Containment and Specificity for Type-2 Fuzzy Sets

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Abstract

We describe fuzzy sets of type-2 and look at the definition of some of the available operations. With the aid of the extension principle we suggest some operations for implementing the concept of containment between fuzzy sets of type-2. We discuss the idea of specificity of a fuzzy set. We suggest a procedure for calculating the specificity of a interval value type-2 fuzzy subset. We show the uniqueness of this procedure. We then extend our method to the case of a general type-2 fuzzy subset.

1. Introduction

Beginning with his fuzzy set based theory of approximate reasoning [1-4] and continuing to his recent work on the generalized theory of uncertainty [5] Zadeh has developed a framework/system for reasoning with information expressed in a natural language. A central requirement of such a system is the ability to make deductions and inferences. One important deduction rule in Zadeh's generalized theory of approximate reasoning is the entailment principle [4-8], this principle is closely related to the binary logic law of addition which allows one to infer A or B from the knowledge of A. Effectively the entailment principle allows one to infer less precise information from more precise information. Central to the formalization and implementation of deduction in the theory of approximate is the fuzzy set operation of containment. In addition the concept of specificity [9], via the principle of minimal specificity [10, 11], plays an important role in the theory of approximate reasoning.

Recent interest has focused on the use of non-standard fuzzy sets as a way of representing linguistic, imprecise and granular information. Particularly notable here is the interest in fuzzy subsets of type-2 and especially the case of type-2 fuzzy subsets with interval valued membership grades [12-18]. Our focus here is to support the use of fuzzy subsets of type-2 within the framework of Zadeh's generalized theory of approximate reason by extending the operation of containment and the concept of specificity to the case of fuzzy subsets oftype-2.

2. The Extension Principle

Central to working with fuzzy subsets of type-2 is Zadeh's extension principle [1, 19-21]. This principle allows us to extend operators defined on elements of a space to act on fuzzy subsets of the space. In the following we define various manifestations of this principle. We shall let X, Y and Z be three sets.

Assume f: X → Y, it maps an element in X to an element in Y, f(x) = y. Let A be a fuzzy subset of X, $A = \underset{x \in X}{\cup} \{\frac{A(x)}{x}\}$. Using the extension principle we obtain f(A) = G where G is the fuzzy subset of Y such that $G = \underset{x \in X}{\cup} \{\frac{A(x)}{f(x)}\}$. An alternative description of G is in terms of its membership function, $G(y) = \underset{\substack{x \text{ s.t.} \\ y = f(x)}}{\text{Max}} [A(x)]$.

A simple example of this is f: R → R where $f(x) = 2x + 3$. In this case $f(A) = \{\frac{A(x)}{2x+3}\}$. Another example is $f(x) = x^2$ here $f(A) = \{\frac{A(x)}{x^2}\}$.

Assume f: X × Y → Z, it maps pairs of (x, y) to point in points of Z, f(x, y) = z. Let A and B be fuzzy subsets of X and Y respectively then using the extension principle F(A,B) is a fuzzy subset G of Z such that $f(A, B) = G = \underset{(x, y) \in X \times Y}{\cup} \{\frac{A(x) \wedge B(y)}{f(x, y)}\}$. Alternatively we express G in terms of its membership function, $G(z) = \underset{\substack{(x, y) \text{ s.t.} \\ y = f(x, y)}}{\text{Max}} [A(x) \wedge B(y)]$. Many binary operators can be extended using the version of the extension principle. This is particularly useful for arithmetic operators [22].

More generally let f: $X_1 \times X_2 \times .. . \times X_n \to Z$, $f(x_1, ..., x_n) = z$. Assume A_i, i= 1 to n, are fuzzy subsets of X_i then $f(A_1, ..., A_n)$ is a fuzzy subset of Z such

$$f(A_1, ..., A_n) = \underset{(x_1, ..., x_n) \in X_1 \times .. \times X_n}{\cup} \{\frac{\text{Min}_i [A_i(x_i)]}{f(x_1, ..., x_n)}\}$$

There exists another example of the extension princi-

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ple which provides for the extension of set operators. Prior to defining this we shall introduce some concepts that are needed. Assume A is a fuzzy subset of X. For any $\alpha \in [0, 1]$ we define $D = \alpha \rightarrow A$ as the fuzzy of X such $D(x) = \alpha\, A(x)$, we multiply the membership grades of A by α. In the special case where A is a crisp subset, a fuzzy subset with membership grades either 1 or 0, then D is a fuzzy subset such $D(x) = \alpha$ if $x \in A$ and $D(x) = 0$ if $x \notin A$.

We now recall the concept of α-level sets. Let A be a fuzzy subset of X the α level set of A is a crisp subset A_α of X such that $A_\alpha = \{x/A(x) \geq \alpha\}$.

Combining the preceding two allows us to obtain the resolution identity. Any fuzzy subset A of X can be expressed as

$$A = \bigcup_{\alpha=0}^{1} \alpha \rightarrow A_\alpha$$

We now can define the set version of the extension principle. Again let X and Y be two sets. Let H be a mapping that takes ordinary subsets of X into Y, H: $2^X \rightarrow Y$. It is a set mapping. We now extend H to act on fuzzy subsets of X. Let A be a fuzzy subset of X we define

$$H(A) = \bigcup_{\alpha=0}^{1} \{\frac{\alpha}{H(A_\alpha)}\}.$$

3. Fuzzy Subsets of Type-2

A fuzzy subset of type-2 is a fuzzy set whose membership grades are fuzzy subsets of the unit interval. In the following we shall denote the unit interval [0, 1] as I. Thus if A is a type-2fuzzy subset of X then for any $x \in X$ its membership grade, A(x), is a fuzzy subset of I. At times, for notational convenience, we shall express the fuzzy subset A(x) as A_X. Using this for any $r \in [0,1]$, $A_X(r)$ is the membership grade of r in the fuzzy subset A_X. An important special class of type-2 memberships grades are interval valued. These have been extensively studied by Mendel and Turksen [14, 18]. A membership grade A_X is said to be interval valued if there exists some interval $[a, b] \in [0, 1]$ such that $A_X(r) = 1$ for $r \in [a, b]$ and $A_X(r) = 0$ for $r \notin [a, b]$. We shall find it convenient to denote such interval valued membership grades simply as [a, b]. There two notable examples of interval valued membership grades. The first, [a, a], corresponds to a precise membership grade of a. In this case A(x) = a. It is a fuzzy subset where $A_X(r) = 1$ for r = a and $A_X(r) = 0$ for $r \neq a$. The other notable case is $A_X = [0, 1]$. This corresponds to the case where the membership grade of x in A is completely unknown. Another interesting interval membership grade is $A_X = [\alpha, 1]$ this is can be expressed as "at last α". Another interesting membership grade is $A_X = [0, \alpha]$ this is "at most α."

It should be clear that an ordinary fuzzy subset can be viewed as a type-2 with membership [A(x), A(x)].

We now look at some operations on type-2 fuzzy sets. Here considerable use will be made of the extension principle. In the following we shall refer to ordinary fuzzy subsets simply as fuzzy subsets and use the qualification type-2 when we want to denote a fuzzy subset of type-2.

Assume H is a fuzzy subset of X, its negation $\overline{H}$ is the fuzzy subset of X such that $\overline{H}(x) = 1 - H(x)$. We extend this operation to type-2 fuzzy subsets using the extension principle. Let A be a type-2 fuzzy subset its negation $\overline{A}$ is a type-2 fuzzy set of X such that

$$\overline{A}(x) = \overline{A}_x = \bigcup_{r=0}^{1} \{\frac{A_x(r)}{1-r}\}.$$

Here the fuzzy subset A_X is the membership grade of x in A and $A_X(r)$ is the membership grade of r in A_X. We note that $\overline{A}_X$ can alternatively be expressed as $\overline{A}_x = \bigcup_{r=0}^{1} \{\frac{A_x(1-r)}{r}\}$

For the case where A_X interval valued, $A_X = [a, b]$, then $\overline{A}_X = [1 - b, 1 - a]$. Thus the negation of an interval valued membership grade is still internal valued. We note the following special cases of negation of interval values:

$A_X = [a, a]$	$\overline{A}_X = [1 - a, 1 - a]$
$A_X = [0, 1]$	$\overline{A}_X = [0, 1]$
$A_X = [a, 1]$	$\overline{A}_X = [0, 1 - a]$
$A_X = [0, a]$	$\overline{A}_X = [1 - a, 1]$

Assume H and G are fuzzy subsets of X their interaction, $T = G \cap H$ is a fuzzy subset of X such that $T(x) = \text{Min}[G(x), H(x)] = G(x) \wedge H(x)$. We now extend this to type-2 fuzzy subsets. Let A and B be type-2 fuzzy subsets there intersection is a type-2fuzzy set $D = A \cap B$ where

$$D(x) = D_x = \bigcup_{r_1, r_2 \in I^2} \{\frac{A_x(r_1) \wedge B_x(r_2)}{r_1 \wedge r_2}\}$$

For the case of interval valued membership grades, $A_X = [L_a, U_a]$ and $B_X = [L_b, U_b]$, we get

$$D_X = [L_a \wedge L_b, U_a \wedge U_b]$$

It's worth noting that if $A_X = [0, 1]$ then $D_X = [0, U_b]$, *at most* U_b, and if $A_X = [L_a, 1]$ then $D_X = [L_a \wedge L_b, U_b]$

For the case of union $S = G \cup H$ with $S(x) = \text{Max}[G(x), H(x)] = G(x) \vee H(x)]$. The extension to type-2 follows in a similar way to the above. That is $E = A \cup B$ is type-2 with

$$E(x) = E_x = \bigcup_{r_1, r_2 \in I^2} \{\frac{A_x(r_1) \wedge B_x(r_2)}{r_1 \vee r_2}\}$$

For the interval case where $A_x = [L_a, U_a]$ and $B_x = [L_b, U_b]$ then $D_x = [L_a \vee L_b, U_a \vee U_b]$. In the special case where $A_x = [0, 1]$ then $D_x = [L_b, 1]$, *at least L_b*. If $A_x = [L_a, 1]$ then $D_x = [L_a \vee L_b, 1]$. If $A_x = a$ then $D_x = [L_b \vee a, U_b \vee a]$

Actually we can provide a more general result that is particularly useful in the case of interval valued type-2 membership grades. Assume G and H are two fuzzy subset of X. Let R be a fuzzy subset of X denoted $R = G \perp H$ where $R(x) = F(G(x), H(x))$. Here we shall assume that F is monotonic and continuous, $F(a_1, b_1) \geq F(a_2, b_2)$ if $a_1 \geq a_2$ and $b_1 \geq b_2$. We can extend this to type–2. If A and B are type-2then the extension is $D = A \perp B$ where

$$D(x) = D_x = \bigcup_{r_1, r_2 \in I^2} \left\{ \frac{A_x(r_1) \wedge B_x(r_2)}{F(r_1, r_2)} \right\}$$

Consider now the case of interval valued membership grades, $A_x = [L_a, U_a]$ and $B_x = [L_b, U_b]$. We see that in this case $D_x = \bigcup_{\substack{r_1 \in [L_a, U_a] \\ r_2 \in [L_b, U_b]}} \left\{ \frac{1}{F(r_1, r_2)} \right\}$. From the monotonicity and continuity of F we get the simple result that

$$D_x = [F(L_a, L_b), F(U_a, U_b)]$$

Furthermore if F is not continuous but just monotonic then while D_x is not necessarily an interval we can say it is a crisp subset such that $D_x \subseteq [F(L_a, L_b), F(U_a, U_b)]$.

We recall t-norms [23] provide a generalization of the Min operation used to define intersection. Thus $P = G \cap H$ can be defined by $P(x) = T(G(x), H(x))$ where T is a t-norm. Similarly t-conorms provide a generation of the Max operator used to define union. Thus $Q = G \cup H$ can be defined by $Q(x) = S(G(x), H(x))$. Since t-norms and t-conorms are by definition monotonic and often are continuous the preceding result allows us easily to implement aggregations using these operators.

Another interesting special case of F can be considered. We shall say F is **mixed monotonic** if it satisfies the conditions

$$F(a, c) \geq F(b, c) \qquad \text{if } a > b$$
$$F(c, a) \leq F(c, b) \qquad \text{if } a > b$$

Here F is monotonically increasing in the first argument and monotonically decreasing in the second argument. We shall additionally assume that F is also continuous. We now consider interval membership grades for this class of operators. Thus here $A_x = [L_a, U_a]$ and $B_x = [L_b, U_b]$. It can easily be shown in this case that

$$D_x = [F(L_a, U_b), F(U_a, L_b)]$$

Here we combine the smallest element in A_x with the largest element in B_x to get the lower bound of D_x and we combine the largest value in A_x with the lowest value in B_x to get the upper bound of D_x. In the case where F is not continuous but still mixed monotonic we get that D_x is still a crisp subset of the unit interval but we can only conclude that $D_x \subseteq [F(L_a, U_b), F(U_a, L_b)]$.

We now consider the special cases where F is some arithmetic operation: addition, subtraction, multiplication and division. When necessary we shall let F_+, F_-, F_x, $F_/$ respectively denote the operation. We now consider the extension to type-2 sets, $G = A \perp B$, where A and B are type-2 and $\perp$ is an arithmetic operation. In this case as in the preceding

$$D(x) = D_x = \bigcup_{r_1, r_2} \left\{ \frac{A_x(r_1) \wedge B_x(r_2)}{F(r_1, r_2)} \right\}$$

Before proceeding we note that extension of arithmetic operations to ordinary fuzzy sets have been investigated by Dubois and Prade [24] in considerable detail. Our particular interest here is in the case where A_x and B_x are interval fuzzy sets, $A_x = [L_a, U_a]$ and $B_x = [L_b, U_b]$

Let us consider the individual arithmetic operators. First we consider addition $\perp = +$ $(F = F_+)$. Since addition is both a monotonic and continuous operator then using our earlier result

$$D(x) = [L_a + L_b, U_a + U_b]$$

In the situation when all the numbers are non-negative, as is the case when the intervals lie in the unit interval, multiplication is also monotonic as well as continuous. Thus if $\perp = *$, $(F = F*)$ then $D(x) = [L_a * L_b, U_a * U_b]$

Subtraction on the other hand is mixed monotonic thus if $A_x - B_x = [L_a, U_a] - [L_b, U_b]$ then $A_x - B_x = [L_a - U_b, U_a, - L_b]$.

Division in the case when all the arguments are non-negative is also mixed monotonic. Thus if $A_x / B_x = [L_a, U_a] / [L_b, U_b] = [L_a / U_b, U_a, / L_b]$

In passing we note that the operation of raising to a number to a power, a^b, when all numbers are non-negative is also monotonic. Thus for the interval case we get

$$(A_x)^{B_x} = [L_a, U_a]^{[L_b, U_b]} = [L_a^{L_a}, U_a^{U_a}]$$

4. Containment of Fuzzy Sets of Type-2

We now consider the issue of the containment relationship betweentype-2 fuzzy subsets. We first recall the situation for regular fuzzy subsets. Assume A and B are two fuzzy subsets of X. In his original paper Zadeh [19, 25] suggested that $A \subseteq B$ if $A(x) \leq B(x)$ for all

$x \in X$. Bandler and Kohout in their classic paper [26, 27] noted the binary nature of Zadeh's definition and suggested an alternative, and more general definition, more in the spirit of the concept of gradualness associated with fuzzy sets. Fundamental to their definition was the realization that the "the degree to which A is a fuzzy subset of B is the degree of membership of A is the power set of B." In [26] the authors operationalized this idea using the following definition.

Definition: Assume A and B are fuzzy subsets of X. The degree to which A is a subset of B, denoted $\prod(A \subseteq B)$, is defined as

$$\Pi(A \subseteq B) = \underset{x \in X}{\mathrm{Min}}[A(x) \to B(x)]$$

Here $\to$ is the ply or implication operation. A prototypical example of $a \to b$ is $\bar{a} \vee b$.

Klir and Yuan [28] as well as Bandler and Kohout [26] investigated these ply operators and described some required and optional properties. In the following we shall denote $a \to b$ as P(a, b). Among the properties required of this operator are

1) Anti-monotonicity in the first argument: $P(a, c) \leq P(b, c)$ if $a > b$
2) Monotonicity in the second argument: $P(c, a) \geq P(c, b)$ if $a > b$
3) $P(a, 1) = 1$
4) $P(0, b) \geq b$

We point our that these operators are not necessarily symmetric, $P(a, b) \neq P(b, a)$.

Many examples of these operators can be described. Some of the more notable examples of this operator are

1. $P_1(a, b) = \mathrm{Max}[\bar{a}, b]$

2. $P_2(a, b) = \mathrm{Min}[1, \bar{a}+b]$

3. $P_3(a, b) = \mathrm{Min}[1, \frac{b}{a}]$

4. $P_4(a, b) = 1$ if $a \leq b$
$P_4(a, b) = 0$ if $a > b$

We now consider the issue of extending the definition of containment provided by Bandler and Kohout to the case of type-2 fuzzy subsets. Here we must first address the issue of extending $A(x) \to B(x)$ to the situation where A(x) and B(x) are fuzzy subsets. In the following we shall just consider the case where we have interval valued type-2 fuzzy subset. Thus here $A_X = a = [L_a, U_a]$ and $B_X = b = [L_b, U_b]$.

Let us first look at the ply operator $P_1(a, b) = \mathrm{Max}(\bar{a}, b)$. Since $\bar{a} = [\overline{U}_a, \overline{L}_a]$ then

$$P_1(a, b)=\mathrm{Max}([\overline{U}_a, \overline{L}_a], [L_b, U_b])=[\overline{U}_a \vee L_b, \overline{L}_a \vee U_b]$$

If a = [0, 1], unknown, then $P_1(a, b) = [\overline{U}_a, 1]$. If both are unknown as we would expect $P_1(a, b)=[0, 1]$.

Let us now consider P_2 when $a = [L_a, U_a]$ and $b = [L_b, u_b]$. Here since $\bar{a} = [\overline{U}_a, \overline{L}_a]$ we get $\bar{a} + b= [\overline{U}_a + L_b, \overline{L}_a + U_b]$ and finally

$$P_2(a, b) = [1 \wedge (\overline{U}_a + L_b), 1 \wedge (\overline{L}_a + U_b)]$$

If a = [0, 1], unknown then $\overline{U}_a = 0$ and $\overline{L}_a = 1$ and hence $P_2(a, b) = [1 \wedge L_b, 1 \wedge 1 + U_b] = [L_b, 1]$

If b = [0, 1] unknown then $P_2(a, b)= [\overline{U}_a, 1]$

Actually a closer look at P_1 and P_2will enable us to provide a unification. We note that the operations $x \vee y$ and $1 \wedge (x + y)$ are examples of t-conorms [29]. This inspires us to provide a unification. If we let S be any continuous t-conorm then

$$P_S([L_a, U_a], [L_b, U_b]) = [S(\overline{U}_a, L_b), S(\overline{L}_a, U_b)]$$

We now consider the case $P_3(a, b) = \mathrm{Min}[1, \frac{b}{a}]$ when $a = [L_a, U_a]$ and $b = [L_b, U_b]$. Since division is mixed monotonic then $\frac{b}{a} \neq [\frac{L_b}{U_a}, \frac{U_b}{L_a}]$. Using this P3(a,

$b \wedge \frac{L_b}{U_a}, 1 \wedge \frac{U_b}{L_a}]$.

In the case when = [0, 1] then $P_3(a, b) = [L_b, 1]$ and when b = [0, 1] then $P_3(a, b) = [0, 1]$.

Finally we consider the extension of P_4. Here we have

$$P_4(a, b) = 1 \quad \text{if } b \geq a$$
$$P_4(a, b) = 0 \quad \text{if } b < a$$

The extension to the case where $a = [L_a, U_a]$ and $b = [L_b, U_b]$ is not self-evident. One possible extension is the following

$$P_4(a, b) = [1, 1] \quad \text{if } L_b \geq U_a$$
$$P_4(a, b) = [0, 0] \quad \text{if } L_a \geq U_b$$
$$P_4(a, b) = [0, 1] \quad \text{otherwise}$$

In the preceding we have suggested a number of different ways of extending $A(x) \to B(x)$for the case where both A(x) and B(x) can be interval values. Using these extension we obtained interval values, thus $A(x) \to B(x) = [L_X, U_X]$. In order to obtain $\prod(A \subseteq B)$ we must calculate $\underset{x \in X}{\mathrm{Min}}[A(x) \to B(x)]$. In this case $\Pi(A \subseteq B) = \underset{x \in X}{\mathrm{Min}}([L_x, U_x]) = [\underset{x \in X}{\mathrm{Min}}(L_x), \underset{x \in X}{\mathrm{Min}}(U_x)]$. Thus here we get an interval value for the degree to which A is contained in B.

We now consider an alternative extension of P(a, b). This extension has the useful feature of giving us a single value instead of an interval value even in the case where a and b are intervals. Using this will enable us to obtain a point value for the degree of containment of A in B even when both A and B are interval value type-2 fuzzy sets. Again let $a = [L_a U_a]$ and $b = [L_b, U_b]$. Figure 1 will be useful in understanding this method. The first step in calculating P(a, b) is to draw a rectangular in the x – y space as shown in figure 1 where the lower left coordinate is(L_a, L_b) and the upper right is (U_a, U_b). The area of this rectangle is $\Delta_a \Delta_b$ where $\Delta_a = U_a - L_a$ and $\Delta_b = U_b - L_b$. We next draw the line y = x and calculate the area of the rectangular that is above the line x = y. This is denoted $\boxed{1}$ in the figure 1. Finally we calculate

$$P(a,b) = \frac{\text{Area in } \boxed{1}}{\Delta_a \Delta_b}$$

Thus P(a, b) is the proportion of the rectangle that is above the line x = y.

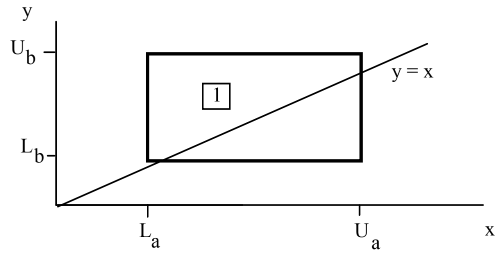

Fig. 1 Basic Calculation of P(a, b)

Some special cases are shown in figure 2 and figure 3. In figure 2 we see the case when $U_a \le L_b$, that is $L_a \le U_a \le L_b \le U_b$. Here area $\boxed{1}$ is the whole rectangular and hence P(a, b) = 1.

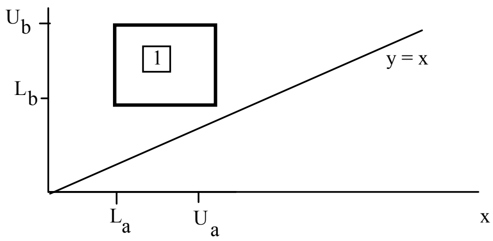

Fig. 2 All area above x = y, P(a, b) =1

At the other extreme is the case where $L_b \le U_b \le L_a \le U_a$. In this case as shown in figure 3 the whole rectangle is below x = y and hence P(a, b) = 0.

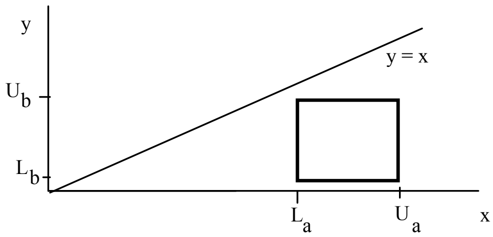

Fig. 3 P(a, b) = 0

Conveniently we can decompose the ordering between L_a, U_b, and L_b and U_b into six distinct cases and determine P(a, b) for each of these. The six cases are:

1) $L_a < U_a \le L_b < U_b$
2) $L_a \le L_a \le U_a \le U_b$
3) $L_a \le L_b \le U_b \le U_a$
4) $L_b \le L_a < U_a \le U_b$
5) $L_b \le L_a \le U_b \le U_a$
6) $L_b < U_a < L_a \le U_b$

Since we have already investigated the first and sixth case let us now consider the four other cases: 2, 3, 4, 5. We can easily show in the second case where $L_a \le L_b \le U_a \le U_b$ that the Area of $\boxed{1} = \Delta_a \Delta_b - \frac{1}{2}(U_a - L_b)^2$ and hence $P(a, b) = 1 - \frac{1}{2}\frac{(U_a - L_b)^2}{\Delta_a \Delta_b}$. In the third case where $L_a \le L_b \le U_b \le U_a$ it can be shown that Area of $\boxed{1} = (L_b - L_a)\Delta_b + \frac{1}{2}\Delta_b^2$ and hence in this case $P(a, b) = \frac{(L_b - L_a) + \frac{1}{2}\Delta_b}{\Delta_a}$. In the fourth case where $L_b \le L_a \le U_a \le U_b$ we can show that the Area of $\boxed{1} = (U_b - U_a)\Delta_a + \frac{1}{2}(\Delta_a)^2$ and hence $P(a, b) = \frac{(U_b - U_a) + \frac{1}{2}\Delta_a}{\Delta_b}$. Finally in the fifth case where $L_b \le L_a \le U_b \le U_a$ we get that Area of $\boxed{1} = \frac{1}{2}(U_b - L_a)^2$ and hence $P(a, b) = \frac{1}{2}\frac{(U_b - L_a)^2}{\Delta_a \Delta_b}$.

In the preceding we implicitly assumed $U_a \ne L_a$ and $U_b \ne U_b$. We must consider the situations in which this not true. If $U_a = L_a = a$ and $U_b = L_b = b$ then P(a, b) = 1 if $b \ge a$ and P(a, b) = 0 otherwise. If $U_a = L_a = a$ and $U_b \ne L_b$ then

$$P(a, b) = 1 \qquad \text{if } a \le L_b$$
$$P(a, b) = \frac{U_b - a}{U_b - L_b} \qquad \text{if } L_b < a < U_b$$
$$P(a, b) = 0 \qquad \text{if } a \ge U_b$$

If $U_a \ne L_a$ and $U_b = L_b = b$ then

$$P(a, b) = 0 \qquad \text{if } b \le L_a$$
$$P(a, b) = \frac{b - L_a}{U_a - L_a} \qquad \text{if } L_a \le b \le U_a$$
$$P(a, b) = 1 \qquad \text{if } b \ge U_a$$

In Table 1 we provide a summary of the calculation of this operator. It can be noted that effectively this operator can be viewed as a probabilistic like operator. In particular we can consider two random variables **x** and **y** such that **x** is uniformly distributed on the range [L_a,U_a] and **y** is uniformly distributed on the range [L_b, U_b]. The our operator can be viewed as

$$P(a, b) = \text{Prob}(\mathbf{y} \ge \mathbf{x})$$

Table 1

1. $L_a = u_a = a$ and $L_b = U_b = b$	
$P(a, b) = 1$	if $b \geq a$
$P(a, b) = 0$	if $b < a$
2. $L_a = U_a = a$ and $L_b \neq U_b$	
$P(a, b) = 1$	if $a \leq L_b$
$P(a, b) = \frac{U_b - a}{U_b - L_b}$	if $L_b \leq a < U_b$
$P(a, b) = 0$	if $a \geq U_b$
3. $L_a \neq U_a$ and $L_b = U_b = b$	
$P(a, b) = 0$	if $b \leq L_a$
$P(a, b) = \frac{b - L_a}{U_a - L_a}$	if $L_a < b < U_b$
$P(a, b) = 1$	if $b \geq u_a$
4. $L_a \neq U_a$ and $L_b \neq U_b$	
$P(a, b) = 1$	$L_a \leq U_a \leq L_b \leq U_b$
$P(a,b) = 1 - \frac{1}{2}\frac{(U_a - L_b)^2}{\Delta_a \Delta_b}$	$L_a \leq L_b \leq U_a \leq U_b$
$P(a,b) = \frac{(L_b - L_a) + \frac{1}{2}\Delta_b}{\Delta_a}$	$L_a \leq L_b \leq U_b \leq U_a$
$P(a,b) = \frac{(U_b - U_a) + \frac{1}{2}\Delta_a}{\Delta_b}$	$L_b \leq L_a \leq U_a \leq U_b$
$P(a,b) = \frac{1}{2}\frac{(U_b - L_a)^2}{\Delta_a \Delta_b}$	$L_b \leq L_a \leq U_b \leq U_a$
$P(a, b) = 0$	$L_b \leq U_b \leq L_a \leq U_a$

5. The Role of Specificity

The concept of specificity [9, 30-34] plays a fundamental role in granular computing by providing a measure of the amount of information contained in a fuzzy subset or possibility distribution. Its role in fuzzy set and possibility theory is analogous to the role that entropy plays in probability theory. The specificity measure evaluates the degree to which a fuzzy subset points to one and only one element as its member. While closely related to the inverse of the cardinality of a set it has an additional degree of sophistication in that it must handle non-normal fuzzy subsets. Klir [35] has discussed a related idea which he calls non-specificity however this is defined for normal fuzzy subsets.

We first emphasize the distinction between specificity and fuzziness. Fuzziness is generally related to the lack of clarity of membership in some set whereas specificity is related to the granularity of the knowledge of some attribute. For example, knowing that the length of a river is between fifty and sixty miles is not fuzzy, for we know with clarity what are the possible values for the length of the river, however it is not specific for we don't the exact length of the river. In most cases these two kinds of uncertainty appear together as in the knowledge that the river is *approximately 50 miles long*. To succinctly distinguish between these two concepts we can say that fuzziness is fundamental to measuring gradualarity while specificity is fundamental to measuring granularity.

Many applications have made use of the measure of specificity. Kacprzyk [36] describes its use in a system for inductive learning. Yager [37, 38] has shown its usefulness as a measure anxiety in making a decision. The more specific the set of choices the easier, the less anxiety provoking the decision. Another important area of application is in the measurement of performance of expert systems and other information proving systems. In this environment the specificity concept plays a role in the determination of the usefulness of the information provided by a system. In this regard we note that an increase in specificity of information provided generally tends to increase the usefulness of the information. Consider an expert system which is used to predict the weather. Assume that the system says that the temperature will be above zero degrees Fahrenheit. While this system will in most cases be correct, the information it provides will not be very useful if the output of the system is to be used to determine what kind of clothes we should wear. This example brings into focus a very fundamental uncertainty principle in information theory which we call **the specificity-correctness tradeoff**. What this principle says is that in providing information we generally must make a tradeoff between being very specific and running the risk of being incorrect or being unspecific and running the risk of providing useless information. In expert and other knowledge based systems we desire both correctness and specificity with its effect of providing more useful information. Thus the performance of systems should be judged by its performance in both these measures, specificity and correctness [39]. More importantly this principle is fundamental to understanding many aspects of human communication where people often choose their words to provide a balance between these conflicting criteria.

Another area where specificity plays a fundamental role is in deductive reasoning systems. In the theory of approximate reasoning (AR), the inference mechanism associated with Zadeh's paradigm of computing with words [40], a central principle used in deduction is what Zadeh called the entailment principle [4]. The principle is a manifestation of the fact that one can always infer less specific information from more specific information.[1] For example this principle allows us to infer from the knowledge that John is 23 years old the conclusion that John is over 20 years old. We note that reasoning the other way, going from less specific to more specific can not be done with complete confidence. We can't conclude from the fact that John is over 20 that he is 23 years old. We note in [41] we investigated issues related to this. The principle of **minimal specificity** introduced by Dubois and Prade [11] is a manifestation of the close relationship between the concept of specificity and the entailment principle. Dubois and Prade using the concept of minimal specificity show the central role of specificity in the theory of approximate reasoning [42].

In many real world applications our available knowledge is not precise enough to allow us to solve the problem at hand. In these situations we must reduce the uncertainty by inferring something that is not totally justified and risk the possibility of concluding something that is not correct. One common motivation for going in this direction opposite of the entailment principle is simplification, to help reduce complexity so as to enable an agent to more effectively and efficiently deal with their environment. The use of default rules is an example of this kind of simplification. In [43] we discussed the role of specificity in manipulating default knowledge.

6. Characterization of Specificity

The concept of specificity was introduced to measure the degree to which a fuzzy subset contains one and only one element. In [32] we provide a comprehensive discussion of this concept. In many applications of fuzzy subsets, especially those based on a possibilistic interpretation of a fuzzy subset, specificity can be seen as measuring the amount of information contained in the fuzzy subset. Consider the following three fuzzy subsets expressing information relating to the age of a person

A_1: 50 years old

A_2: about 50

A_3: middle age

It should be clear A_1 provides more information than A_2 which in turn provides more information than A_3. As noted the concept of specificity plays a role in possibility theory comparable to the concept entropy in probability theory. Both of these measure the amount of information contained in the associated distribution by calculating the degree to which the distribution points to one and only one element as its manifestation.

In the following we provide a characterization of the measure of specificity over a finite universe X. In this definition we shall let A be a fuzzy subset of X and let a_j be the j^{th} largest membership grade in A.

Definition: A measure Sp: $I^X \rightarrow I$ ($I = [0, 1]$) is called a *measure of specificity* if it has the following properties:
1) Sp(A) = 1 if and only if A is a singleton set, A = {x}
2) Sp(Ø) = 0
3) i. $\frac{\partial S_P(A)}{\partial a_1} > 0$

   ii. $\frac{\partial S_P(A)}{\partial a_j} \leq 0$ for all $j \geq 2$.

We see that condition one requires that the specificity is maximal, equal one, for only sets that are singletons. Condition two provides the second boundary condition, a measure of specificity assumes its minimal value for the null set. It should be noted unlike the first property we have not required this to be the only case for this to happen. The third condition imposes the characteristic that specifically increases as the largest membership grade increases and it decreases as any of the other membership grades increase.

From the above definition a number of basic properties about measures of specificity can be easily obtained.

Theorem: Assume A and B are two normal fuzzy subsets of X, they have at least one element with membership grade one. Let a_j and b_j be the ordered membership grades in these sets. If $a_j \geq b_j$ for all j then Sp(B) ≥Sp(A).

Proof: Since starting from B we can obtain A by increasing the j = 2 to n membership grades and since a measure of specificity is non-increasing with respect to changes in these elements we obtain the result.

From this theorem we get some other basic properties of specificity.

Corollary 1: If A and B are normal fuzzy subsets of X and $A \subset B$ then Sp(B) ≥ Sp(A).

Corollary 2: If A and B are two non-null crisp subsets of X where Card(A) ≥ Card(B)then Sp(B) ≥ Sp(A).

Our characterization of specificity allows for many different realizations. A particular realization maybe useful for a particular application. In discussing particular realizations of the specificity measure we shall find the following definitions useful.

Definition: Assume Sp and $\widehat{Sp}$ are two different measures of specificity on the space X. We shall say that Sp is a stricter measure of specificity then $\widehat{Sp}$, de-

[1] Essentially this principle is a generalization of a basic classical logic rule of reasoning called the law of addition which states from the knowledge that P is true we can always conclude P or Q is true.

noted $Sp \leq \widehat{Sp}$, if for any fuzzy subset A we have $Sp(A) \leq \widehat{Sp}(A)$.

Definition: A measure of specificity will be called ***regular*** if for all fuzzy subsets in which the membership grade is constant, A(x) = C for all x, we have Sp(A) = 0.

Specificity plays an important role in procedures and algorithms for manipulating and reducing uncertainty. With this measure we have a tool which can guide us in the correct direction by telling us when information content is increasing or decreasing. In most cases the actual value of the degree of uncertainty is not as important as the relative uncertainty. This situation gives us considerable freedom in selecting the actual form of the measure to be used. One important consideration in the selection of a measure of specificity for a particular application is the ease with which we can manipulate the specificity measure under the operations needed in that application. Simple measures are always desirable.

In [43] Yager looked at a number of different measures of specificity and introduced a class of specificity measures which he called linear specificity measures.

Definition: Assume X is a finite set of cardinality *n* and let F be a fuzzy subset of X. A linear specificity measure is defined as

$$Sp(F) = a_1 - \sum_{j=2}^{n} w_j a_j$$

where a_j is the j^{th} largest membership grade in F and the w_j's are a set of weights satisfying:

1. $w_j \in [0, 1]$
2. $\sum_{j=2}^{n} w_j = 1$
3. $w_j \geq w_i$ for $j < i$.

A notable member of this class is the specificity measure. $Sp(F) = a_1 - \frac{1}{n-1} \sum_{j=2}^{n} a_j$ This can be seen simply as expressing specificity as

Sp(F) = *largest membership grade - average of the others.*

It can be shown that this is the least strict of the linear specificity measures.

7. Specificity of Interval Valued Type-2Fuzzy Sets

In the preceding we indicated that the measure of specificity of a fuzzy subset F can be calculated as $Sp(F) = a1 - \sum_{j=2}^{n} w_j a_j$ where a1 = Max_x [F(x)] is the largest membership grade in F and a_j is the j^{th} largest membership grade. Furthermore the w_j are a collection of non–negative weights that sum to one where $w_j \geq w_i$ if j < i. We indicated an important special case is $w_j = \frac{1}{n-1}$ for j = 2 to n. In this case if we index the x_j's so that x_1 has the maximal membership grade then Sp(F) $= F(x_1) - \frac{1}{n-1} \sum_{j=2}^{n} F(x_j)$. We emphasize in this case we have no need to order the other membership grades.

We now turn to the calculation of the specificity of a fuzzy subset of type-2. We shall use the specificity measure where $w_j = \frac{1}{n-1}$ for j = 2 to n. We shall initially focus on the interval valued type fuzzy subsets of type-2 extensively used by Mendel [18].

Here we assume that X = $\{x_1, x_2, ..., x_n\}$ and F is a type-2fuzzy subset such that the membership grade F(x) is an interval, F(x) = [L(x), U(x)] where $0 \leq L(x) \leq U(x) \leq 1$. We note that this type membership grade can be viewed as a fuzzy subset of the unit interval, actually it is a crisp subset.

We note when L(x) = U(x) this implies that F(x) is precise, F(x) = U(x) = L(x). An important case is where L(x) = U(x) =1, in this case F(x) = 1. We shall say that F is **normal** if there exists at least one $x \in X$ such that L(x) =1, here of course we have F(x) = 1. We shall say that F is **regular** if there exists at least one $x \in X$ such that U(x)= 1. We shall say that F is a **strict** type-2 interval fuzzy subset if there exist at least one $x \in X$ such that $U(x) - L(x) \neq 0$.

We now suggest the following procedure for calculating the specificity of an interval valued type-2 fuzzy subset:

I. Determine the biggest of the lower bounds, $\alpha = Max_x[L(x)]$. (We shall assume this occurs at x*, that is $\alpha = L(x^*)$).
II. For all x calculate $G(x) = \alpha \wedge U(x)$.
III. Calculate the specificity of F as

$$Sp(F) = \alpha - \frac{1}{n-1} \sum_{x \neq x^*} G(x)$$

Example: Assume X = $\{x_1, x_2, x_3, x_4, x4\ _5\}$ and F is an intervaltype-2 fuzzy set with

$F(x_1) = [0.3, 0.4]$, $F(x_2) = [0.6, 1]$, $F(x_3) = [0.4, 0.9]$, $F(x_4) = [0.2, 0.3]$, $F(x_5) = [0, 0.5]$

In this case $\alpha = Max_x[L(x)] = 0.6$, this occurs at x_2. Using this we calculate $G(x) = \alpha \wedge U(x)$

$$G(x_1) = 0.6 \wedge 0.4 = 0.4$$
$$G(x_3) = 0.6 \wedge 0.9 = 0.6$$
$$G(x_4) = 0.6 \wedge 0.3 = 0.3$$
$$G(x_5) = 0.6 \wedge 0.5 = 0.5$$

Using this we obtain

$$Sp(F) = 0.6 - \frac{1}{4}(0.4 + 0.6 + 0.3 + 0.5) = 0.6 - \frac{(1.8)}{4} = 0.15$$

We can easily show that $Sp(F) \in [0, 1]$. Let us observe some other features of this approach to determining the specificity of F. First we note that for all x, $G(x) \in [L(x), U(x)]$. To verify this we first observe that $G(x) = \alpha \wedge U(x) \leq U(x)$. Furthermore since $\alpha \wedge U(x) \geq \alpha \wedge L(x)$ and since $\alpha = Max_X[L(x)]$ then for all x, $\alpha \wedge U(x) \geq L(x)$. In the special case where F is a type-1 fuzzy subset, $L(x) = U(x) = F(x)$, we observe that this reduces to the usual measure of specificity. In this case $\alpha = Max_X[L(x)] = Max_X[F(x)]$ and therefore $G(x) = \alpha \wedge U(x) = \alpha \wedge F(x) = Max_Z[F(z)] \wedge F(x) = F(x)$ and hence $Sp(F) = F(x^*) - \sum_{x \neq x^*} F(x)$.

An important special case is a normal interval type-2, there exists some x* such that L(x*) = 1. Here of course L(x*)= U(x*) = F(x*) = 1. In the case $\alpha = Max_X[L(x)] = 1$. With $\alpha = 1$ we get $G(x) = \alpha \wedge U(x) = U(x)$ and hence $Sp(F) = 1 - \frac{1}{n-1}\sum_{x \neq x^*} U(x)$. In this case the specificity it is one minus the average o fall the upper bounds except that of x*.

Another property of our definition is that for any **strict** type-2 fuzzy subset F we have Sp(F) < 1.We verify this as follows. First we observe that if $\alpha < 1$ then it must be the case that Sp(F) < 1. Consider now we have $\alpha = 1$. In this case since L(x*) = 1 then it must be true that F(x*) = L(x*)= U(x*) = 1 Let $\hat{x}$ be an element where $U(\hat{x}) > L(\hat{x})$. > For this element as for all elements $G(\hat{x}) = \alpha \wedge U(\hat{x}) = U(\hat{x})$. Since $U(\hat{x}) > L(\hat{x})$ then $U(\hat{x}) > 0$. Since $G(\hat{x}) > 0$ then $\sum_{x \neq x^*} G(x) > 0$ and hence

$$S_P(F) = 1 - \frac{1}{n-1}\sum_{x \neq x^*}^{x} G(x) < 1.$$

Another interesting special case is when all the elements have the same membership grade. Here then for all x we have F(x)= [L(x), U(x)] = [a, b]. In this case $\alpha = Max_X[L(x)] = a$. Using this for all x that $G(x) = \alpha \wedge b = a \wedge b = a$. From this we see $Sp(F) = a - \frac{1}{n-1}(n-1)\,a = 0$. In this case we get the minimal specificity.

We note that this approach can be extended to the case where X = [c, d] and F is a type-2 interval fuzzy subset of X. Here we again let $\alpha = \underset{x \in X}{Max}[L(x)]$ and $G(x) = \alpha \wedge U(x)$. Using this we calculate

$$S_P(F) = \alpha - \frac{1}{d-c}\int_c^d G(x)\,dx$$

In the following we shall confine ourselves to situation where X is a discrete set.

8. Uniqueness of Specificity Measure

In the preceding we have suggested an approach for obtaining the specificity of a fuzzy subset of type-2 with interval valued membership grades. Other formulations can be considered. For example we can calculate $\tilde{F}(x) = \frac{1}{2}[L(x) + U(x)]$ for all x and then obtain $\tilde{F}(x^*) = Max_X[\hat{F}(x)]$. Using this we can express the specificity of F as $S_P(F) = \tilde{F}(x^*) - \frac{1}{n-1}\sum_{x \neq x^*}\tilde{F}(x)$.

More generally we can let $Q_F(x)$ be such that $Q_F(x)$ is any arbitrary value in [L(x), U(x)]. Using this we can calculate $Q_F(x^*) = Max_X[Q_F(x)]$and then calculate

$$S_P(F) = Q_F(x^*) - \frac{1}{n-1}\sum_{x \neq x^*} Q_F(x).$$

Essentially we see that any function $Q_F(x)$ as described above can be seen as a manifestation of a possible type-1 fuzzy subset that is consistent with the original type-2 interval valued fuzzy subset F. Thus Q_F can be seen as an arbitrary manifestation of the imprecise fuzzy subset of type-2by a type-1 fuzzy subset. Thus one way to obtain the specificity of an interval valued type-2 fuzzy subset is to select some type-1manifestation and then calculate its specificity using the standard definition.

An important special manifestation is one which we shall denote as $\hat{Q}_F$. For this manifestation we let $\alpha = Max_X[L(x)] = L(x^*)$. Using this we define

$$\hat{Q}_F(x^*) = \alpha$$

$$\hat{Q}_F(x) = \alpha \wedge U(x) = G(x) \qquad \text{for } x \neq x^*$$

This is essentially the manifestation that we used earlier in defining our original measure of specificity for interval valuedtype-2 fuzzy sets.

We prove an important theorem related to the uniqueness of $\hat{Q}_F$.

Theorem: Let F be a type-2fuzzy subset with interval membership grades, F(x) = [L(x), U(x)]. Let Q_F be any arbitrary manifestation of F, Q_F is a type 1 fuzzy subset such that $Q_F(x) \in [L(x), U(x)]$ then $Sp(Q_F) \geq Sp(\hat{Q}_F)$.

Proof: $S_P(\hat{Q}_F) = \alpha - \frac{1}{n-1}\sum_{x \neq x^*} \alpha \wedge U(x)$. Consider now an arbitrary Q_F with $Q_F(x) \in [L(x), U(x)]$. Let $Max_X[Q(x)] = Q_F(x^{**}) = \beta$. We first observe that $\beta \geq \alpha$. This follows since $Q_F(x^*) \geq L(x^*) \geq \alpha$ and the fact that $\beta = Max_X[Q_F(x)]$. Hence

$$S_P(Q_F) = \beta - \frac{1}{n-1}\sum_{x \neq x^{**}} Q_F(x) = \beta - \frac{Q_F(x^*)}{n-1} - \frac{1}{n-1}\sum_{x \neq x^*, x^{**}} Q_F(x)$$

$$S_P(Q_F) = \frac{1}{n-1}\sum_{x \neq x^*, x^{**}} (\beta - Q_F(x)) + \frac{1}{n-1}(\beta - Q_F(x^*))$$

while

$$S_P(\hat{Q}_F) = \alpha - \frac{1}{n-1}\sum_{x \neq x^*} \alpha \wedge U(x)$$

$$= \alpha - \frac{\alpha \wedge U(x^{**})}{n-1} - \frac{1}{n-1}\sum_{x \neq x^*, x^{**}} \alpha \wedge U(x)$$

$$S_P(\hat{Q}_F) = \frac{1}{n-1}\sum_{x \neq x^*, x^{**}} (\alpha - (\alpha \wedge U(x))) + \frac{1}{n-1}(\alpha - (\alpha \wedge U(x^{**})))$$

Consider any $x \neq x^*, x^{**}$

1) If $U(x) \geq \alpha$ then $\alpha - (\alpha \wedge U(x)) = 0$ and hence $\beta - Q_F(x) \geq \alpha - (\alpha \wedge U(x))$
2) If $U(x) < \alpha$ then $\alpha \wedge U(x) = U(x)$. Since $Q_F(x) \in [L(x), U(x)]$ then $Q_F(x) \leq U(x)$ and hence $\beta - Q_F(x) \geq \alpha - U(x)$

Thus for any $x \neq x^*, x^*$ we have $\alpha - \alpha \wedge U(x) \leq \beta - Q(x)$.

Consider now the term $\alpha - (\alpha \wedge U(x^{**}))$ since $U(x^{**}) \geq Q_F(x^{**}) \geq \beta$ then $\alpha \wedge U(x^{**}) = \alpha$ and hence $\alpha - (\alpha \wedge U(x^{**})) = \alpha - \alpha = 0$. Since $\beta - Q_F(x^*) \geq 0 \geq \alpha - \alpha \wedge u(x^*)$ the result follows.

Thus our early definition for the specificity of the type-2fuzzy set F is equal to the smallest specificity of any ordinary fuzzy subset that can be manifested from the interval valuedtype-2 set F.

9. Measure with the Largest Specificity

We now consider a related problem. Again let F be a type-2fuzzy subset with interval valued membership grades, [L(x), U(x)]. Here we are interested in finding the manifestation Q_F of F with the largest specificity. In anticipation of solving this we introduce the concept of a vortex subset.

Definition: Let F be an interval valued type-2 fuzzy subset of X. A type-1 fuzzy subset Q of X is called a **vortex** subset associated with F if Q is such that there exists one element $x^* \in X$ such that $Q(x^*) = U(x^*)$ and for all $x \neq x^*$ is $Q(x) = L(x)$.

We further observe that if $X = \{x_1, ..., x_n\}$ then there are exactly *n* vortex sets associated with F. We shall denote Q_j as the vortex set where $Q_j(x_j) = U(x_j)$ and $Q_j(x) = L(x)$ for $x \neq x_j$.

We show the following theorem about vortex sets.

Theorem: If Q is any manifestation of F, $Q(x) \in [L(x), U(x)]$there exists some vortex set Q_k such that $S_P(Q_k) \geq S_P(Q)$.

Proof: Let Q be such that $Max_X[Q(x)] = Q(x_k)$. In this case $S_P(Q) = Q(x_k) - \frac{1}{n-1}\sum_{j \neq k} Q(x_j)$

Consider now the vortex set Q_k, here $Q_k(x_k) = U(x_k)$ and $Q_k(x_j) = L(x_j)$ for $j \neq k$. It is easy to see that

$$S_P(Q_k) = U(x_k) - \frac{1}{n-1}\sum_{j \neq k} L(x_j) \geq S_P(Q).$$

Thus we see that the manifestation with the largest specificity must be a vortex subset. Let us now compare two vortex sets with respect to their specificity. Consider without loss of generality Q_1 and Q_2. Here then

$$S_P(Q_1) = U(x_1) - \frac{1}{n-1}\sum_{j \neq x_1} L(x_j)$$

$$S_P(Q_2) = U(x_2) - \frac{1}{n-1}\sum_{j \neq x_2} L(x_j)$$

Taking the difference we get

$$S_P(Q_1) - S_P(Q_2) = (U(x_1) - \frac{L(x_2)}{n-1}) - (U(x_2) - \frac{L(x_1)}{n-1})$$

$$S_P(Q_1) - S_P(Q_2) = (U(x_1) + \frac{1}{n-1}L(x_1)) - (U(x_2) + \frac{1}{n-1}L(x_2))$$

Thus we see if we define $D(x_j) = U(x_j) + \frac{1}{n-1}L(x_j)$ then the manifestation of F with the largest specificity is the vortex set Q_j such that $D(x_j) = Max_X[D(x)]$.

Using Q^+ to denote the vortex set with the largest specificity we can make some observations. We first note that if for all x_j we have $U(x_j) = 1$ then the maximal vortex set is the Q_j where $L(j) = Max_X[L(x)]$. It is the vortex set with the largest lower bound. We also observe that as *n* gets larger the calculation of D(x) and in turn the selection of Q^+ is more and more determined by the upper bounds. We also note that if $D(x_j) = D(x_k)$ then we have more than one possible manifestation for Q^+. We further note that if $L(x) = 1$ then $Q(x) = 1$ and hence D(x) assumes its maximal possible value, $\frac{n}{n-1}$. Thus we see that if $L(x_j) = 1$ then it is one of the possible maximal vortex sets.

10. Specificity of General Type-2 Fuzzy Sets

We now consider the issue of defining the specificity of a generaltype-2 fuzzy subset F. In this case each of the membership grades F(x) is a fuzzy subset of the unit interval. To obtain the specificity in this case we proceed as follows. For each $x \in X$ we calculate

$$F(x)_\alpha = \{x/F(x) \geq \alpha\}$$

the collection of level sets of its membership grade. We now define fuzzy lower bound of F(x) as

$$\tilde{L}(x)=\{\frac{\alpha}{Min[F(x)_\alpha]}\}$$

Here $Min[F(x)_\alpha]$ is the smallest element in the α level set of F(x). Using this we calculate the effective lower bound $\hat{L}(x)=\int_0^1 Min[F(x)_\alpha]\, d\alpha$.

From these effective lower bounds we now calculate $\beta = Max_X[\hat{L}(x)]$. We let x* be such that $\beta = \hat{L}(x^*)$, it is the element that attains the largest effective lower bound.

For each x we now calculate the upper bound of its membership grade F(x). This is the fuzzy subset

$$\tilde{U}(x)=\{\frac{\alpha}{Max[F(x)_\alpha]}\}$$

We proceed to then calculate

$$\hat{U}(x)=\int_0^1 Max[F(x)_\alpha]\, d\alpha$$

Using this we obtain $\hat{G}(x) = \beta \wedge \hat{U}(x)$. From this we calculate the specificity of F as

$$S_P(F) = \beta - \frac{1}{n-1}\sum_{x \neq x^*} \hat{G}(x)$$

We observe that if F(x) is an interval, F(x) = [a, b], then $F(x)_\alpha = [a, b]$ for all $\alpha > 0$. In this case, $Min[F(x)_\alpha] = a$ for α and hence $\hat{L}(x)=\int_0^1 a\, d\alpha = a$. Similarly in this case $Max[F(x)_\alpha] = b$ for all α and $\tilde{U}(x) = \{\frac{1}{b}\}$ and hence $\hat{U}(x) = b$. In this case G(x) = $L(x^*) \wedge b$. This results in the same value for specificity as we had in the case formulated particularly for interval type-2fuzzy sets.

11. Conclusion

We discussed fuzzy sets of type-2 and looked at some of the available operations. With the aid of the extension principle we suggested a method for implementing the concept of containment between fuzzy sets of type-2. We suggested a procedure for calculating the specificity of a interval value type-2 fuzzy subset. We showed the uniqueness of this procedure. We then extended our method to the case of a general type-2 fuzzy subset.

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