}_i+\mathbf{R}_{ji}-a\mathbf{S}_{ji} & a\mathbf{A}_j\mathbf{X}+a\mathbf{B}_j\mathbf{N}_i+a\mathbf{R}_{ji}-\mathbf{S}_{ji} \\ -\mathbf{R}_{ji}+a\mathbf{A}_j\mathbf{X}+a\mathbf{B}_j\mathbf{M}_i+a\mathbf{S}_{ji} & -a\mathbf{R}_{ji}+\mathbf{A}_j\mathbf{X}+\mathbf{B}_j\mathbf{M}_i+\mathbf{S}_{ji} \end{bmatrix}\le\mathbf{V}_{ij}+\mathbf{V}_{ij}{}^{\mathrm{T}}$$
$$j=1,2,\ldots,p;\ i<j \tag{17}$$

Remark 8: It should be noted that the stability condition is less relaxed by choosing $\begin{bmatrix} \mathbf{X} & a\mathbf{X} \\ a\mathbf{X} & \mathbf{X} \end{bmatrix}=\mathbf{P}^{-1}$. However, under such choice, it can convert the feedback gains as decision variables in the LMI conditions of (16) and (17) to ease the feedback gain design.

B. System Performance Design

The proposed fuzzy controller is designed subject to the following scalar performance index which is commonly used in the traditional optimum control problem [14].

$$J=\int_0^{\infty}\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2{}^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix}\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix}dt \tag{18}$$

where $\mathbf{J}_1=\mathbf{J}_1{}^{\mathrm{T}}\in\Re^{n\times n}>0$, $\mathbf{J}_2\in\Re^{n\times m}$, $\mathbf{J}_3=\mathbf{J}_3{}^{\mathrm{T}}\in\Re^{m\times m}>0$ and $\begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2{}^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix}>0$ are constant matrices to be determined. From (6), (18) and recalling that $\mathbf{G}_i=\mathbf{N}_i\mathbf{X}^{-1}$, $\hat{\mathbf{G}}_i=\mathbf{M}_i\mathbf{X}^{-1}$, we have,

$$J=\int_0^{\infty}\sum_{i=1}^{p}\sum_{j=1}^{p}w_iw_j\begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{G}}_i & \mathbf{G}_i-\hat{\mathbf{G}}_i \end{bmatrix}^{\mathrm{T}}\times\begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2{}^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix}\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{G}}_j & \mathbf{G}_j-\hat{\mathbf{G}}_j \end{bmatrix}\begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix}dt \tag{19}$$

Let

$$J < \eta \int_0^\infty \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{X} & a\mathbf{X} \\ a\mathbf{X} & \mathbf{X} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{X} & a\mathbf{X} \\ a\mathbf{X} & \mathbf{X} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} dt$$

where η is a non-zero positive scalar. The performance index J is attenuated to a prescribed level η. From (19), we have,

$$\begin{aligned} &\int_0^\infty \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{X} & a\mathbf{X} \\ a\mathbf{X} & \mathbf{X} \end{bmatrix}^{-1} \\ &\times \left(\begin{bmatrix} \mathbf{X} & a\mathbf{X} \\ (1-a)\mathbf{M}_i + a\mathbf{N}_i & (a-1)\mathbf{M}_i + \mathbf{N}_i \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix} \times \begin{bmatrix} \mathbf{X} & a\mathbf{X} \\ (1-a)\mathbf{M}_j + a\mathbf{N}_j & (a-1)\mathbf{M}_j + \mathbf{N}_j \end{bmatrix} - \eta \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \right) \\ &\times \begin{bmatrix} \mathbf{X} & a\mathbf{X} \\ a\mathbf{X} & \mathbf{X} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} dt < 0 \end{aligned} \tag{20}$$

It can be seen that (20) holds when the following inequality holds.

$$\begin{aligned} &\left(\sum_{i=1}^{p} w_i \begin{bmatrix} \mathbf{X} & a\mathbf{X} \\ (1-a)\mathbf{M}_i + a\mathbf{N}_i & (a-1)\mathbf{M}_i + \mathbf{N}_i \end{bmatrix} \right)^{\mathrm{T}} \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix} \\ &\times \left(\sum_{j=1}^{p} w_j \begin{bmatrix} \mathbf{X} & a\mathbf{X} \\ (1-a)\mathbf{M}_j + a\mathbf{N}_j & (a-1)\mathbf{M}_j + \mathbf{N}_j \end{bmatrix} \right) - \eta \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} < 0 \end{aligned} \tag{21}$$

By Schur complement, the inequality of (21) is equivalent to the following inequality.

$$\sum_{i=1}^{p} w_i \mathbf{W}_i < 0 \tag{22}$$

where

$$\mathbf{W}_i = \begin{bmatrix} -\eta\mathbf{I} & \mathbf{0} & \mathbf{X} & (1-a)\mathbf{M}_i^{\mathrm{T}} + a\mathbf{N}_i^{\mathrm{T}} \\ \mathbf{0} & -\eta\mathbf{I} & a\mathbf{X} & (a-1)\mathbf{M}_i^{\mathrm{T}} + \mathbf{N}_i^{\mathrm{T}} \\ \mathbf{X} & a\mathbf{X} & -\begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix}^{-1} & \\ (1-a)\mathbf{M}_i + a\mathbf{N}_i & (a-1)\mathbf{M}_i + \mathbf{N}_i & & \end{bmatrix},$$

$i = 1, 2, ..., p$ (23)

If $\mathbf{W}_i < 0$ for all i, the inequality of (22) will be satisfied. In order to constrain the element values of $\mathbf{R}_{ij}$ and $\mathbf{S}_{ij}$ to prevent them going to large values which make the auxiliary system impractical, the following constraints are imposed.

$$\mathbf{R}_{ij}^{\mathrm{T}}\mathbf{R}_{ij} \le \sigma_1 \mathbf{I},\ i, j = 1, 2, ..., p \tag{24}$$

$$\mathbf{S}_{ij}^{\mathrm{T}}\mathbf{S}_{ij} \le \sigma_2 \mathbf{I},\ i, j = 1, 2, ..., p \tag{25}$$

where σ_1 and σ_2 are non-zero positive scalars to be determined. By Schur complement, (24) and (25) are equivalent to the following LMIs,

$$\begin{bmatrix} -\sigma_1\mathbf{I} & \mathbf{R}_{ij}^{\mathrm{T}} \\ \mathbf{R}_{ij} & -\mathbf{I} \end{bmatrix} \le 0,\ i, j = 1, 2, ..., p \tag{26}$$

$$\begin{bmatrix} -\sigma_2\mathbf{I} & \mathbf{S}_{ij}^{\mathrm{T}} \\ \mathbf{S}_{ij} & -\mathbf{I} \end{bmatrix} \le 0,\ i, j = 1, 2, ..., p \tag{27}$$

The LMI conditions of $\mathbf{W}_i < 0$ for all i, (26) and (27) are the performance conditions. It should be noted that the performance conditions govern the system performance only. The system stability will not be affected by dissatisfaction of the performance conditions. The design of feedback gains and system performance subject to the system stability can be cast as a GEVP (generalized eigenvalue problem) and are summarized in the following theorem.

Theorem 4: The fuzzy-model-based control system form by the nonlinear system in the form of (2) and the ASD fuzzy controller of (6) is asymptotically stable if there exist a scalar $0 \le a < 1$, non-zero positive scalars η, σ_1 and σ_2, and matrices $\mathbf{X} \in \Re^{n\times n}$, $\mathbf{V}_{ij} \in \Re^{2n\times 2n}$, $\mathbf{J}_1 \in \Re^{n\times n}$, $\mathbf{J}_2 \in \Re^{n\times m}$ and $\mathbf{J}_3 \in \Re^{m\times m}$ such that the following GEVP is satisfied.

$\underset{\mathbf{X},\mathbf{M}_j,\mathbf{N}_j,\mathbf{R}_{ij},\mathbf{S}_{ij},\mathbf{V}_{ij}}{\text{minimize}}\ \eta$ subject to

$\mathbf{X} = \mathbf{X}^{\mathrm{T}} > 0$; $\begin{bmatrix} \mathbf{X} & a\mathbf{X} \\ a\mathbf{X} & \mathbf{X} \end{bmatrix} > 0$, (16), (17); $\mathbf{J}_1 = \mathbf{J}_1^{\mathrm{T}} > 0$; $\mathbf{J}_3 = \mathbf{J}_3^{\mathrm{T}} > 0$; $\begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_2^{\mathrm{T}} & \mathbf{J}_3 \end{bmatrix} > 0$; $\mathbf{W}_i < 0$, i = 1, 2, ..., p; (26) and (27).

Remark 9: It should be noted that the values of a, σ_1, σ_2, $\mathbf{J}_1$, $\mathbf{J}_2$ and $\mathbf{J}_3$ have to be determined prior to applying Theorem 4. To determine the value of a, searching algorithm combined with convex programming technique can be employed.

5. Simulation Examples

Two examples are given in this section to illustrate the effectiveness of the proposed approaches.

A. Example 1: Numerical Example

The same numerical example used in [8] is employed to show the merits of the proposed approach. A simple fuzzy model with the following two fuzzy rules is considered.

Rule i: IF $x_1(t)$ is M_1^i THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)$, $i = 1, 2$ (28)

where $\mathbf{A}_1 = \begin{bmatrix} 2 & -10 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{A}_2 = \begin{bmatrix} r & -10 \\ 1 & 1 \end{bmatrix}$; $\mathbf{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{B}_1 = \begin{bmatrix} s \\ 0 \end{bmatrix}$. The feedback gains $\mathbf{G}_1$ and $\mathbf{G}_2$ are designed under the same parallel distributed compensation (PDC) design criteria of [8] such that the eigenvalues of $\mathbf{A}_1 + \mathbf{B}_1\mathbf{G}_1$ and $\mathbf{A}_2 + \mathbf{B}_2\mathbf{G}_2$ are located at –1 and –15. The stability conditions of Theorem 1 to Theorem 3 are employed to testify the stability of the closed-loop system. Fig. 1 shows the stable regions (indicated by “o”) for the stability conditions of Theorem 1 to Theorem 3 respectively for $2 \le r \le 6$ and $5 \le s \le 16$. For the proposed approach, the feedback gains $\hat{\mathbf{G}}_j$ are set to zero. As a result, the ASD fuzzy controller becomes $u(t) = \sum_{j=1}^{2} w_j \mathbf{G}_j \hat{\mathbf{x}}(t)$. The matrices $\mathbf{E}_{ij}$ and $\mathbf{F}_{ij}$ are designed heuristically such that the eigenvalues of $\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j + \mathbf{E}_{ij}$ and $\mathbf{A}_i + \mathbf{F}_{ij}$ for all i and j are located at –150 and –200 respectively. It can be seen from Fig. 1 that Theorem 3 provides larger stable region than those of Theorem 1 and Theorem 2. The relaxation in the proposed stability conditions are granted by the freely design matrices $\mathbf{E}_{ij}$ and $\mathbf{F}_{ij}$ of the auxiliary system.

Furthermore, the stability conditions can be furthered relaxed if the ASD fuzzy controller is designed as $u(t) = \sum_{j=1}^{2} w_j \left(\mathbf{G}_j \hat{\mathbf{x}}(t) + \hat{\mathbf{G}}_j \mathbf{e}(t) \right)$. The stable region offered by this ASD fuzzy controller is shown in Fig. 1. Under this fuzzy controller, we choose the same $\mathbf{G}_j$, $\mathbf{E}_{ij}$ and $\mathbf{F}_{ij}$. The feedback gain $\hat{\mathbf{G}}_j$ is chosen heuristically such that the eigenvalues of $\mathbf{A}_i + \mathbf{B}_i \hat{\mathbf{G}}_i$ for all i are located at –1 and –20 respectively. It can be seen that the stable region can be furthered enlarged.

In this simple numerical example, it can be seen that the proposed ASD approach is able to produce stability conditions relaxed than those of the existing ones [6], [8]-[10]. The relaxation of stability conditions is mainly due to the introduction of the freely designed matrices of $\mathbf{E}_{ij}$ and $\mathbf{F}_{ij}$ in the auxiliary system which provides enriched information to enhance the stabilization ability of the fuzzy controller. Comparing the proposed ASD stability conditions with those in Theorem 1 and Theorem 2, it can be seen that the ASD stability conditions are complicated comparatively. However, the merits of the ASD control approach is shown when the existing stability conditions in Theorem 1 or Theorem 2 fail to provide a stable design for fuzzy controller.

B. Example 2: Inverted Pendulum

An application example on stabilizing a cart-pole typed inverted pendulum [15] using the proposed fuzzy controller is given in this section.

Step I) The dynamic equations of the inverted pendulum on the cart [15] is given by,

$$\dot{x}_1(t) = x_2(t) \tag{29}$$

$$\dot{x}_2(t) = \frac{\begin{pmatrix} -F_1(M+m)x_2(t) - m^2l^2x_2(t)^2 \sin x_1(t) \cos x_1(t) \\ +F_0 mlx_4(t)\cos x_1(t) + (M+m)mgl \sin x_1(t) \\ -ml\cos x_1(t)u(t) \end{pmatrix}}{(M+m)(J+ml^2) - m^2l^2(\cos x_1(t))^2} \tag{30}$$

$$\dot{x}_3(t) = x_4(t) \tag{31}$$

$$\dot{x}_4(t) = \frac{\begin{pmatrix} F_1 mlx_2(t)\cos x_1(t) + (J+ml^2)mlx_2(t)^2 \sin x_1(t) \\ -F_0(J+ml^2)x_4(t) - m^2gl^2 \sin x_1(t)\cos x_1(t) \\ +(J+ml^2)u(t) \end{pmatrix}}{(M+m)(J+ml^2) - m^2l^2(\cos x_1(t))^2} \tag{32}$$

where $x_1(t)$ and $x_2(t)$ denote the angular displacement (rad) and the angular velocity (rad/s) of the pendulum from vertical respectively, $x_3(t)$ and $x_4(t)$ denote the displacement (m) and the velocity (m/s) of the cart respectively, $g = 9.8$ m/s^2 is the acceleration due to gravity, $m = 0.22$ kg is the mass of the pendulum, $M = 1.3282$ kg is the mass of the cart, $l = 0.304$ m is the length from the center of mass of the pendulum to the shaft axis, $J = ml^2/3$ kgm^2 is the moment of inertia of the pendulum round the center of mass, $F_0 = 22.915$ N/m/s and $F_1 = 0.007056$ N/rad/s are the friction factors of the cart and the pendulum respectively, and u(t) is the force (N) applied to the cart. The objective is to design a fuzzy controller to close the feedback loop such that $x_1(t) = x_3(t) = 0$ at steady state. The nonlinear plant can be represented by a fuzzy model with two fuzzy rules [15]. The i-th rule is given by,

$$\text{Rule } i: \text{ IF } x_1(t) \text{ is } \mathrm{M}_1^i \text{ THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t) \quad \text{for } i = 1, 2 \tag{33}$$

The system dynamics are described by,

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{2} w_i(x_1)\left(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i u(t)\right) \tag{34}$$

where $\mathbf{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{bmatrix}^{\mathrm{T}}$;

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ (M+m)mgl/a_1 & -F_1(M+m)/a_1 & 0 & F_0 ml/a_1 \\ 0 & 0 & 1 & 0 \\ -m^2gl^2/a_1 & F_1 ml/a_1 & 0 & -F_0(J+ml^2)/a_1 \end{bmatrix},$$

$$\mathbf{A}_2=\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{3\sqrt{3}}{2\pi}(M+m)mgl/a_2 & -F_1(M+m)/a_2 & 0 & F_0 ml\cos(\pi/3)/a_2 \\ 0 & 0 & 1 & 0 \\ -\frac{3\sqrt{3}}{2\pi}m^2gl^2\cos(\pi/3)/a_2 & F_1 ml\cos(\pi/3)/a_2 & 0 & -F_0(J+ml^2)/a_1 \end{bmatrix},$$

$$\mathbf{B}_1=\begin{bmatrix} 0 \\ -ml/a_1 \\ 0 \\ (J+ml^2)/a_1 \end{bmatrix}, \quad \mathbf{B}_2=\begin{bmatrix} 0 \\ -ml\cos(\pi/3)/a_2 \\ 0 \\ (J+ml^2)/a_2 \end{bmatrix};$$

$$a_1=(M+m)(J+ml^2)-m^2l^2,$$

$$a_2=(M+m)(J+ml^2)-m^2l^2\cos(\pi/3)^2.$$

$w_i(x_1)=\dfrac{\mu_{\mathrm{M}_1^i}(x_1(t))}{\sum_{l=1}^{2}\mu_{\mathrm{M}_1^l}(x_1(t))}$. The membership functions are defined as $\mu_{\mathrm{M}_1^1}(x_1(t))=\left(1-\dfrac{1}{1+e^{-7(x_1(t)-\pi/6)}}\right)\dfrac{1}{1+e^{-7(x_1(t)+\pi/6)}}$ and $\mu_{\mathrm{M}_1^2}(x_1(t))=1-\mu_{\mathrm{M}_1^1}(x_1(t))$.

Step II) From (6), a two-rule ASD fuzzy controller is proposed to handle the nonlinear plant described by (29) to (32). The fuzzy rules are of the following format.

Rule j: IF $x_1(t)$ is M_1^j

THEN $u(t)=\mathbf{G}_j\hat{\mathbf{x}}(t)$, j = 1, 2 (35)

where $\hat{\mathbf{x}}(t)$ is the system state vector of the auxiliary system. The fuzzy controller is described by,

$$u(t)=\sum_{j=1}^{2}w_i(x_1(t))\mathbf{G}_j\hat{\mathbf{x}}(t) \tag{36}$$

Step III) From (7), the auxiliary system is defined as follows,

$$\dot{\hat{\mathbf{x}}}(t)=\sum_{i=1}^{2}\sum_{j=1}^{2}w_i(x_1(t))w_j(x_1(t))\left(\left(\mathbf{A}_i+\mathbf{B}_i\mathbf{G}_j+\mathbf{E}_{ij}\right)\hat{\mathbf{x}}(t)-\mathbf{F}_{ij}\mathbf{e}(t)\right) \tag{37}$$

where $\hat{\mathbf{x}}(t)=\left[\hat{x}_1(t) \quad \hat{x}_2(t) \quad \hat{x}_3(t) \quad \hat{x}_4(t)\right]^{\mathrm{T}}$.

Step IV) To apply the stability conditions in Theorem 4, we choose a = 0.9, $\sigma_1=\sigma_2=5\times10^{-2}$, $\mathbf{J}_1=\begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{J}_2=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{J}_3$ = 1. With the help of the LMI toolbox, we have the following solution such that the LMI-based stability and performance conditions in Theorem 4 are satisfied: $\eta=2.3744\times10^{-3}$, $\mathbf{G}_1$ = [152.6526 138.4046 9.3624 27.2715], $\mathbf{G}_2$ = [285.0681 210.4750 9.5582 31.2629],

$$\mathbf{X}=\begin{bmatrix} 4.5914\times10^{-4} & -5.2382\times10^{-4} & -7.1517\times10^{-6} & 2.6796\times10^{-4} \\ -5.2382\times10^{-4} & 9.5049\times10^{-4} & 2.4964\times10^{-5} & -9.6155\times10^{-4} \\ -7.1517\times10^{-6} & 2.4964\times10^{-5} & 4.8573\times10^{-4} & -3.6079\times10^{-4} \\ 2.6796\times10^{-4} & -9.6155\times10^{-4} & -3.6079\times10^{-4} & 4.2584\times10^{-3} \end{bmatrix},$$

$$\mathbf{E}_{11}=\begin{bmatrix} -807.3512 & -611.5968 & -58.3201 & 171.8416 \\ -514.5580 & -565.6713 & -65.2635 & 110.3249 \\ -32.9824 & -54.8844 & -296.5217 & -7.9792 \\ -63.1965 & -96.6423 & -36.6932 & -0.3827 \end{bmatrix},$$

$$\mathbf{E}_{12}=\mathbf{E}_{21}=\begin{bmatrix} -379.3553 & -659.9579 & -38.0220 & 319.6500 \\ -276.4073 & -542.3499 & -39.8362 & 231.3937 \\ -16.1258 & -48.4644 & -145.8163 & 11.7949 \\ -31.5599 & -84.6200 & -19.2550 & 28.5140 \end{bmatrix},$$

$$\mathbf{E}_{22}=\begin{bmatrix} -720.2976 & -1157.6048 & -57.4796 & 471.6978 \\ -518.5854 & -948.0628 & -61.7292 & 293.6989 \\ -29.2501 & -80.0927 & -296.0778 & 1.8052 \\ -55.8863 & -148.4248 & -35.8437 & 18.2086 \end{bmatrix},$$

$$\mathbf{F}_{11}=\begin{bmatrix} -883.5120 & -594.1793 & -25.3353 & -204.0899 \\ -573.2861 & -555.2208 & -34.4713 & -223.7528 \\ -39.6470 & -49.4909 & -293.6082 & -42.2598 \\ -75.1817 & -98.4760 & -31.7760 & -58.8412 \end{bmatrix},$$

$$\mathbf{F}_{12}=\mathbf{F}_{21}=\begin{bmatrix} -611.8769 & -446.6596 & -8.6968 & -122.4928 \\ -436.2836 & -367.4054 & -13.9369 & -129.8017 \\ -32.6150 & -28.5988 & -143.3728 & -23.7871 \\ -62.1085 & -57.9209 & -15.2341 & -31.7313 \end{bmatrix},$$

$$\mathbf{F}_{22}=\begin{bmatrix} -1119.4936 & -675.8572 & -30.9725 & -61.2020 \\ -785.9431 & -569.9311 & -39.8144 & -115.0003 \\ -55.9992 & -45.3529 & -294.0496 & -36.3434 \\ -106.0169 & -90.4394 & -32.6168 & -46.5168 \end{bmatrix}.$$

It can be shown that the nonlinear system can be stabilized by the proposed ASD fuzzy controller. Furthermore, in order to show the effect of the performance conditions, the feedback gains and the system parameters of the auxiliary system are obtained with $\mathbf{J}_1=\begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{J}_1=\begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{J}_1=\begin{bmatrix} 1000 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ respectively with other parameters being unchanged. It can be seen that these $\mathbf{J}_1$s put different weights on $x_1(t)$ and $x_3(t)$. Fig. 2 shows the system state responses and the control signals of the fuzzy-model-based control systems under the initial system state conditions of $\mathbf{x}(0)=\hat{\mathbf{x}}(0)=\begin{bmatrix}\frac{7\pi}{18} & 0 & 0 & 0\end{bmatrix}^T$. Referring to this figure, it can be seen that the fuzzy control system with $\mathbf{J}_1=\begin{bmatrix}1000 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1000 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$, which places heaviest weights on $x_1(t)$ and $x_3(t)$, offers the best performance in terms of transient response and settling time.

Through this example, it can be seen that the proposed ASD fuzzy controller can stabilize the nonlinear plant with the used of the derived stability conditions. Furthermore, the system performance can be designed by properly assigning the values of $\mathbf{J}_1$, $\mathbf{J}_2$ and $\mathbf{J}_3$. In this example, for seek of simple structural complexity, the ASD fuzzy controller of (36) does not use the error state $\mathbf{e}(t)$. It is revealed in example 1 that the ASD fuzzy controller with $\mathbf{e}(t)$ can further enhance its stabilization ability. Hence, when stable design of ASD fuzzy controller of (36) cannot be achieved, the ASD fuzzy controller with $\mathbf{e}(t)$ should be employed.

For comparison purpose, the obtained feedback gains, $\mathbf{G}_j$, of the ASD fuzzy controller are put to the stability conditions in Theorem 2 to verify the system stability. It can be shown that no feasible solution is achieved. As a result, it can be seen that the stabilization ability of the proposed fuzzy controller is not only granted by the feedback gains but also from the information given by the auxiliary system. The effectiveness of the proposed control approach and stability conditions can thus be reflected. In the following, to differentiate the feedback gains in Theorem 2 and Theorem 3, the feedback gains in Theorem 2 are denoted as $\mathbf{K}_j$. In order to apply Theorem 2 to design the feedback gains automatically, the feedback gains are designed as $\mathbf{K}_j=\mathbf{N}_j\mathbf{X}^{-1}$, $j=1, 2$. By putting this design into Theorem 2, the feedback gains can be determined by the convex programming techniques. Furthermore, putting $a=1$ to the performance conditions in Theorem 3, they become the performance conditions for Theorem 2. The fuzzy controller for Theorem 2 is defined as $u(t)=\sum_{j=1}^{2} w_j\mathbf{K}_j\mathbf{x}(t)$ [9]-[10]. Considering $\mathbf{J}_1=\begin{bmatrix}10 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 10 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$ and all other parameter values unchanged, the feedback gains are obtained as $\mathbf{K}_1=[3918.9683 \quad 310.4852 \quad 54.6481 \quad 289.3873]$ and $\mathbf{K}_2=[1988.0786 \quad 155.1525 \quad 24.5723 \quad 140.5409]$ by solving the solution of the stability conditions in Theorem 2 and the corresponding performance conditions. Fig. 3 shows the system state responses and control signal. It can be seen that this fuzzy controller provides a better transient response at the cost of large magnitude of control signal with peak magnitude around 2054.4922 N. Compared with that of the proposed control approach, the ASD controller produces less control power to realize the control process.

6. Conclusion

An auxiliary-state-driven fuzzy controller has been proposed to handle nonlinear systems. With a proper design, the system stability of the fuzzy-model-based control system is implied by the auxiliary system and the error system. The auxiliary system offers some freely designed matrices contributed to the system stability. With these freely designed matrices, the stability conditions can be relaxed compared with some existing ones. Based on the Lyapunov-based approach, LMI-based stability and performance conditions have been derived. Simulation examples have been given to illustrate the merits of the proposed approach.

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7. References

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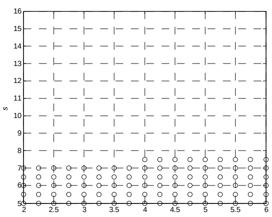

Fig. 1(a).

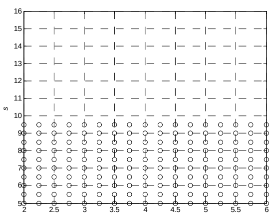

Fig. 1(b).

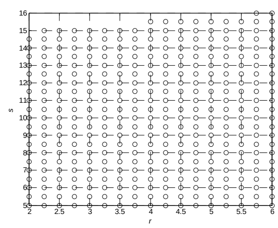

Fig. 1(c).

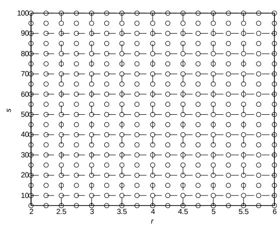

Fig. 1(d).

Fig. 1. Stable regions of different theorems. (a). Theorem 1. (b). Theorem 2. (c). Theorem 3 without $\hat{\mathbf{G}}_j$. (d) Theorem 3 with $\hat{\mathbf{G}}_j$.

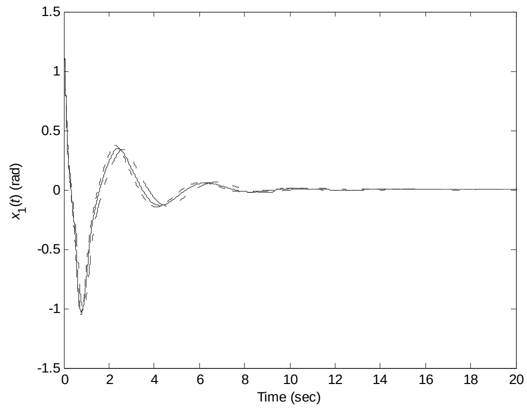

Fig. 2(a). $x_1(t)$.

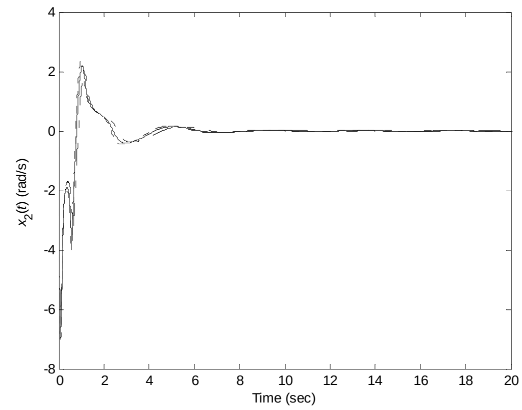

Fig. 2(b). $x_2(t)$.

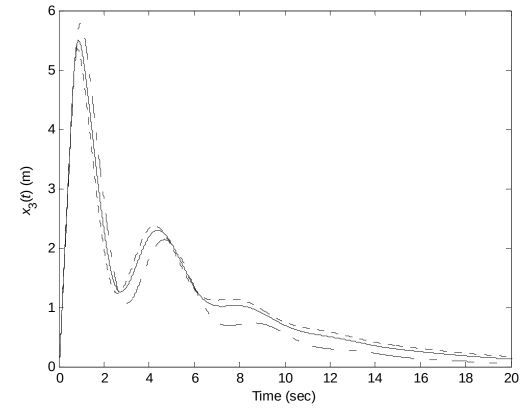

Fig. 2(c). $x_3(t)$.

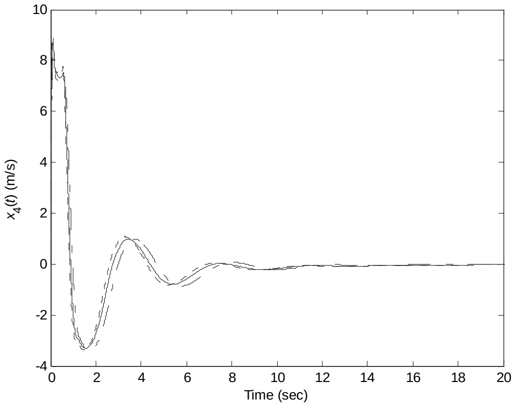

Fig. 2(d). $x_4(t)$.

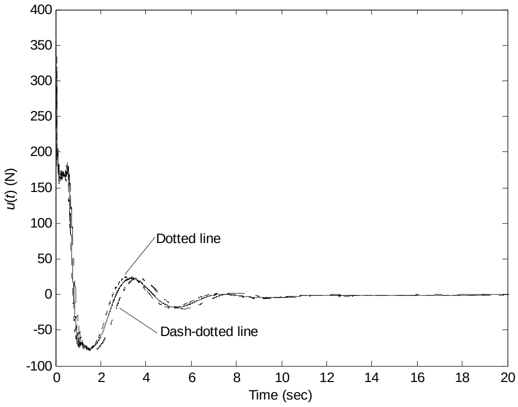

Fig. 2(e). $u(t)$.

Fig. 2. System responses of the ASD fuzzy control system (solid lines) and the auxiliary system (dotted lines), and the control signal of the ASD fuzzy controller under $\mathbf{J}_1 = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (dotted lines), $\mathbf{J}_1 = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (solid lines) and $\mathbf{J}_1 = \begin{bmatrix} 1000 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (dash-dotted lines) with the same $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{J}_3 = 1$.

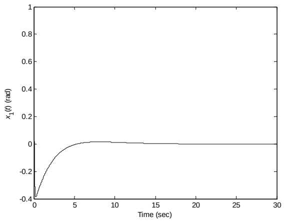

Fig. 3(a). $x_1(t)$.

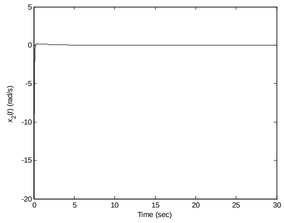

Fig. 3(b). $x_2(t)$.

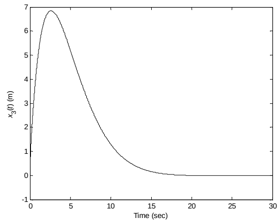

Fig. 3(c). $x_3(t)$.

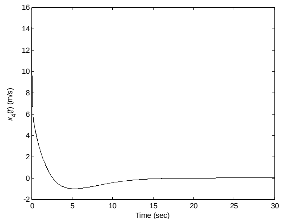

Fig. 3(d). $x_4(t)$.

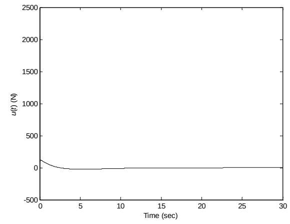

Fig. 3(e). $u(t)$.

Fig. 3. System responses and control signal for the nonlinear system with the published fuzzy controller under $\mathbf{J}_1 = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{J}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{J}_3 = 1$.

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