# An Approach to Centroids of Fuzzy Numbers

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## Abstract

**The paper considers the centroids of fuzzy numbers in the general condition. It suggests a pair of formulae for determining the centroid of a fuzzy number. The proposed formulae are correct in the sense described by Wangs' paper. They are available for all fuzzy numbers and can be applied to ranking fuzzy numbers.**


***Keywords: Centroid, fuzzy numbers, ranking.***

## 1. Introduction

The centroids of fuzzy numbers have been examined recently. Cheng [2] used a centroid-based distance method to rank fuzzy numbers in 1998. Then, Chu and Tsao [3] utilized the area between the centroid point and the origin to rank fuzzy numbers in 2002. Abbasbandy and Asady [1] suggested a sign distance method for ranking fuzzy numbers in 2006. However, the above papers are found to have incorrect centroids described in a paper by Wang et al. [8]. Wang et al. also suggested a pair of centroid formulae for trapezoidal fuzzy numbers, satisfying two fundamental properties of correct centroid formulae [8]. This paper extends their work to a more general case and presents a pair of correct centroid formulae for all fuzzy numbers. The proposed formulae are developed based on the premise that the coordinate on the vertical axis is as important as the coordinate on the horizontal axis and should be determined naturally as the way of determining the coordinate on the horizontal axis. The paper is organized as follows. Section 2 first introduces several centroid formulae for trapezoidal fuzzy numbers, then the paper suggests a pair of formulae for determining the centriod of a fuzzy number in the general condition. Section 3 discusses the relationship of the centroids between two same shaped fuzzy numbers and justifies the presented formulae.

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## 2. Centroids of fuzzy numbers

A fuzzy number is a convex fuzzy subset of real line $R$ and is completely defined by its membership function. Let $A$ be a fuzzy number, whose membership function $A(x)$ can generally be defined as [4]:

$$A(x)=\begin{cases} A_L(x), & a\le x<b \\ \omega, & b\le x\le c \\ A_R(x), & c<x\le d \\ 0, & o.w. \end{cases} \qquad (1)$$

where $0<\omega\le 1$ is a constant, $A_L:[a,b)\to[0,\omega]$ is monotonic increasing continuous from the right function, and $A_R:(c,d]\to[0,\omega]$ is monotonic decreasing continuous from the left function. If $\omega=1$, then $A$ is a normal fuzzy number, otherwise, it is said to be a non-normal fuzzy number. If the membership function $A(x)$ is piecewise linear and continuous, then $A$ is referred to as a trapezoidal fuzzy number and is usually denoted by $A=(a,b,c,d;\omega)$ or $A=(a,b,c,d)$ if $\omega=1$. In this case, $A_L(x)=\frac{\omega}{b-a}(x-a), a\le x<b$ and $A_R(x)=\frac{\omega}{c-d}(x-d), c<x\le d$. In particular, when $b=c$, the trapezoidal fuzzy number is reduced to a triangular fuzzy number denoted by $A=(a,b,d;\omega)$ or $A=(a,b,d)$ if $\omega=1$.

The papers [2], [3], [5], [6], [8] only considered trapezoidal fuzzy numbers and determined the centroid point $(\bar{x}_0,\bar{y}_0)$ of a trapezoidal fuzzy number $A$ as follows.

Wang et al. [8] proposed the following centroid formulae:

$$\bar{x}_0(A)=\frac{\int_{-\infty}^{\infty} xA(x)dx}{\int_{-\infty}^{\infty} A(x)dx} = \frac{\int_a^b \frac{\omega}{b-a}(x-a)xdx+\int_b^c \omega xdx+\int_c^d \frac{\omega}{c-d}(x-d)xdx}{\int_a^b \frac{\omega}{b-a}(x-a)dx+\int_b^c \omega dx+\int_c^d \frac{\omega}{c-d}(x-d)dx} \qquad (2)$$

$$\bar{y}_0(A)=\frac{\int_0^{\omega} y[A_R^{-1}(y)-A_L^{-1}(y)]dy}{\int_0^{\omega}[A_R^{-1}(y)-A_L^{-1}(y)]dy} \qquad (3)$$

where $A_L^{-1}:[0,\omega]\to[a,b]$ and $A_R^{-1}:[0,\omega]\to[c,d]$

are the inverse functions of $A_L$ and $A_R$, respectively. They proved that (2) and (3) satisfy the following two properties of correct centroid formulae [8]:

*(p1) If A and B are fuzzy numbers with their membership functions A(x) and B(x) have the relation of B(y) = A(x), where y = x+ b, then*

$$\bar{x}_0(B) = \bar{x}_0(A) + b, \quad \bar{y}_0(B) = \bar{y}_0(A).$$

*(p2) If A and B are fuzzy numbers with their membership functions A(x) and B(x) have the relation of*

$$B(x) = \omega\, A(x) \quad \textit{for all} \quad x \in R, \textit{then} \quad \bar{x}_0(B) = \bar{x}_0(A).$$

Cheng [2] provided the following cetroid formulae:

$$\bar{x}_0(A) = \frac{\int_a^b xA_L(x)dx + \int_b^c xdx + \int_c^d xA_R(x)dx}{\int_a^b A_L(x)dx + \int_b^c dx + \int_c^d A_R(x)dx} \qquad (4)$$

$$\bar{y}_0(A) = \frac{\omega[\int_0^1 y(A_L^{-1}(y) + A_R^{-1}(y))dy]}{\int_0^1 A_L^{-1}(y) + A_R^{-1}(y)dy}. \qquad (5)$$

Chu and Tsao [3] defined the centroid formulae by

$$\bar{x}_0(A) = \frac{\int_a^b xA_L(x)dx + \int_b^c xdx + \int_c^d xA_R(x)dx}{\int_a^b A_L(x)dx + \int_b^c dx + \int_c^d A_R(x)dx} \qquad (6)$$

$$\bar{y}_0(A) = \frac{\int_0^\omega y(A_L^{-1}(y) + A_R^{-1}(y))dy}{\int_0^\omega A_L^{-1}(y) + A_R^{-1}(y)dy}. \qquad (7)$$

Formulae (6) and (7) were also adopted by Pan and Yeh [5], [6]. However, Wang et al. [8] found that (4)-(7) are incorrect which do not satisfy (p1) and (p2). Compared to (4)-(7), (2) and (3) are correct but they are restricted to be available under the condition of trapezoidal fuzzy numbers. Notably, Wang et al. only addressed the centoid points of trapezoidal fuzzy numbers, so that (2) can be expressed explicitly, and $A_L^{-1}$ and $A_R^{-1}$ exist in (3). For this sake, the paper suggests the following centroid formulae for all fuzzy numbers.

$$\bar{x}_0(A) = \frac{\int_{-\infty}^\infty xA(x)dx}{\int_{-\infty}^\infty A(x)dx} \qquad (8)$$

$$\bar{y}_0(A) = \frac{\int_0^\omega \alpha\, |A^\alpha|\, d\alpha}{\int_0^\omega |A^\alpha|\, d\alpha} \qquad (9)$$

where $A$ is a fuzzy number with $\sup_{x\in R} A(x) = \omega$ and $|A^\alpha|$ is the length of the $\alpha$-cut $A^\alpha$, $0 < \alpha \le 1$. If $A$ is a crisp set with $A(x_0) = \omega$ and $A(x) = 0$ if $x \ne x_0$, then its centroid is defined by $(x_0, \omega)$. It is clear that $(x_0, \omega)$ satisfies (p1) and (p2).

Formula (8) is a natural definition of the coordinate on the horizontal axis. Formula (9) is defined based on the following property.

**Proposition 1.** Let $A$ be a fuzzy number with $\sup_{x\in R} A(x) = \omega$, then $\int_{-\infty}^\infty A(x)dx = \int_0^\omega |A^\alpha|\, d\alpha.$

**Proof.** Without loss of generality, let the membership function of $A$ be defined as (1). Let $\overline{A}_R^{-1}(y) = \sup\{x \mid A_R(x) = y\}$ and $\underline{A}_L^{-1}(y) = \inf\{x \mid A_L(x) = y\}$. Since $A_R(x)$ is decreasing, $\overline{A}_R^{-1}(y) = \sup\{x \mid x \in A^y\}$. Similarly, since $A_L(x)$ is increasing, $\underline{A}_L^{-1}(y) = \inf\{x \mid x \in A^y\}$. Note that $A^\alpha = [\underline{A}_L^{-1}(\alpha), \overline{A}_R^{-1}(\alpha)]$ is a closed interval, for all $0 < \alpha \le 1$. We have $\overline{A}_R^{-1}(\alpha) - \underline{A}_L^{-1}(\alpha) = |A^\alpha|$. That is

$$\int_{-\infty}^\infty A(x)dx = \int_0^\omega [\overline{A}_R^{-1}(\alpha) - \underline{A}_L^{-1}(\alpha)]d\alpha = \int_0^\omega |A^\alpha|\, d\alpha.$$

**Remark 1.** The functions $A_L$ and $A_R$ in (1) need not be invertable, for instance,

$$A = 0.5/1 + 0.5/2 + 1/3 + 0.5/4,$$

where $A_L(x) = 0.5,\ x = 1,\ 2$ is not invertable. So that $\overline{A}_R^{-1}$ and $\underline{A}_L^{-1}$ are defined above instead of $A_R^{-1}$ and $A_L^{-1}$, respectively.

It will be seen that (8) and (9) satisfy (p1) and (p2) in Section 3. Therefore, they are correct centriod formulae. It is noticed that (3), (5), and (7) are determined by the inverse function of the membership function. This is the case that the membership function must be invertable. Compared to (3), (5), and (7), (9) is determined in a more natural and simple way, it is available in any cases.

## 3. Main results

**Definition 1.** Two fuzzy numbers $A$ and $B$ are said to have the same shape if their membership functions satisfy the following relation [7]:

$$B(y) = A(x) \quad \text{for all } y = ax + b \qquad (10)$$

where $a \neq 0$ and $a, b \in R$ are constants.

**Lemma 1** [7]. If fuzzy numbers *A* and *B* satisfy (10), then $B^{\alpha} = aA^{\alpha} + b,$ for all $0 < \alpha \leq 1$.

**Theorem 1.** If fuzzy numbers *A* and *B* satisfy (10), then $\bar{x}_0(B) = a\bar{x}_0(A) + b, \quad \bar{y}_0(B) = \bar{y}_0(A).$

**Proof.** Let $y = ax+b$.

$\int_{-\infty}^{\infty} B(y)dy = \int_{-\infty}^{\infty} aA(x)dx = |a| \int_{-\infty}^{\infty} A(x)dx$ if $a > 0$.

$\int_{-\infty}^{\infty} B(y)dy = \int_{\infty}^{-\infty} aA(x)dx = |a| \int_{-\infty}^{\infty} A(x)dx$ if $a < 0$.

Thus, $\int_{-\infty}^{\infty} B(y)dy = |a| \int_{-\infty}^{\infty} A(x)dx.$

Similarly, $\int_{-\infty}^{\infty} yB(y)dy = |a| \int_{-\infty}^{\infty} (ax+b)A(x)dx$.

Therefore,

$$\bar{x}_0(B) = \frac{\int_{-\infty}^{\infty} yB(y)dy}{\int_{-\infty}^{\infty} B(y)dy} = \frac{|a| \int_{-\infty}^{\infty} (ax+b)A(x)dx}{|a| \int_{-\infty}^{\infty} A(x)dx}$$

$$= a\frac{\int_{-\infty}^{\infty} xA(x)dx}{\int_{-\infty}^{\infty} A(x)dx} + b = a\bar{x}_0(A) + b.$$

Conversely, for any interval [*s*, *t*], we have $[s,t] + b = [s+b, t+b]$ and $a \cdot [s,t] = [as, at]$ if $a > 0$, $a \cdot [s,t] = [at, as]$ if $a < 0$. According to Lemma 1, $|B^{\alpha}| = |aA^{\alpha} + b| = |a||A^{\alpha}|$ for all $0 < \alpha \leq 1$.

Therefore,

$$\bar{y}_0(B) = \frac{\int_0^{\omega} \alpha |B^{\alpha}| d\alpha}{\int_0^{\omega} |B^{\alpha}| d\alpha} = \frac{|a| \int_0^{\omega} \alpha |A^{\alpha}| d\alpha}{|a| \int_0^{\omega} |A^{\alpha}| d\alpha} = \bar{y}_0(A).$$

By Theorem 1, the following corollary holds immediately.

**Corollary 1.** Formulae (8) and (9) satisfy (p1).

Next, we justify the property (p2). Consider a general fuzzy number *B* with its membership function $B(x) = \omega\, A(x)$, where *A* is assumed to be a normal fuzzy number and $0 < \omega < 1$. According to (8), we have

$$\bar{x}_0(B) = \frac{\int_{-\infty}^{\infty} xB(x)dx}{\int_{-\infty}^{\infty} B(x)dx} = \frac{\omega\int_{-\infty}^{\infty} xA(x)dx}{\omega\int_{-\infty}^{\infty} A(x)dx} = \bar{x}_0(A). \quad (11)$$

Since

$$B^{\alpha} = \{x \mid B(x) \geq \alpha\} = \{x \mid \omega A(x) \geq \alpha\}$$

$$= \{x \mid A(x) \geq \alpha/\omega\}$$

$$= A^{\frac{\alpha}{\omega}} \qquad \text{for all } 0 < \alpha < 1,$$

$$\int_0^{\omega} |B^{\alpha}| d\alpha = \int_0^{\omega} |A^{\frac{\alpha}{\omega}}| d\alpha = \omega\int_0^1 |A^{\beta}| d\beta$$

where $\frac{\alpha}{\omega} = \beta$, and

$$\int_0^{\omega} \alpha |B^{\alpha}| d\alpha = \int_0^{\omega} \alpha |A^{\frac{\alpha}{\omega}}| d\alpha = \omega^2 \int_0^1 \beta |A^{\beta}| d\beta.$$

Therefore,

$$\bar{y}_0(B) = \frac{\int_0^{\omega} \alpha |B^{\alpha}| d\alpha}{\int_0^{\omega} |B^{\alpha}| d\alpha} = \frac{\omega^2 \int_0^1 \beta |A^{\beta}| d\beta}{\omega\int_0^1 |A^{\beta}| d\beta}.$$

$$= \omega\bar{y}_0(A). \qquad (12)$$

By (11) and (12), the following theorem holds immediately.

**Theorem 2.** If fuzzy numbers *A* and *B* satisfy the relation of $B(x) = \omega\, A(x)$ for all $x \in R$, then $\bar{x}_0(B) = \bar{x}_0(A)$ and $\bar{y}_0(B) = \omega\bar{y}_0(A)$, where $0 < \omega < 1$.

By Theorem 2, the following corollary holds immediately.

**Corollary 2.** Formulae (8) and (9) satisfy (p2).

Clearly, (8) does not change the centriod coordi nate of a fuzzy number on the horizontal axis when it changes its membership degrees on the vertical axis and (9) does not cause any changes of its centriod on the vertical axis when a fuzzy number moves along the horizontal axis. Therefore, they are correct centroid formulae.

## 4. Centroids of trapezoidal fuzzy numbers

Formulae (2) and (3) have the properties of (p1) and (p2) only in the cases of trapezoidal and triangular fuzzy numbers [8]. We have seen that (8) and (9) have the properties of (p1) and (p2) for all fuzzy numbers, so that the results of [8] are included by this paper. For comparing, we also calculate the centroid of a trapezoidal fuzzy number by using (8) and (9).

**Example 1.** Consider a general trapezoidal fuzzy number $A = [a, b, c, d; \omega]$, whose membership function is defined as

$$A(x)=\begin{cases}\omega\dfrac{x-a}{b-a}, & a\le x<b\\ \omega, & b\le x\le c\\ \omega\dfrac{d-x}{d-c}, & c<x\le d\\ 0, & o.w.\end{cases} \tag{13}$$

where $0<\omega\le 1$. It has been determined by using (2) and (3) that

$$\bar{x}_0(A)=\tfrac{1}{3}[a+b+c+d-\frac{dc-ab}{(d+c)-(a+b)}] \tag{14}$$

$$y_0(A)=\frac{\omega}{3}[1+\frac{c-b}{(d+c)-(a+b)}] \tag{15}$$

respectively [8]. Now, we determine $\bar{x}_0(A)$ and $\bar{y}_0(A)$ by using (8) and (9). Since (8) is the same as (2), they have the same coordinate on the horizontal axis. We only need to determine $\bar{y}_0(A)$. Note that

$A^\alpha=[a+\frac{\alpha}{\omega}(b-a),\ d-\frac{\alpha}{\omega}(d-c)]$ and

$|A^\alpha|=(d-a)-\frac{\alpha}{\omega}(d-c+b-a)$. We have

$$\int_0^\omega |A^\alpha|\,d\alpha=\int_0^\omega (d-a)-\frac{\alpha}{\omega}(d-c+b-a)d\alpha$$
$$=\frac{\omega}{2}[(d+c)-(b+a)],$$
$$\int_0^\omega \alpha|A^\alpha|\,d\alpha=\int_0^\omega (d-a)\alpha-\frac{\alpha^2}{\omega}(d-c+b-a)d\alpha$$
$$=\frac{\omega^2}{6}[(d+c)-(b+a)+(c-b)].$$

Therefore,

$$\bar{y}_0(A)=\frac{\int_0^\omega \alpha|A^\alpha|\,d\alpha}{\int_0^\omega |A^\alpha|\,d\alpha}=\frac{\omega}{3}[1+\frac{c-b}{(d+c)-(a+b)}]$$

which coincides with (15). Compared to (3), $\bar{y}_0(A)$ is obtained more easily by using (9).

In the papers [1-3], [8] the authors used the centroid-based distance method to rank fuzzy numbers. It is noticed that Theorem 1 also suggests a method for ranking the same shaped fuzzy numbers. We can rank the same shaped fuzzy numbers directly by their centriod coordinates on the horizontal axis since they all have the same coordinate on the vertical axis.

## 5. Conclusions

This paper presents a pair of correct centroid formulae for fuzzy numbers. The proposed formulae are simple and have consistent expressions on the horizontal and vertical axis. Those centroid formulae proposed by [1-3], [8] all used the inverse function of the membership function to determine the centorid on the vertical axis. This is a more restrictive condition that the membership functions must be invertable. The proposed formulae are unlimited, so that they are available for all fuzzy numbers. This is a significant improvement on the formulae proposed by [1-3], [8]. Since the centoid points are applied to ranking fuzzy numbers [2], [3] how to find the correct centoid formulae becomes an important issue. Therefore, the proposed formulae are of great value to ranking approaches.

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